

Averaging approximation to singularly perturbed nonlinear stochastic wave equations

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Abstract

An averaging method is applied to derive effective approximation to the following singularly perturbed nonlinear stochastic damped wave equation

$$\nu u_{tt} + u_t = \Delta u + f(u) + \nu^\alpha \dot{W}$$

on an open bounded domain $D \subset \mathbb{R}^n$, $1 \leq n \leq 3$. Here $\nu > 0$ is a small parameter characterising the singular perturbation, and ν^α , $0 \leq \alpha \leq 1/2$, parametrises the strength of the noise. Some scaling transformations and the martingale representation theorem yield the following effective approximation for small ν ,

$$u_t = \Delta u + f(u) + \nu^\alpha \dot{W}$$

to an error of $o(\nu^\alpha)$.

Keywords stochastic nonlinear wave equations, averaging, tightness, martingale.

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1 Introduction

Wave motion is one of the most commonly observed physical phenomena, and typically described by hyperbolic partial differential equations. Nonlinear wave equations also have been studied a great deal in many modern problems such as sonic booms, bottlenecks in traffic flows, nonlinear optics and quantum field theory [13, 20, e.g.]. However, for many problems, such as wave propagation through the atmosphere or the ocean, the presence of turbulence causes random fluctuations. More realistic models must account for such random fluctuations. Hence we study stochastic wave equations [7, 9, e.g.].

Here we study an effective approximation, in the sense of distribution, for the following nonlinear wave stochastic partial differential equation (SPDE). The SPDE is a singularly perturbed problem on a bounded open domain $D \subset \mathbb{R}^n$, $1 \leq n \leq 3$,

$$\nu u_{tt}^\nu + u_t^\nu = \Delta u^\nu + f(u^\nu) + \nu^\alpha \dot{W}, \quad u^\nu(0) = u_0, \quad u_t^\nu(0) = u_1, \quad (1)$$

with zero Dirichlet boundary on D . Here ν^α with $0 < \nu \leq 1$ and $0 \leq \alpha \leq 1/2$ parametrises the strength of noise, and Δ is the Laplace operator in \mathbb{R}^n . The noise $W(t)$ is an infinite dimensional Q-Wiener process which is detailed in section 2. The SPDE (1) also describes the motion of a small particle with mass ν and an infinite number of degrees of freedom [4, 5]. We are concerned with the effective approximation of the solution to the SPDE (1) for small $\nu > 0$.

For $\alpha = 1/2$, the limit of the random dynamics of SPDE (1) as $\nu \rightarrow 0$ has been studied by Lv and Wang [11, 18]. The random attractor and measure attractor of SPDE (1) are approximated by those of the deterministic PDE

$$u_t = \Delta u + f(u) \quad (2)$$

as $\nu \rightarrow 0$ in the almost sure sense [11] and weak topology [18] respectively.

The important case of $\alpha = 0$, which is an infinite dimensional version of the Smolukowski–Kramers approximation, is studied by analysing the structure of solution of linear stochastic wave equations [4, 5]. For any $T > 0$, the solution $u(t)$ to the SPDE (1) is approximated in probability by that of the stochastic system

$$u_t = \Delta u + f(u) + \dot{W}$$

as $\nu \rightarrow 0$ in space $C(0, T; L^2(D))$.

Here we extend the approximating result to the case when $0 \leq \alpha \leq 1/2$ and derive a higher order approximation in the sense of distribution. Recently, the stochastic averaging approach was developed to study the effective

approximation to slow-fast SPDEs [6, 19] in the following form

$$\begin{aligned} u_t^\nu &= \Delta u^\nu + f(u^\nu, v^\nu) + \sigma_1 \dot{W}_1, \\ v_t^\nu &= \frac{1}{\nu} [\Delta v^\nu + g(u^\nu, v^\nu)] + \frac{\sigma_2}{\sqrt{\nu}} \dot{W}_2, \end{aligned}$$

where f and g are nonlinear terms, σ_1 and σ_2 are some constants, and W_1 and W_2 are Wiener processes. Notice that upon introducing $v^\nu = u_t^\nu$, the SPDE (1) is rewritten as

$$\begin{aligned} u_t^\nu &= v^\nu, \quad u^\nu(0) = u_0, \\ v_t^\nu &= \frac{1}{\nu} [-v^\nu + \Delta u^\nu + f(u^\nu)] + \frac{1}{\nu^{1-\alpha}} \dot{W}, \quad v^\nu(0) = u_1, \end{aligned}$$

which are also in the form of slow-fast SPDEs. Then we can follow the stochastic averaging approach to derive an effective averaging approximation of u^ν , the solution of SPDE (1) as $\nu \rightarrow 0$ for all $0 \leq \alpha \leq 1/2$. Here the case $\alpha = 1/2$ is the most important case because all cases of $\alpha \in [0, 1/2]$ can be transformed to the case $\alpha = 1/2$, see section 4 and section 5. By an averaging approach and martingale representation theorem, we prove that for small $\nu > 0$ with $0 \leq \alpha \leq 1/2$ the solution of SPDE (1) is approximated in the sense of distribution by \bar{u}^ν which solves

$$\bar{u}_t^\nu = \Delta \bar{u}^\nu + f(\bar{u}^\nu) + \nu^\alpha \dot{\bar{W}}, \quad \bar{u}^\nu(0) = u_0, \quad (3)$$

where $\bar{W}(t)$ is a Wiener process distributes same as $W(t)$. This result shows that for any small $\nu > 0$ with $0 \leq \alpha \leq 1/2$ the term $\nu u_t^\nu(t)$ is a higher order term than the random force term $\nu^\alpha W(t)$.

