ON THE VANISHING IDEAL OF AN ALGEBRAIC TORIC SET AND ITS PARAMETERIZED LINEAR CODES

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Abstract. Let K be a finite field and let X be a subset of a projective space, over the field K, which is parameterized by monomials arising from the edges of a clutter. We show some estimates for the degree-complexity, with respect to the revlex order, of the vanishing ideal $I(X)$ of X. If the clutter is uniform, we classify the complete intersection property of $I(X)$ using linear algebra. We show an upper bound for the minimum distance of certain parameterized linear codes along with certain estimates for the algebraic invariants of $I(X)$.

1. Introduction

Let $K=\mathbb{F}_q$ be a finite field with q elements and let $y^{v_1},\ldots,y^{v_s}$ be a finite set of square-free monomials with $s\geq 2$. As usual if $v_i=(v_{i1},\ldots,v_{in})\in\mathbb{N}^n$, then we set

$$y^{v_i}=y_1^{v_{i1}}\cdots y_n^{v_{in}},\qquad i=1,\ldots,s,$$

where $y_1,\ldots,y_n$ are the indeterminates of a ring of polynomials with coefficients in K. We shall always assume that $\mathcal{A}=\{v_1,\ldots,v_s\}$ is the set of all characteristic vectors of the edges of a clutter (see Definitions 3.1 and 3.2). In particular this means that the entries of v_i are in $\{0,1\}$ for all i. Consider the following set parameterized by these monomials

$$X:=\{[(x_1^{v_{11}}\cdots x_n^{v_{1n}},\ldots,x_1^{v_{s1}}\cdots x_n^{v_{sn}})]\in\mathbb{P}^{s-1}\,|\,x_i\in K^*\text{ for all }i\},$$

where $K^*=K\setminus\{0\}$ and $\mathbb{P}^{s-1}$ is a projective space over the field K. Following [18] we call X an *algebraic toric set* parameterized by $y^{v_1},\ldots,y^{v_s}$. Let $S=K[t_1,\ldots,t_s]=\oplus_{d=0}^{\infty}S_d$ be a polynomial ring over the field K with the standard grading, let $[P_1],\ldots,[P_m]$ be the points of X, and let $f_0(t_1,\ldots,t_s)=t_1^d$. The *evaluation map*

$$\text{(1.1)}\qquad \mathrm{ev}_d\colon S_d=K[t_1,\ldots,t_s]_d\to K^{|X|},\qquad f\mapsto\left(\frac{f(P_1)}{f_0(P_1)},\ldots,\frac{f(P_m)}{f_0(P_m)}\right)$$

defines a linear map of K-vector spaces. The image of ev_d, denoted by $C_X(d)$, defines a *linear code*. Following [17] we call $C_X(d)$ a *parameterized linear code* of order d. As usual by a *linear code* we mean a linear subspace of $K^{|X|}$. The *dimension* and *length* of $C_X(d)$ are given by $\dim_K C_X(d)$ and $|X|$ respectively. The dimension and length are two of the *basic parameters* of a linear code. A third basic parameter is the *minimum distance* which is given by

$$\delta_d=\min\{\|v\|\colon 0\neq v\in C_X(d)\},$$

where $\|v\|$ is the norm of the Hamming distance, i.e., $\|v\|$ is the number of non-zero entries of v. The basic parameters of $C_X(d)$ are related by the Singleton bound for the minimum distance

$$\delta_d\leq|X|-\dim_K C_X(d)+1.$$

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Parameterized linear codes are a nice family of *evaluation codes* (the notion of an evaluation code is introduced in Section 2). They were introduced and studied in [17]. Some other families of evaluation codes have been studied extensively [3, 4, 10, 22].

The *vanishing ideal* of X, denoted by $I(X)$, is the ideal of S generated by the homogeneous polynomials of S that vanish on X. The contents of this paper are as follows. In Section 2 we introduce the preliminaries and explain the connection between the invariants of the vanishing ideal of X and the parameters of $C_X(d)$.

The ideal $I(X)$ is called a *complete intersection* if it can be generated by $s-1$ homogeneous polynomials of S. In [19] it is shown that $I(X)$ is a complete intersection if and only if X is a projective torus in $\mathbb{P}^{s-1}$ (see Definition 2.4). If the clutter has all its edges of the same cardinality, in Section 3 we classify the complete intersection property of $I(X)$ using linear algebra (see Theorem 3.9).

Let $\succ$ be the reverse lexicographical order on the monomials of S. Recall that the ideal $I(X)$ has a unique reduced Gröbner basis with respect to $\succ$. The *degree-complexity* of $I(X)$, with respect to $\succ$, is the maximum degree in the reduced Gröbner basis of $I(X)$. In Section 4 we study the structure of the reduced Gröbner basis of $I(X)$ and show an upper bound for the degree-complexity of $I(X)$ (see Theorem 4.1). This means that the algebraic methods of [17] to compute the invariants of $I(X)$ will probably work better using the revlex order.

In Section 5 we show upper bounds for the minimum distance of $C_X(d)$ for a certain family of algebraic toric sets X arising from normal edge ideals (see Theorem 5.1(b)). For this family we also show estimates for the algebraic invariants of $I(X)$. The bounds on the minimum distance indicate that the codes $C_X(d)$ that emerge from unicyclic connected graphs are especially attractive from the point of view of their error-correcting capacity and so are the codes $C_X(d)$ with $d = 1$ (see Remark 5.3). We give examples, within our family, of parameterized codes having a large minimum distance relative to $|X|$ (see Example 5.4). Such examples of linear codes with large minimum distance are essential, as they show that our construction is attractive in the context of coding theory. The codes $C_X(d)$ are only interesting when d lies within a certain range because $\delta_d = 1$ for $d \gg 0$. This range is determined by $\operatorname{reg}(S/I(X))$, the index of regularity of $S/I(X)$ (see Proposition 2.3). This is one of the motivations to study the index of regularity. Another motivation comes from commutative algebra because, in our situation, $\operatorname{reg}(S/I(X))$ is equal to the Castelnuovo-Mumford regularity which is an algebraic invariant of central importance in the area [5]. The problem of finding a good decoding algorithm for our family of parameterized codes is not considered here. The reader is referred to [2, Chapter 9], [15, 27] and the references there for some available decoding algorithms for some families of linear codes.

For all unexplained terminology and additional information we refer to [6] (for the theory of binomial ideals), [1, 23] (for the theory of Gröbner bases and Hilbert functions), and [16, 24, 26] (for the theory of error-correcting codes and algebraic geometric codes).

2. Preliminaries

We continue to use the notation and definitions used in the introduction. In this section we introduce the basic algebraic invariants of $S/I(X)$ and recall their connection with the basic parameters of parameterized linear codes. Then, we present a result on complete intersections that will be needed later.

Recall that the *projective space* of dimension $s-1$ over K, denoted by $\mathbb{P}^{s-1}$, is the quotient space

$$(K^s \setminus \{0\})/\sim$$

where two points α, β in $K^s \setminus \{0\}$ are equivalent if $\alpha = \lambda\beta$ for some $\lambda \in K$. We denote the equivalence class of α by $[\alpha]$. Let $X \subset \mathbb{P}^{s-1}$ be an algebraic toric set parameterized by $y^{v_1}, \ldots, y^{v_s}$ and let $C_X(d)$ be a parameterized code of order d. The kernel of the evaluation map ev_d, defined in Eq. (1.1), is precisely $I(X)_d$ the degree d piece of $I(X)$. Therefore there is an isomorphism of K-vector spaces

$$S_d/I(X)_d \simeq C_X(d). \tag{2.1}$$

Two of the basic parameters of $C_X(d)$ can be expressed using Hilbert functions of standard graded algebras [17, 23], as we explain below. Recall that the *Hilbert function* of $S/I(X)$ is given by

$$H_X(d) := \dim_K\,(S/I(X))_d = \dim_K\; S_d/I(X)_d.$$

The unique polynomial $h_X(t) = \sum_{i=0}^{k-1} c_i t^i \in \mathbb{Z}[t]$ of degree $k-1 = \dim(S/I(X))-1$ such that $h_X(d) = H_X(d)$ for $d \gg 0$ is called the *Hilbert polynomial* of $S/I(X)$. The integer $c_{k-1}(k-1)!$, denoted by $\deg(S/I(X))$, is called the *degree* or *multiplicity* of $S/I(X)$. In our situation $h_X(t)$ is a non-zero constant because $S/I(X)$ has dimension 1. Furthermore:

Proposition 2.1. ([14, Lecture 13], [8]) $h_X(d) = |X|$ *for* $d \geq |X|-1$.

This result means that $|X|$ is equal to the *degree* of $S/I(X)$. From Eq. (2.1), we get the equality $H_X(d) = \dim_K C_X(d)$. Thus, we have:

Proposition 2.2. [8, 12] $H_X(d)$ *and* $\deg(S/I(X))$ *are equal to the dimension and the length of* $C_X(d)$ *respectively.*

There are algebraic methods, based on elimination theory and Gröbner bases, to compute the dimension and the length of $C_X(d)$ [17].