Section 3 gives the approximation for the important case that $\alpha = 1/2$. Previous research [11] gives an approximation which is a deterministic equation. However, our new approximation shows that for small $\nu \neq 0$, there is a small fluctuation which distributes same as $\sqrt{\nu}W(t)$, see (3). This gives a more effective approximation.

Section 6 explores a parameter regime where a nonlinear coordinate transformation underlies the existence of a stochastic slow manifold for the case $\alpha = 0$. The stochastic slow manifolds of both the SPDE (1) and the model (3) have the same evolution in the parameter regime and so provide evidence of the stronger result of pathwise approximation therein.

2 Preliminaries

Let $D \subset \mathbb{R}^n$, $1 \leq n \leq 3$, be a regular domain with boundary Γ . Denote by $L^2(D)$ the Lebesgue space of square integrable real valued functions on D ,

which is a Hilbert space with inner product

$$\langle u, v \rangle = \int_D u(x)v(x) dx, \quad u, v \in L^2(D).$$

Write the norm on $L^2(D)$ by $\|u\|_0 = \langle u, u \rangle^{1/2}$. Define the following abstract operator

$$Au = -\Delta u, \quad u \in \text{Dom}(A) = \{u \in L^2(D) : Au \in L^2(D), u|_\Gamma = 0\}.$$

Denote by $\{\lambda_k\}$ the eigenvalues of A with $0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots$, $\lambda_k \rightarrow \infty$ as $k \rightarrow \infty$. For any $s \in \mathbb{R}$, introduce the space $H_0^s(D) = \text{Dom}(A^{s/2})$ endowed with the norm

$$\|u\|_s = \|A^{s/2}u\|_0, \quad u \in H_0^s(D).$$

Consider the following singularly perturbed stochastic wave equation with cubic nonlinearity on D :

$$\nu u_{tt}^\nu + u_t^\nu = \Delta u^\nu + \beta u^\nu - (u^\nu)^3 + \nu^\alpha \dot{W}(t), \quad (4)$$

$$u^\nu(0) = u_0, \quad u_t^\nu(0) = u_1, \quad (5)$$

$$u^\nu|_\Gamma = 0, \quad (6)$$

with $0 < \nu < 1$ and $0 \leq \alpha \leq 1/2$. Here $\{W(t)\}_{t \in \mathbb{R}}$ is an $L^2(D)$ -valued two sided Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with covariance operator Q such that

$$Qe_k = b_k e_k, \quad k = 1, 2, \dots,$$

where $\{e_k\}$ is a complete orthonormal system in $L^2(D)$, b_k is a bounded sequence of non-negative real numbers. Then

$$W(t) = \sum_{k=1}^{\infty} \sqrt{b_k} e_k w_k(t),$$

where w_k are real mutually independent Brownian motions [12]. Further, we assume

$$B_0 = \sum_{k=1}^{\infty} b_k < \infty \quad \text{and} \quad B_1 = \sum_{k=1}^{\infty} \lambda_k b_k < \infty. \quad (7)$$

Then by a standard method [8], for any $(u_0, u_1) \in H_0^{s+1}(D) \times H^s(D)$, $s \in \mathbb{R}$, there is a unique solution u^ν to (4)–(6),

$$u^\nu \in L^2(\Omega, C(0, T; H_0^{s+1}(D))), \quad (8)$$

$$u_t^\nu \in L^2(\Omega, C(0, T; H^s(D))). \quad (9)$$

In the following we write $f(u) = \beta u - u^3$ and $F(u) = \int_0^u f(r) dr$.

For our purpose we need the following lemma.

Lemma 1 (Simon [17]). *Assume E , E_0 and E_1 be Banach spaces such that $E_1 \Subset E_0$, the interpolation space $(E_0, E_1)_{\theta,1} \subset E$ with $\theta \in (0,1)$ and $E \subset E_0$ with \subset and \Subset denoting continuous and compact embedding respectively. Suppose $p_0, p_1 \in [1, \infty]$ and $T > 0$, such that*

$$\mathcal{V} \text{ is a bounded set in } L^{p_1}(0, T; E_1), \quad \text{and}$$

$$\partial\mathcal{V} := \{\partial v : v \in \mathcal{V}\} \text{ is a bounded set in } L^{p_0}(0, T; E_0).$$

Here ∂ denotes the distributional derivative. If $1 - \theta > 1/p_\theta$ with

$$\frac{1}{p_\theta} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1},$$

then \mathcal{V} is relatively compact in $C(0, T; E)$.

In the following, for any $T > 0$, we denote by C_T a generic positive constant which is independent of ν .