The *index of regularity* of $S/I(X)$, denoted by $\mathrm{reg}(S/I(X))$, is the least integer $p \geq 0$ such that $h_X(d) = H_X(d)$ for $d \geq p$. The degree and the index of regularity can be read off the Hilbert series as we now explain. The Hilbert series of $S/I(X)$ can be written as

$$F_X(t) := \sum_{d=0}^{\infty} H_X(d)t^d = \frac{h_0 + h_1 t + \cdots + h_r t^r}{1-t},$$

where $h_0, \ldots, h_r$ are positive integers. Indeed $h_i = \dim_K(S/(I(X), t_s))_i$ for $0 \leq i \leq r$ and $\dim_K(S/(I(X), t_s))_i = 0$ for $i > r$. This follows from the fact that $I(X)$ is a Cohen-Macaulay lattice ideal [17] and by observing that $\{t_s\}$ is a regular system of parameters for $S/I(X)$ (see [23]). The number r is equal to the index of regularity of $S/I(X)$ and the degree of $S/I(X)$ is equal to $h_0 + \cdots + h_r$ (see [23] or [29, Corollary 4.1.12]).

A good parameterized code should have large $|X|$ and with $\dim_K C_X(d)/|X|$ and $\delta_d/|X|$ as large as possible. The following result gives an indication of where to look for non-trivial parameterized codes. Only the codes $C_X(d)$ with $1 \leq d < \mathrm{reg}(S/I(X))$ have the potential to be good linear codes.

Proposition 2.3. $\delta_d = 1$ *for* $d \geq \mathrm{reg}(S/I(X))$.

Proof. Since $H_X(d)$ is equal to the dimension of $C_X(d)$ and $H_X(d) = |X|$ for $d \geq \mathrm{reg}(S/I(X))$, by a direct application of the Singleton bound we get that $\delta_d = 1$ for $d \geq \mathrm{reg}(S/I(X))$. □

The definition of $C_X(d)$ can be extended to any finite subset $X \subset \mathbb{P}^{s-1}$ of a projective space over a field K [10, 12]. In this generality—the resulting linear code—$C_X(d)$ is called an *evaluation code* associated to X (see for instance [10]). It is also called a *projective Reed-Muller code* over the set X (see [4, 12]). In this paper we will only deal with parameterized codes over finite fields.

The parameters of evaluation codes associated to X have been computed in a number of cases. If $X = \mathbb{P}^{s-1}$, the parameters of $C_X(d)$ are described in [22, Theorem 1]. If X is the image of the affine space $\mathbb{A}^{s-1}$ under the map $x \mapsto [(1, x)]$, the parameters of $C_X(d)$ are described in [3, Theorem 2.6.2]. If X is a projective torus, the parameters of $C_X(d)$ are described in [4] and [19]. In this paper we give upper bounds for the parameters of certain parameterized codes.

As seen above, parameterized codes are a special type of evaluation codes. What makes a parameterized code interesting is the fact that the vanishing ideal of X is a binomial ideal [17], which allows the computation of the dimension and length using the computer algebra system *Macaulay*2 [13]. The index of regularity of $S/I(X)$ can also be computed using *Macaulay*2, which is useful to find genuine parameterized codes (see Proposition 2.3).

Definition 2.4. The set $\mathbb{T} = \{[(x_1, \ldots, x_s)] \in \mathbb{P}^{s-1} \,|\, x_i \in K^* \text{ for all } i\}$ is called a *projective torus* in $\mathbb{P}^{s-1}$.

An algebraic toric set is a multiplicative group under componentwise multiplication. Thus, a projective torus is a multiplicative group. For future reference we recall the following result on complete intersections.

Proposition 2.5. [11, Theorem 1, Lemma 1] *If* $\mathbb{T}$ *is a projective torus in* $\mathbb{P}^{s-1}$*, then*

(a) $I(\mathbb{T}) = (\{t_i^{q-1} - t_1^{q-1}\}_{i=2}^{s})$.
(b) $F_{\mathbb{T}}(t) = (1 - t^{q-1})^{s-1}/(1 - t)^s$.
(c) $\operatorname{reg}(S/I(\mathbb{T})) = (s - 1)(q - 2)$ *and* $\deg(S/I(\mathbb{T})) = (q - 1)^{s-1}$.

3. The complete intersection property of $I(X)$

We continue to use the notation and definitions used in the introduction and in the preliminaries. The main result of this section is a structure theorem for the complete intersection property of $I(X)$.

Definition 3.1. A *clutter* $\mathcal{C}$ is a family E of subsets of a finite ground set $Y = \{y_1, \ldots, y_n\}$ such that if $f_1, f_2 \in E$, then $f_1 \not\subset f_2$. The ground set Y is called the *vertex set* of $\mathcal{C}$ and E is called the *edge set* of $\mathcal{C}$, they are denoted by $V_\mathcal{C}$ and $E_\mathcal{C}$ respectively.

Clutters are special hypergraphs and are sometimes called *Sperner families* in the literature. One important example of a clutter is a graph with the vertices and edges defined in the usual way for graphs.

Definition 3.2. Let $\mathcal{C}$ be a clutter with vertex set $V_\mathcal{C} = \{y_1, \ldots, y_n\}$ and let f be an edge of $\mathcal{C}$. The *characteristic vector* of f is the vector $v = \sum_{y_i \in f} e_i$, where e_i is the *ith* unit vector in $\mathbb{R}^n$.

Throughout this paper we assume that $\mathcal{A} := \{v_1, \ldots, v_s\}$ is the set of all characteristic vectors of the edges of a clutter $\mathcal{C}$.

Definition 3.3. If $a \in \mathbb{R}^s$, its *support* is defined as $\operatorname{supp}(a) = \{i \,|\, a_i \neq 0\}$. Note that $a = a^+ - a^-$, where a^+ and a^- are two non negative vectors with disjoint support called the *positive* and *negative* part of a respectively.

Lemma 3.4. *Let $\mathcal{C}$ be a clutter and let $f \neq 0$ be a homogeneous binomial of $I(X)$ of the form $t_i^b - t^c$ with $b \in \mathbb{N}$, $c \in \mathbb{N}^s$ and $i \notin \operatorname{supp}(c)$. Then*

(a) $\deg(f) \geq q-1$.

(b) *If $\deg(f) = q-1$, then $f = t_i^{q-1} - t_j^{q-1}$ for some $j \neq i$.*

Proof. For simplicity of notation assume that $f = t_1^b - t_2^{c_2} \cdots t_r^{c_r}$, where $c_j \geq 1$ for all j and $b = c_2 + \cdots + c_r$. Then

$$(x_1^{v_{11}} \cdots x_n^{v_{1n}})^b = (x_1^{v_{21}} \cdots x_n^{v_{2n}})^{c_2} \cdots (x_1^{v_{r1}} \cdots x_n^{v_{rn}})^{c_r} \quad \forall\, (x_1, \ldots, x_n) \in (K^*)^n, \tag{3.1}$$

where $v_i = (v_{i1}, \ldots, v_{in})$. Let β be a generator of the cyclic group $(K^*, \cdot)$.

(a) We proceed by contradiction. Assume that $b < q-1$. First we claim that if $v_{1k} = 1$ for some $1 \leq k \leq n$, then $v_{jk} = 1$ for $j = 2, \ldots, r$. To prove the claim assume that $v_{1k} = 1$ and $v_{jk} = 0$ for some $j \geq 2$. Then, making $x_i = 1$ for $i \neq k$ in Eq. (3.1), we get $(x_k^{v_{1k}})^b = x_k^b = x_k^m$, where $m = v_{2k}c_2 + \cdots + v_{rk}c_r < b$. Then $x_k^{b-m} = 1$ for $x_k \in K^*$. In particular $\beta^{b-m} = 1$. Hence $b - m$ is a multiple of $q-1$ and consequently $b \geq q-1$, a contradiction. This completes the proof of the claim. Therefore $\operatorname{supp}(v_1) \subset \operatorname{supp}(v_j)$ for $j = 2, \ldots, r$. Since $\mathcal{C}$ is a clutter we get that $v_1 = v_j$ for $j = 2, \ldots, r$, a contradiction because $v_1, \ldots, v_r$ are distinct. Hence $b \geq q-1$.

(b) It suffices to show that $r = 2$. Assume $r \geq 3$. We claim that if $v_{2k} = 1$ for some $1 \leq k \leq n$, then $v_{jk} = 1$ for $j \geq 3$. Otherwise, if $v_{2k} = 1$ and $v_{jk} = 0$ for some $j \geq 3$, making $x_i = 1$ for $i \neq k$ and $b = q-1$ in Eq. (3.1) we get $1 = x_k^m$ for any $x_k \in K^*$, for some $0 < m < q-1$. A contradiction because $\beta^m \neq 1$. This proves the claim. Therefore $\operatorname{supp}(v_2) \subset \operatorname{supp}(v_j)$ for $j \geq 3$. As in part (a) we get $v_2 = v_j$ for $j \geq 3$, a contradiction. Hence $r = 2$. □

The complete intersection property of $I(X)$ was first studied in [19]. We complement the following result by showing a characterization of this property—valid for uniform clutters—using linear algebra (see Theorem 3.9).

Theorem 3.5. [19] *Let $\mathcal{C}$ be a clutter with s edges and let $\mathbb{T}$ be a projective torus in $\mathbb{P}^{s-1}$. The following are equivalent:*

(c_1) *$I(X)$ is a complete intersection.*

(c_2) $I(X) = (t_1^{q-1} - t_s^{q-1}, \ldots, t_{s-1}^{q-1} - t_s^{q-1})$.

(c_3) $X = \mathbb{T} \subset \mathbb{P}^{s-1}$.