3 The case of $\alpha = 1/2$

We first consider the special case of $\alpha = 1/2$ which was recently studied by a direct approximation method [11, 18]. Here we apply an averaging method to give more effective approximation to equation (4)–(6). We rewrite (4)–(6) in the form of slow-fast SPDEs:

$$du^\nu = v^\nu dt, \quad u^\nu(0) = u_0, \quad (10)$$

$$dv^\nu = -\frac{1}{\nu} [v^\nu - \Delta u^\nu - f(u^\nu)] dt + \frac{1}{\sqrt{\nu}} dW(t), \quad v^\nu(0) = u_1. \quad (11)$$

Notice that the slow part u^ν and fast part v^ν are linearly coupled. For simplicity we consider $(u_0, u_1) \in (H^2(D) \cap H_0^1(D)) \times H^1(D)$. Then (4)–(6) has a unique solution in $L^2(\Omega, C(0, T; (H^2(D) \cap H_0^1(D)) \times H^1(D)))$.

3.1 Tightness of solutions

Let (u^ν, v^ν) be a solution to (10)–(11) with $\nu > 0$. In order to pass to the limit $\nu \rightarrow 0$ in the averaging approach, we need some a priori estimates on the solutions.

Theorem 1. *Assume $B_1 < \infty$. For any $T > 0$, there is a positive constant C_T such that*

$$\mathbb{E} \left[\max_{0 \leq t \leq T} \|u^\nu(t)\|_2^2 + \max_{0 \leq t \leq T} \|v^\nu(t)\|_0^2 \right] \leq C_T, \quad (12)$$

and for any integer $m > 0$

$$\mathbb{E} \int_0^T \|u^\nu(t)\|_1^{2m} dt \leq C_T.$$

Proof. The result on $\|u^\nu(t)\|_2$ is found by a simple energy estimate [18]. Now we give the estimate on $\|v^\nu(t)\|_0$. By equation (11),

$$\begin{aligned} v^\nu(t) &= e^{-t/\nu} u_1 + \frac{1}{\nu} \int_0^t e^{-(t-s)/\nu} [\Delta u^\nu(s) + f(u^\nu(s))] ds \\ &\quad + \frac{1}{\sqrt{\nu}} \int_0^t e^{-(t-s)/\nu} dW(s). \end{aligned}$$

Noticing assumption (7), by the estimate on $\|u^\nu(t)\|_2$ and maximal inequality of stochastic convolution [12, Lemma 7.2],

$$\mathbb{E} \left[\max_{0 \leq t \leq T} \|v^\nu(t)\|_0^2 \right] \leq C_T$$

for some positive constant C_T . The last inequality of the theorem is obtained by the same method [18] and Poincaré inequality. This completes the proof. \square

Now by the above estimates and Lemma 1, we have the following theorem.

Theorem 2. *For any $T > 0$, $\{\mathcal{L}(u^\nu)\}_{0 < \nu \leq 1}$ the distribution of u^ν is tight in the space $C(0, T; H_0^1(D))$.*

By the above tightness result, to determine the limit of u^ν we can pass to the limit $\nu \rightarrow 0$ in a weak sense; that is, for any $\varphi \in C_0^\infty(D)$, we consider the limit of $u^{\nu, \varphi}(t) = \langle u^\nu(t), \varphi \rangle$ in the space $C(0, T)$ as $\nu \rightarrow 0$.

3.2 Limit of $u^{\nu, \varphi}$ in $C(0, T)$

Now we pass to the limit $\nu \rightarrow 0$ in $\{u^{\nu, \varphi}\}$ in the space $C(0, T)$ for any $T > 0$. First, by equations (10)–(11),

$$du^{\nu, \varphi} = v^{\nu, \varphi} dt, \tag{13}$$

$$dv^{\nu, \varphi} = -\frac{1}{\nu} [v^{\nu, \varphi} + \langle \nabla u^\nu, \nabla \varphi \rangle - \langle f(u^\nu), \varphi \rangle] dt + \frac{1}{\sqrt{\nu}} dW^\varphi(t), \tag{14}$$

with $u^{\nu, \varphi}(0) = \langle u_0, \varphi \rangle$ and $v^{\nu, \varphi}(0) = \langle u_1, \varphi \rangle$ where $v^{\nu, \varphi} = \langle v^\nu, \varphi \rangle$ and $W^\varphi(t) = \langle W(t), \varphi \rangle$. In the following we also write $v^{\nu, \varphi}$ as $v^{\nu, \varphi, u(t)}$ which shows the dependence of $v^{\nu, \varphi}$ on the slow part u^ν .

Second, for any fixed $u \in H^2(D) \cap H_0^1(D)$ we consider the fast equation

$$dv^{\nu,u} = -\frac{1}{\nu} [v^{\nu,u} - \Delta u - f(u)] dt + \frac{1}{\sqrt{\nu}} dW(t). \quad (15)$$

Equation (15) has a unique stationary solution $\bar{v}^{\nu,u}$. Moreover, the stationary solution $\bar{v}^{\nu,u}$ is exponentially mixing and the distribution of $\bar{v}^{\nu,u}$ is the normal distribution $\mathcal{N}(\Delta u + f(u), Q/2)$ [4].