For use below recall that the *toric ideal* associated to $\mathcal{A} = \{v_1, \ldots, v_s\}$, denoted by $I_{\mathcal{A}}$, is the prime ideal of $S = K[t_1, \ldots, t_s]$ given by (see [25]):

$$I_{\mathcal{A}} = \left(t^a - t^b \middle|\, a = (a_i), b = (b_i) \in \mathbb{N}^s, \textstyle\sum_i a_i v_i = \sum_i b_i v_i\right) \subset S. \tag{3.2}$$

A clutter is called *uniform* if all its edges have the same number of elements.

Proposition 3.6. *Let $\mathcal{C}$ be a uniform clutter. If $I(X)$ is a complete intersection and $q \geq 3$, then $v_1, \ldots, v_s$ are linearly independent.*

Proof. To begin with we claim that if $f = t^{a^+} - t^{a^-}$ is any non-zero homogeneous binomial in the lattice ideal $I(X)$, then

$$a = a^+ - a^- \;\equiv\; 0 \bmod (q-1),$$

that is, any entry of a is a multiple of $q-1$. By Theorem 3.5 the degree of f is at least $q-1$. To show the claim we proceed by induction on $\deg(f)$. If $\deg(f) = q-1$, then by Theorem 3.5 and Lemma 3.4(b) it is seen that $f = t_i^{q-1} - t_j^{q-1}$ for some i, j, i.e., $a = (q-1)e_i - (q-1)e_j$. Assume that $\deg(f) > q-1$. By Theorem 3.5 we obtain that t^{a^+} and t^{a^-} are divisible by some t_i^{q-1} and t_j^{q-1} respectively. Then, $a_i^+ \geq q-1$ and $a_j^- \geq q-1$ for some $i \in \text{supp}(a^+)$ and $j \in \text{supp}(a^-)$. Therefore using that $f \in I(X)$ and the fact that $(K^*, \cdot)$ is a cyclic group of order $q-1$, it follows readily that the binomial

$$f' = \frac{t^{a^+}}{t_i^{q-1}} - \frac{t^{a^-}}{t_j^{q-1}}$$

is homogeneous, of degree $\deg(f) - (q-1)$, and belongs to $I(X)$. Hence by induction hypothesis the vector $(a^+ - (q-1)e_i) - (a^- - (q-1)e_j)$ is a multiple of $q-1$, and so is $a = a^+ - a^-$. This completes de proof of the claim.

To show that $v_1, \ldots, v_s$ are linearly independent we proceed by contradiction. Assume that $v_1, \ldots, v_s$ are linearly dependent. As $\mathcal{C}$ is uniform, there is a non-zero homogeneous binomial $f = t^{a^+} - t^{a^-}$ of least degree in the toric ideal $I_\mathcal{A}$. This means that the degree of f is equal to the initial degree of $I_\mathcal{A}$ [29, p. 110]. Since $I_\mathcal{A} \subset I(X)$ we obtain that $a = a^+ - a^-$ is a multiple of $q-1$. Then, we can write $a^+ = (q-1)b^+$, $a^- = (q-1)b^-$ for some b^+, b^- in $\mathbb{N}^s$. We set $u = t^{b^+}$, $v = t^{b^-}$, $g = u - v$, $h = u^{q-2} + u^{q-3}v + \cdots + v^{q-2}$. From the equality $f = gh$ we obtain that $g \in I_\mathcal{A}$ or $h \in I_\mathcal{A}$ because $I_\mathcal{A}$ is a prime ideal and $q \geq 3$, a contradiction to the choice of f because g and h have degree less than that of f. □

Definition 3.7. For an ideal $I \subset S$ and a polynomial $h \in S$ the *saturation* of I with respect to h is the ideal

$$(I\colon h^\infty) := \{f \in S \mid fh^m \in I \text{ for some } m \geq 1\}.$$

We will only deal with the case where $h = t_1 \cdots t_s$.

We call $\mathcal{A}$ *homogeneous* if $\mathcal{A}$ lies on an affine hyperplane not containing the origin. Notice that if $\mathcal{C}$ is uniform, then $\mathcal{A}$ is homogeneous. Given $\Gamma \subset \mathbb{Z}^n$, the subgroup of $\mathbb{Z}^n$ generated by Γ will be denoted by $\mathbb{Z}\Gamma$.