Now for any $u \in H^2(D) \cap H_0^1(D)$ define

$$H^\nu(u, t) = \nu [v^{\nu,u}(t) - v^{\nu,u}(0)] + \int_0^t [v^{\nu,u}(s) - \Delta u - f(u)] ds.$$

Then $u^{\nu,\varphi}$ solves the following equation

$$\begin{aligned} u^{\nu,\varphi}(t) &= \langle u_0, \varphi \rangle - \int_0^t [\langle \nabla u^\nu(s), \nabla \varphi \rangle - \langle f(u^\nu(s)), \varphi \rangle] ds \\ &\quad + \langle H^\nu(u^\nu(t), t), \varphi \rangle - \nu \langle v^{\nu,u}(t) - v^{\nu,u}(0), \varphi \rangle. \end{aligned} \quad (16)$$

Third, we study the behaviour of $\langle H^\nu(u^\nu(t), t), \varphi \rangle$ for small ν . Let $H^{\nu,\varphi}(u, t) = \langle H^\nu(u, t), \varphi \rangle$, then define

$$M_t^{\nu,\varphi} = \frac{1}{\sqrt{\nu}} H^{\nu,\varphi}(u^\nu(t), t). \quad (17)$$

By the definition of $H^{\nu,\varphi}(u, t)$ and equation (15), $M_t^{\nu,\varphi}$ is a martingale with respect to $\{\mathcal{F}_t : t \geq 0\}$, and the quadratic covariance is $\langle M^{\nu,\varphi} \rangle_t = t \langle Q \varphi, \varphi \rangle$.

Now define $R^{\nu,\varphi}(t) = -\langle v^{\nu,u}(t) - v^{\nu,u}(0), \varphi \rangle$, then rewrite (16) as

$$u^{\nu,\varphi}(t) = \langle u_0, \varphi \rangle - \int_0^t [\langle \nabla u^\nu(s), \nabla \varphi \rangle - \langle f(u^\nu(s)), \varphi \rangle] ds + \sqrt{\nu} M_t^{\nu,\varphi} + \nu R^{\nu,\varphi}(t). \quad (18)$$

Invoking Theorem 1,

$$\lim_{\nu \rightarrow 0} \mathbb{E} \left[\max_{0 \leq t \leq T} \sqrt{\nu} |R^{\nu,\varphi}(t)| \right] = 0. \quad (19)$$

Then define the process

$$\mathcal{M}_t^{\nu,\varphi} = \frac{1}{\sqrt{\nu}} \left\{ u^{\nu,\varphi}(t) - \langle u_0, \varphi \rangle + \int_0^t [\langle \nabla u^\nu(s), \nabla \varphi \rangle - \langle f(u^\nu(s)), \varphi \rangle] ds \right\}. \quad (20)$$

By the definition of $H^{\nu,\varphi}(u, t)$ and (19) we have the tightness of $\mathcal{M}_t^{\nu,\varphi}$ in space $C(0, T)$ for any $T > 0$. Let P be a limit point of the family of probability measures $\{\mathcal{L}(\mathcal{M}_t^{\nu,\varphi})\}_{0 < \nu \leq 1}$ and denote by \mathcal{M}_t^φ , a $C(0, T)$ -valued random variable with distribution P . Let Ψ be a continuous bounded function on $C(0, T)$. Set $\Psi^\nu(s) = \Psi(u^{\nu,\varphi}(s))$, then noticing (19),

$$\mathbb{E}[(\mathcal{M}_t^{\nu,\varphi} - \mathcal{M}_s^{\nu,\varphi})\Psi^\nu(s)] = \mathbb{E}[\sqrt{\nu}(R^{\nu,\varphi}(t) - R^{\nu,\varphi}(s))\Psi^\nu(s)] \rightarrow 0, \quad \nu \rightarrow 0,$$

which yields that the process $\{\mathcal{M}_t^\varphi\}_{0 \leq t \leq T}$ is a P -martingale with respect to the Borel σ -filter of $C(0, T)$.

We consider the quadratic covariation of the martingale \mathcal{M}_t^φ . By the definition of $\mathcal{M}_t^{\nu,\varphi}$, passing to the limit $\nu \rightarrow 0$ in (20), we derive \mathcal{M}_t^φ is a square integrable martingale with the associated quadratic covariation process is $\langle Q\varphi, \varphi \rangle t$. Then by the representation theorem for martingales [10], without changing the distributions of $\mathcal{M}_t^{\nu,\varphi}$ and \mathcal{M}_t^φ , one extends the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and chooses a new Wiener process $\hat{W}^\varphi(t)$ such that $\mathcal{M}_t^\varphi = \sqrt{Q}\hat{W}^\varphi(t)$, which is unique in the sense of distribution.

By the definition of $\mathcal{M}_t^{\nu,\varphi}$, \hat{W}^φ can be chosen as $\langle \hat{W}, \varphi \rangle$ where \hat{W} is a cylindrical Wiener process. Then from (20) we have in the sense of distribution

$$\begin{aligned} & \langle u^\nu(t), \varphi \rangle \\ &= \langle u_0, \varphi \rangle - \int_0^t [\langle \nabla u^\nu(s), \nabla \varphi \rangle - \langle f(u^\nu(s)), \varphi \rangle] ds + \sqrt{\nu} \mathcal{M}_t^\varphi + o(\sqrt{\nu}) \\ &= \langle u_0, \varphi \rangle - \int_0^t [\langle \nabla u^\nu(s), \nabla \varphi \rangle - \langle f(u^\nu(s)), \varphi \rangle] ds + \sqrt{\nu} \sqrt{Q} \langle \hat{W}, \varphi \rangle + o(\sqrt{\nu}) \end{aligned}$$

for any $\varphi \in C_0^\infty(D)$. Then by discarding the higher order term and the tightness of u^ν , we have the following approximating equation

$$d\bar{u}^\nu = [\Delta \bar{u}^\nu + f(\bar{u}^\nu)] dt + \sqrt{\nu} d\bar{W}^Q, \quad (21)$$

where \bar{W}^Q is some an $L^2(D)$ valued Q-Wiener process.

Theorem 3. *Assume $B_1 < \infty$ and $\alpha = 1/2$. For small $\nu > 0$, there is a new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, an extension of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that for any $T > 0$, the solution u^ν to (10)–(11) is approximated by \bar{u}^ν which solves (21), to an error of $o(\sqrt{\nu})$, in the space $C(0, T; H_0^1(D))$ for almost all $\omega \in \bar{\Omega}$.*

The above SPDE (21) is more effective than the limit PDE (2) [11] as it incorporates fluctuations for small $\nu > 0$. This result also implies that the singular term $\nu u_t^\nu(t)$ is a higher order term than $\sqrt{\nu}W(t)$ for small $\nu > 0$ in the sense of distribution at least. The following two sections show that $\nu u_t^\nu(t)$ is always a higher order term than $\nu^\alpha W(t)$ for any $0 \leq \alpha \leq 1/2$.

4 The case of $\alpha = 0$

Next we consider the case of $\alpha = 0$; that is, consider the following SPDE

$$\nu u_{tt}^\nu + u_t^\nu = \Delta u^\nu + \beta u^\nu - (u^\nu)^3 + \dot{W}(t), \quad (22)$$

$$u^\nu(0) = u_0, \quad u_t^\nu(0) = u_1, \quad (23)$$

$$u^\nu|_\Gamma = 0. \quad (24)$$

First we have the following a priori estimates on u^ν in the space $C(0, T; H_0^1(D))$.

Theorem 4 (Cerrai & Freidlin [5]). *Assume $B_1 < \infty$. For any $T > 0$, there is a positive constant C_T such that*

$$\mathbb{E} \left[\max_{0 \leq t \leq T} \|u^\nu(t)\|_1^2 \right] \leq C_T.$$

We follow the approach for the case of $\alpha = 1/2$. For this we introduce the scalings $\tilde{u}^\nu = \sqrt{\nu}u^\nu$ and $\tilde{v}^\nu = \sqrt{\nu}u_t^\nu$. Then

$$\begin{aligned} d\tilde{u}^\nu &= \tilde{v}^\nu dt, \quad \tilde{u}^\nu(0) = \sqrt{\nu}u_0, \\ d\tilde{v}^\nu &= -\frac{1}{\nu} \left[\tilde{v}^\nu - \Delta \tilde{u}^\nu - \sqrt{\nu} f \left(\frac{\tilde{u}^\nu}{\sqrt{\nu}} \right) \right] dt + \frac{1}{\sqrt{\nu}} dW(t), \quad \tilde{v}^\nu(0) = \sqrt{\nu}u_1. \end{aligned}$$

By standard energy estimates [18], by a similar discussion to that in Section 3, and by Theorem 4, we have the following theorem.

Theorem 5. *Assume $B_1 < \infty$. For any $T > 0$, there is a positive constant C_T such that*

$$\mathbb{E} \left[\max_{0 \leq t \leq T} \|\tilde{u}^\nu(t)\|_2^2 + \max_{0 \leq t \leq T} \|\tilde{v}^\nu(t)\|_0^2 \right] \leq C_T,$$

and for any integer $m > 0$

$$\mathbb{E} \int_0^T \|\tilde{u}^\nu(t)\|_1^{2m} dt \leq C_T.$$

Moreover, the distribution of \tilde{u}^ν is tight in space $C(0, T; H_0^1(D))$.

We consider the asymptotic approximation of \tilde{u}^ν for small $\nu > 0$. For any $\varphi \in C_0^\infty(D)$, let $\tilde{u}^{\nu, \varphi} = \langle \tilde{u}^\nu, \varphi \rangle$, $\tilde{v}^{\nu, \varphi} = \langle \tilde{v}^\nu, \varphi \rangle$ and $W^\varphi(t) = \langle W(t), \varphi \rangle$. Then

$$\begin{aligned} d\tilde{u}^{\nu, \varphi} &= \tilde{v}^{\nu, \varphi} dt, \\ d\tilde{v}^{\nu, \varphi} &= -\frac{1}{\nu} \left[\tilde{v}^{\nu, \varphi} + \langle \nabla \tilde{u}^\nu, \nabla \varphi \rangle - \sqrt{\nu} \langle f(\tilde{u}^\nu / \sqrt{\nu}), \varphi \rangle \right] dt + \frac{1}{\sqrt{\nu}} dW^\varphi(t), \end{aligned}$$

with $\tilde{u}^{\nu,\varphi}(0) = \langle \tilde{u}^\nu(0), \varphi \rangle$ and $\tilde{v}^{\nu,\varphi}(0) = \langle \tilde{v}^\nu(0), \varphi \rangle$.