Theorem 3.8. [17, Theorem 2.6] *Let $K = \mathbb{F}_q$ be a finite field, let $\mathcal{A} = \{v_1, \ldots, v_s\} \subset \mathbb{Z}^n$, and let $\phi\colon \mathbb{Z}^n/L \to \mathbb{Z}^n/L$ be the multiplication map $\phi(\overline{a}) = (q-1)\overline{a}$, where $L = \mathbb{Z}\{v_i - v_1\}_{i=2}^s$. If $\mathcal{A}$ is homogeneous, then*

$$((I_\mathcal{A} + (t_2^{q-1} - t_1^{q-1}, \ldots, t_s^{q-1} - t_1^{q-1}))\colon (t_1 \cdots t_s)^\infty) \subset I(X) \tag{3.3}$$

with equality if and only if the map ϕ is injective.

We come to the main result of this section, a structure theorem for complete intersections via linear algebra.

Theorem 3.9. *Let $\phi\colon \mathbb{Z}^n/L \to \mathbb{Z}^n/L$ be the multiplication map $\phi(\overline{a}) = (q-1)\overline{a}$, where L is the subgroup generated by $\{v_i - v_1\}_{i=2}^s$. If $\mathcal{C}$ is a uniform clutter and $q \geq 3$, then $I(X)$ is a complete intersection if and only if $v_1, \ldots, v_s$ are linearly independent and the map ϕ is injective.*

Proof. $\Rightarrow$) By Proposition 3.6 the vectors $v_1, \ldots, v_s$ are linearly independent. Then $I_\mathcal{A} = (0)$ and by Theorem 3.5 we get the equality $I(X) = (\{t_1^{q-1} - t_i^{q-1}\}_{i=2}^s)$. Hence, we have equality in Eq. (3.3). Therefore using Theorem 3.8 we conclude that ϕ is injective.

$\Leftarrow$) As the map ϕ is injective and $\mathcal{C}$ is uniform, using Theorem 3.8, we get the equality

$$((I_\mathcal{A} + (t_2^{q-1} - t_1^{q-1}, \ldots, t_s^{q-1} - t_1^{q-1}))\colon (t_1 \cdots t_s)^\infty) = I(X).$$

Since $\mathcal{A}$ is linearly independent one has that $I_\mathcal{A} = (0)$. Hence, the equality above becomes $(\{t_1^{q-1} - t_i^{q-1}\}_{i=2}^s) = I(X)$, i.e., $I(X)$ is a complete intersection. □

A graph with only one cycle is called *unicyclic.*

Corollary 3.10. *Let $\mathcal{C}$ be a unicyclic connected graph with n vertices. If the only cycle of $\mathcal{C}$ is odd, then $X = \mathbb{T}$ is a projective torus in $\mathbb{P}^{n-1}$.*

Proof. Assume that $\mathcal{C}$ is an odd cycle of length n. Let $y_1, \ldots, y_n$ be the vertices of $\mathcal{C}$. The characteristic vectors of the edges of $\mathcal{C}$ are

$$v_1 = e_1 + e_2,\, v_2 = e_2 + e_3, \ldots, v_{n-1} = e_{n-1} + e_n,\, v_n = e_n + e_1,$$

where e_i is the *ith* unit vector in $\mathbb{N}^n$. The vectors $v_1, \ldots, v_n$ are linearly independent because n is odd. It is not hard to see that the quotient group $\mathbb{Z}^n/\mathbb{Z}\{v_i - v_1\}_{i=2}^n$ is torsion-free. Hence, by Theorem 3.9, $I(X)$ is a complete intersection. Then, $X = \mathbb{T}$ is a projective torus in $\mathbb{P}^{n-1}$ by Theorem 3.5. If $\mathcal{C}$ is not an odd cycle, then it has a vertex of degree 1 and the proof follows by induction because removing this vertex results in a graph that is connected and has a unique odd cycle. □

The next result shows that the index of regularity of complete intersections associated to clutters provides an upper bound for the index of regularity of $S/I(X)$.

Proposition 3.11. [19] $\mathrm{reg}(S/I(X)) \leq (q-2)(s-1)$, *with equality if $I(X)$ is a complete intersection associated to a clutter with s edges.*

Remark 3.12. In Theorem 5.1(c) we provide another upper bound for the index of regularity of $S/I(X)$ valid for a certain family of algebraic toric sets.

4. The degree-complexity of $I(X)$

We continue to use the notation and definitions used in the introduction. The main result of this section is an upper bound for the degree-complexity of $I(X)$.

In what follows we shall assume that $\succ$ is the *reverse lexicographical order* (*revlex order* for short) on the monomials of S. This order is given by $t^b \succ t^a$ if and only if the last non-zero entry of $b - a$ is negative. As usual, if g is a polynomial of S, we denote the leading term of g by $\mathrm{in}(g)$ and the leading coefficient of g by $\mathrm{lc}(g)$.

According to [1, Proposition 6, p. 91] the ideal $I(X)$ has a unique reduced Gröbner basis. We refer to [1] for the theory of Gröbner bases. The *degree-complexity* of $I(X)$, with respect to $\succ$, is the maximum degree of the polynomials in the reduced Gröbner basis of $I(X)$. Next we study the reduced Gröbner basis and the degree-complexity of $I(X)$.