We also consider the following fast SPDE for fixed ν and $\tilde{u} \in H^2(D) \cap H_0^1(D)$:

$$d\tilde{v}^{\nu,\tilde{u}} = -\frac{1}{\nu} [\tilde{v}^{\nu,\tilde{u}} - \Delta\tilde{u} - \sqrt{\nu}f(\tilde{u}/\sqrt{\nu})] dt + \frac{1}{\sqrt{\nu}} dW(t). \quad (25)$$

For fixed $\nu \in (0, 1]$ and $\tilde{u} \in H^2(D) \cap H_0^1(D)$, SPDE (25) has a unique stationary solution with the normal distribution $\mathcal{N}(\Delta\tilde{u} + \sqrt{\nu}f(\tilde{u}/\sqrt{\nu}), Q/2)$ [4]. Now for any $\tilde{u} \in H^2(D) \cap H_0^1(D)$ define

$$\tilde{H}^\nu(\tilde{u}, t) = \nu [\tilde{v}^{\nu,\tilde{u}}(t) - \tilde{v}^{\nu,\tilde{u}}(0)] + \int_0^t [\tilde{v}^{\nu,\tilde{u}}(s) - \Delta\tilde{u} - \sqrt{\nu}f(\tilde{u}/\sqrt{\nu})] ds.$$

Thus we can follow the same discussion in last section for the case of $\alpha = 1/2$. We write

$$\begin{aligned} \tilde{u}^{\nu,\varphi}(t) &= \sqrt{\nu}\langle u_0, \varphi \rangle - \int_0^t \langle \nabla\tilde{u}^\nu(s), \nabla\varphi \rangle ds + \sqrt{\nu} \int_0^t \langle f(\tilde{u}^\nu(s)/\sqrt{\nu}), \varphi \rangle ds \\ &\quad + \sqrt{\nu}\tilde{\mathcal{M}}_t^{\nu,\varphi}, \end{aligned} \quad (26)$$

where $\sqrt{\nu}\tilde{\mathcal{M}}_t^{\nu,\varphi}$ is the remainder term. By a similar discussion to that of the last section, $\tilde{\mathcal{M}}_t^{\nu,\varphi}$ is tight in space $C(0, T)$ for any $T > 0$. Let \tilde{P} be a limit point of the family of probability measures $\mathcal{L}\{\tilde{\mathcal{M}}_t^{\nu,\varphi}\}_{0 < \nu \leq 1}$ in space $C(0, T)$. Let $\tilde{\mathcal{M}}_t^\varphi$ be a $C(0, T)$ -valued random variable with distribution \tilde{P} . Then we have the following lemma.

Lemma 2. *For any $\varphi \in C_0^\infty(D)$, the process $\tilde{\mathcal{M}}_t^\varphi$ defined on the probability space $(C(0, T), \mathcal{B}(C(0, T)), \tilde{P})$ is a square integrable martingale with the associated quadratic covariation process $\langle Q\varphi, \varphi \rangle t$.*

By the representation theorem for martingales [10], without changing the distributions of $\tilde{\mathcal{M}}_t^{\nu,\varphi}$ and $\tilde{\mathcal{M}}_t^\varphi$ one can extend the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and choose a new cylindrical Wiener process $\tilde{W}(t)$ such that $\tilde{\mathcal{M}}_t^\varphi = \sqrt{Q}\langle \tilde{W}, \varphi \rangle$, which is unique in the sense of distribution.

Then in the sense of distribution by (26) we write out

$$\begin{aligned} \langle \tilde{u}^\nu(t), \varphi \rangle &= \sqrt{\nu}\langle u_0, \varphi \rangle - \int_0^t \langle \nabla\tilde{u}^\nu(s), \nabla\varphi \rangle ds + \sqrt{\nu} \int_0^t \langle f(\tilde{u}^\nu(s)/\sqrt{\nu}), \varphi \rangle ds \\ &\quad + \sqrt{\nu}\tilde{\mathcal{M}}_t^{\nu,\varphi} + o(\sqrt{\nu}) \\ &= \sqrt{\nu}\langle u_0, \varphi \rangle - \int_0^t \langle \nabla\tilde{u}^\nu(s), \nabla\varphi \rangle ds + \sqrt{\nu} \int_0^t \langle f(\tilde{u}^\nu(s)/\sqrt{\nu}), \varphi \rangle ds \\ &\quad + \sqrt{\nu}\sqrt{Q}\langle \tilde{W}, \varphi \rangle + o(\sqrt{\nu}) \end{aligned} \quad (27)$$

for any $\varphi \in C_0^\infty(D)$. Then we have, noticing that $\tilde{u}^\nu = \sqrt{\nu}u^\nu$, the following approximating SPDE for small $\nu > 0$:

$$d\bar{u}^\nu = [\Delta\bar{u}^\nu + f(\bar{u}^\nu)]dt + d\bar{W}^Q, \quad \bar{u}^\nu(0) = u_0, \quad (28)$$

where \bar{W}^Q is some $L^2(D)$ valued Q-Wiener process. Then we infer the following result.

Theorem 6. *Assume $B_1 < \infty$ and $\alpha = 0$. Then for small $\nu > 0$, there is a new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ which is an extension of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $T > 0$, the solution u^ν to (22)–(24) is approximated by \bar{u}^ν which solves (28), to an error of $o(1)$, in the space $C(0, T; H_0^1(D))$ for almost all $\omega \in \bar{\Omega}$.*

5 The case of $0 < \alpha < 1/2$

Now we consider the case of $0 < \alpha < 1/2$; that is, consider the following SPDE

$$\nu u_{tt}^\nu + u_t^\nu = \Delta u^\nu + \beta u^\nu - (u^\nu)^3 + \nu^\alpha \dot{W}(t), \quad (29)$$

$$u^\nu(0) = u_0, \quad u_t^\nu(0) = u_1, \quad (30)$$

$$u^\nu|_\Gamma = 0. \quad (31)$$

First, by the same analysis as Theorem 4, we also have the following result on the a priori estimates on u^ν .

Theorem 7 (Cerrai & Freidlin [5]). *Assume $B_1 < \infty$. For any $T > 0$, there is a positive constant C_T such that*

$$\mathbb{E} \left[\max_{0 \leq t \leq T} \|u^\nu(t)\|_1^2 \right] \leq C_T.$$

We also apply the method in Section 3. Make the following scaling transformation $\tilde{u}^\nu = \nu^{1/2-\alpha}u^\nu$ and $\tilde{v}^\nu = \nu^{1/2-\alpha}v^\nu$. Then

$$\begin{aligned} d\tilde{u}^\nu &= \tilde{v}^\nu dt, \\ d\tilde{v}^\nu &= -\frac{1}{\nu} \left[\tilde{v}^\nu - \Delta\tilde{u}^\nu - \nu^{1/2-\alpha} f \left(\frac{\tilde{u}^\nu}{\nu^{1/2-\alpha}} \right) \right] dt + \frac{1}{\sqrt{\nu}} dW(t), \\ \tilde{u}^\nu(0) &= \nu^{1/2-\alpha}u_0, \quad \tilde{v}^\nu(0) = \nu^{1/2-\alpha}u_1. \end{aligned}$$

By a direct energy estimate or the scaling transformation and Theorem 7 we deduce the following theorem.

Theorem 8. Assume $B_1 < \infty$. For any $T > 0$, there is a positive constant C_T such that

$$\mathbb{E} \left[\max_{0 \leq t \leq T} \|\tilde{u}^\nu(t)\|_2^2 + \max_{0 \leq t \leq T} \|\tilde{v}^\nu(t)\|_0^2 \right] \leq C_T,$$

and for any integer $m > 0$

$$\mathbb{E} \int_0^T \|\tilde{u}^\nu(t)\|_1^{2m} dt \leq C_T.$$

Moreover, the distribution of \tilde{u}^ν is tight in space $C(0, T; H_0^1(D))$.

Then we can follow the same discussion of Section 4 and have the following result.

Theorem 9. Assume $B_1 < \infty$ and $0 < \alpha < 1/2$. For small $\nu > 0$, there is a new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ which is an extension of the original probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $T > 0$, the solution u^ν to (29)–(31) is approximated by \bar{u}^ν which solves

$$d\bar{u}^\nu = [\Delta \bar{u}^\nu + f(\bar{u}^\nu)]dt + \nu^\alpha d\bar{W}^Q, \quad \bar{u}^\nu(0) = u_0, \quad (32)$$

to an error of $o(\nu^\alpha)$, in the space $C(0, T; H_0^1(D))$ for almost all $\omega \in \bar{\Omega}$.

6 A stochastic slow manifold compares the SPDEs for the case of $\alpha = 0$

This section shows the long time effectiveness of the averaged model by comparing it to the original via their stochastic slow manifolds.

We compare the SPDE (22) and its model SPDE (28) in a parameter regime where both have an accessible stochastic slow manifold. Consider the SPDE (22) restricted to one spatial dimension as

$$\nu u_{tt} + u_t = u_{xx} + f(u) + \sigma \dot{W} \quad \text{where} \quad f = (1 + \beta')u - u^3. \quad (33)$$

Consider this SPDE on the non-dimensional domain $D = (0, \pi)$ with boundary conditions $u = 0$ on $x = 0, \pi$. The parameter σ here explicitly measures the overall size of the Q-Wiener process $W(t)$ which by (7) is finite. The small parameter β' measures the distance from the stochastic bifurcation that occurs near $\beta' = 0$. In this domain there will be a stochastic slow manifold of the SPDE (33) that matches the slow dynamics in the approximating SPDE (28). This section compares the stochastic slow manifolds.

The SPDE (33) has a technically challenging spectrum. However, the construction of its stochastic slow manifold is easiest by embedding the SPDE (33) as the $\gamma = 1$ case of the following slow-fast system of SPDEs

$$u_t = u_{xx} + u + v, \quad (34)$$

$$\nu v_t = -v - \gamma\nu(\partial_{xx} + 1)u_t + \beta'u - u^3 + \sigma\dot{W}. \quad (35)$$

The parameter γ controls the homotopy: from a tractable base when $\gamma = 0$ as then all linear modes in the very fast v equation (35) decay at the same rate $1/\nu$ (and the slow u modes of $\sin kx$ have decay rates $1 - k^2$); to the original SPDE (33) when $\gamma = 1$ (upon eliminating v).

A stochastic slow manifold appears On the non-dimensional interval $(0, \pi)$, with Dirichlet boundary conditions on u , the eigenmodes must be proportional to $\sin kx$ for integer wavenumber k . Neglecting noise temporarily, $\sigma = 0$ in this sentence, for all $\nu < 1$ and all homotopy parameter $0 \leq \gamma \leq 1$ there is one zero eigenvalue and all the rest of the eigenvalues have negative real part; the slow subspace corresponding to the neutral mode is spanned by $(u, v) \propto (\sin x, 0)$ (local in (u, v, σ) , but global in ν and γ). By stochastic center manifold theory [1, 3], and supported by stochastic normal form transformations [2, 15, 16], when the noise spectrum truncates and the nonlinearity is small enough, the dynamics of the SPDEs (34)–(35) are essentially finite dimensional and a stochastic slow manifold exists which is exponentially quickly attractive to all nearby trajectories.