We come to one of the main results of this section.

Theorem 4.1. *Let $\mathcal{C}$ be a clutter and let $\succ$ be the revlex order on the monomials of S. If $\mathcal{G}$ is the reduced Gröbner basis of the ideal $I(X)$, then $t_i^{q-1} - t_s^{q-1} \in \mathcal{G}$ for $i = 1, \ldots, s-1$ and $\deg_{t_i}(g) \leq q-1$ for $g \in \mathcal{G}$ and $1 \leq i \leq s$.*

Proof. The reduced Gröbner basis of $I(X)$ consists of homogeneous binomials [17]. As $I(X)$ is a lattice ideal [17], it is seen that each binomial $t^a - t^b \in \mathcal{G}$ satisfies that $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$, this follows using that each variable t_i is not a zero-divisor of $S/I(X)$. Since $t_i^{q-1} - t_s^{q-1}$ is in $I(X)$ for $i = 1, \ldots, s-1$, there is $g_i \in \mathcal{G}$ such that $g_i = t_i^{b_i} - t^{c_i}$, $b_i \leq q-1$, $c_i \in \mathbb{N}^s$, $i \notin \mathrm{supp}(c_i)$, and $\mathrm{in}(g_i) = t_i^{b_i}$. Then, by Lemma 3.4, the binomial g_i has the form $g_i = t_i^{q-1} - t_{j_i}^{q-1}$ for some $i < j_i$. As $\mathcal{G}$ is a reduced Gröbner basis we get that $g_i = t_i^{q-1} - t_s^{q-1}$ for $i = 1, \ldots, s-1$. Let $g \in \mathcal{G} \setminus \{g_1, \ldots, g_{s-1}\}$. Using that $\mathcal{G}$ is reduced we get that $\deg_{t_i}(g) \leq q-2$ for $i = 1, \ldots, s-1$. To complete the proof we need only show $\deg_{t_s}(g) \leq q-1$. Assume that $a_s = \deg_{t_s}(g) > q-1$. After permuting $t_1, \ldots, t_{s-1}$ we may assume that $\mathrm{in}(g) = t_1^{a_1} \cdots t_r^{a_r}$ and $g = t_1^{a_1} \cdots t_r^{a_r} - t_{r+1}^{a_{r+1}} \cdots t_s^{a_s}$, where $r < s$. Consider the polynomial

$$\begin{aligned} h &= t_2^{a_2} \cdots t_r^{a_r} g_1 - t_1^{q-1-a_1} g \\ &= t_s^{q-1}\left(-t_2^{a_2} \cdots t_r^{a_r} + t_1^{q-1-a_1} t_{r+1}^{a_{r+1}} \cdots t_{s-1}^{a_{s-1}} t_s^{a_s-(q-1)}\right) = t_s^{q-1} h_1. \end{aligned}$$

Since $h \in I(X)$ and using that $I(X)$ is a lattice ideal, we get that the binomial

$$h_1 = -t_2^{a_2} \cdots t_r^{a_r} + t_1^{q-1-a_1} t_{r+1}^{a_{r+1}} \cdots t_{s-1}^{a_{s-1}} t_s^{a_s-(q-1)}$$

belongs to $I(X)$. As $\mathrm{in}(h_1) = t_2^{a_2} \cdots t_r^{a_r}$, we obtain that $\mathrm{in}(g) \in (\mathrm{in}(\mathcal{G} \setminus \{g\})$, a contradiction. Thus $\deg_{t_s}(g) \leq q-1$. $\Box$

The next result is interesting because it shows that the Hilbert functions of $S/I(X)$ and $S/I_{\mathcal{A}}$ are equal up to degree $q-2$.

Proposition 4.2. *Let $\mathcal{C}$ be a clutter. If $f = t^{a^+} - t^{a^-}$ is a non-zero homogeneous binomial of $I(X)$ and $\deg(f) \leq q-2$, then $f \in I_{\mathcal{A}}$.*