Computer algebra constructs the stochastic slow manifold We seek the stochastic slow manifold as a systematic perturbation of the slow subspace $u = a \sin x$. The intricate algebra necessary to handle the multitude of nonlinear noise interactions is best left to a computer [14, 16, e.g.]. However, the following expressions may be checked by substituting into the governing SPDEs (34)–(35) and confirming the order of the residuals is as small as quoted—albeit tedious, this check is considerably easier than the derivation. The evolution on the stochastic slow manifold may be written

$$\begin{aligned} \dot{a} = & \beta'a - \frac{3}{4}a^3 + \left[1 - 2\nu\beta' + \frac{9}{2}\nu a^2 - \frac{9}{1024}a^4\right] b_1\dot{w}_1 \\ & + \left[\left(\frac{3}{32} + \frac{3}{128}\beta'\right)a^2 - \frac{21}{1024}a^4\right] b_3\dot{w}_3 + \frac{5}{1024}a^4 b_5\dot{w}_5 + o(\nu^2 + \beta'^2 + a^4, \sigma) \end{aligned} \quad (36)$$

The stochastic slow manifold itself involves Ornstein–Uhlenbeck processes written as convolutions over the past history of the noise processes: define

$e^{-\mu t} \star \dot{w} = \int_{-\infty}^t \exp[-\mu(t-s)] dw_s$ for decay rates $\mu_k = k^2 - 1$ characteristic of the k th mode. Then the stochastic slow manifold is

$$\begin{aligned}
u &= a \sin x + \frac{1}{32} a^3 \sin 3x - \frac{3}{32} a^2 [b_3 e^{-8t} \star w_3 \sin x + b_1 e^{-8t} \star w_1 \sin 3x] \\
&+ \sum_{k \geq 2} b_k [1 + \mu_k \nu + \gamma \nu (\mu_k - \mu_k^2 e^{-\mu_k t} \star)] e^{-\mu_k t} \star \dot{w}_k \sin kx \\
&- \sum_{k \geq 1} b_k e^{-t/\nu} \star \dot{w}_k \sin kx + \beta' \sum_{k \geq 2} b_k e^{-\mu_k t} \star e^{-\mu_k t} \star \dot{w}_k \sin kx \\
&+ \frac{3}{4} \sum_{k \geq 2} \{ b_{k+2} e^{-\mu_k t} \star e^{-\mu_{k+2} t} \star \dot{w}_{k+2} \sin kx - 2b_k e^{-\mu_k t} \star e^{-\mu_k t} \star \dot{w}_k \sin kx \\
&\quad + b_k e^{-\mu_{k+2} t} \star e^{-\mu_k t} \star \dot{w}_k \sin[(k+2)x] \} + \mathcal{O}(\nu^2 + \beta'^2 + a^4, \sigma^2), \quad (37)
\end{aligned}$$

and a correspondingly complicated expression for the field $v(x, t)$. Observe that the slow SDE (36) does not contain any fast time convolutions from the Ornstein–Uhlenbeck processes: it would be incongruous to have such fast processes in a supposedly slow model. We keep fast time convolutions out of the slow SDE (36) by introducing carefully crafted terms in the slow mode $\sin x$ in the parametrization of the stochastic slow manifold (37): here the amplitude of the slow mode $\sin x$ is approximately $a - \frac{3}{32} a^2 b_3 e^{-8t} \star w_3 - b_1 e^{-t/\nu} \star \dot{w}_1$. Other methods which do not adjust the slow mode either average over such adjustments and so are weak models, or invoke fast processes in the slow model.

Note that the homotopy parameter γ affects the stochastic slow manifold shape (37), but only weakly. To this order the homotopy has no effect on the evolution on the stochastic slow manifold (36).

Compare with SPDE (28) The corresponding stochastic slow manifold of the SPDE (28), in this parameter regime, is straightforward to construct, via the web server [16] for example. For stochastic slow manifold $\bar{u} \approx \bar{a} \sin x$ one finds the corresponding slow SDE

$$\begin{aligned}
\dot{\bar{a}} &= \beta' \bar{a} - \frac{3}{4} \bar{a}^3 + [1 - \frac{9}{1024} a^4] \bar{b}_1 \dot{\bar{w}}_1 \\
&+ [(\frac{3}{32} + \frac{3}{128} \beta') \bar{a}^2 - \frac{21}{1024} a^4] \bar{b}_3 \dot{\bar{w}}_3 + \frac{5}{1024} a^4 \bar{b}_5 \dot{\bar{w}}_5 + o(\beta'^2 + \bar{a}^4, \sigma). \quad (38)
\end{aligned}$$

This slow SDE is symbolically identical with the SDE (36), one just removes the overbars. We conclude that these stochastic slow manifolds confirm the modeling of the SPDE (22) by its model SPDE (28); at least in the regime of one space dimension with small amplitude a , bifurcation parameter β' , and finite truncation to the noise.

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