Proof. We may assume that $a^+ = (a_1, \ldots, a_r, 0, \ldots, 0)$ and $a^- = (0, \ldots, 0, a_{r+1}, \ldots, a_m, 0, \ldots, 0)$ and $a_i \geq 1$ for $i = 1, \ldots, m$. Then

$$(x_1^{v_{11}} \cdots x_n^{v_{1n}})^{a_1} \cdots (x_1^{v_{r1}} \cdots x_n^{v_{rn}})^{a_r} = (x_1^{v_{r+1,1}} \cdots x_n^{v_{r+1,n}})^{a_{r+1}} \cdots (x_1^{v_{m,1}} \cdots x_n^{v_{m,n}})^{a_m} \tag{4.1}$$

for all $(x_1, \ldots, x_n) \in (K^*)^n$, where $v_i = (v_{i1}, \ldots, v_{in}) = (v_{i,1}, \ldots, v_{i,n})$. To show that $f \in I_{\mathcal{A}}$ we need only show that $Aa^+ = Aa^-$, where A is the incidence matrix of $\mathcal{C}$, i.e., A is the matrix with column vectors $v_1, \ldots, v_s$. Equivalently we need only show the equality

$$v_{1,k}a_1 + \cdots + v_{r,k}a_r = v_{r+1,k}a_{r+1} + \cdots + v_{m,k}a_m \tag{4.2}$$

for $1 \leq k \leq n$. If both sides of Eq. (4.2) are zero there is nothing to show. We proceed by contradiction assuming:

$$v_{1,k}a_1 + \cdots + v_{r,k}a_r > v_{r+1,k}a_{r+1} + \cdots + v_{m,k}a_m \geq 0. \tag{4.3}$$

Making $x_i = 1$ for $i \neq k$ in Eq. (4.1), we get

$$x_k^{v_{1,k}a_1 + \cdots + v_{r,k}a_r} = x_k^{v_{r+1,k}a_{r+1} + \cdots + v_{m,k}a_m}$$

for any $x_k \in K^*$. In particular making $x_k = \beta$, where β is a generator of the cyclic group $(K^*, \cdot)$, we get that

$$(v_{1,k}a_1 + \cdots + v_{r,k}a_r) - (v_{r+1,k}a_{r+1} + \cdots + v_{m,k}a_m) \equiv \ 0 \bmod (q-1). \tag{4.4}$$

Consequently $v_{1,k}a_1 + \cdots + v_{r,k}a_r \geq q-1$, a contradiction because

$$q-2 \geq \deg(f) = a_1 + \cdots + a_r \geq v_{1,k}a_1 + \cdots + v_{r,k}a_r.$$

Hence equality in Eq. (4.2) holds for $1 \leq k \leq n$ and the proof is complete. $\Box$

Proposition 4.3. *Let A be the matrix with column vectors $v_1, \ldots, v_s$. Then*

$$I(X) = (\{t^{a^+} - t^{a^-} \,|\, Aa^+ \equiv Aa^- \bmod (q-1) \text{ and } |a^+| = |a^-|\}).$$

Proof. The inclusion "$\subset$" follows from Eq. (4.4) and from the fact that $I(X)$ is a lattice ideal [17]. To show the inclusion "$\supset$" take $f = t^{a^+} - t^{a^-}$ such that $Aa^+ \equiv Aa^- \bmod (q-1)$ and $|a^+| = |a^-|$. From the first condition it is seen that f vanishes on X and from the second condition f is homogeneous in the standard grading of S. Thus $f \in I(X)$. $\Box$

5. Upper bounds for the minimum distance

We continue to use the notation and definitions used in the introduction and in the preliminaries. Let $\mathcal{C}$ be a clutter with vertex set $V_{\mathcal{C}} = \{y_1, \ldots, y_n\}$. Throughout this section we assume that $\mathcal{A} = \{v_1, \ldots, v_s\}$ is the set of all characteristic vectors of the edges of a uniform clutter $\mathcal{C}$.

The set $(K^*)^n$ is called an *affine algebraic torus* of dimension n and is denoted by $\mathbb{T}^*$. The torus $\mathbb{T}^*$ is a multiplicative group under the product operation $(\alpha_i)(\alpha_i') = (\alpha_i\alpha_i')$, where (α_i) really means $(\alpha_1, \ldots, \alpha_n)$. Clearly, the algebraic toric set:

$$X := \{[(x_1^{v_{11}} \cdots x_n^{v_{1n}}, \ldots, x_1^{v_{s1}} \cdots x_n^{v_{sn}})] \,|\, x_i \in K^* \text{ for all } i\} \subset \mathbb{P}^{s-1}$$

is also a multiplicative group with the product operation.

Let I be the ideal of $R = K[y_1, \ldots, y_n]$ generated by $y^{v_1}, \ldots, y^{v_s}$. The ideal I is called the *edge ideal* of $\mathcal{C}$ and the matrix A whose columns are $v_1, \ldots, v_s$ is called the *incidence matrix* of $\mathcal{C}$. Recall that the *integral closure* of I^i, denoted by $\overline{I^i}$, is the ideal of R given by

$$\overline{I^i} = (\{y^a \in R \,|\, \exists\, p \in \mathbb{N} \setminus \{0\}; (y^a)^p \in I^{pi}\}), \tag{5.1}$$

see for instance [29, Proposition 7.3.3]. The ideal I is called *normal* if $I^i = \overline{I^i}$ for $i \geq 1$. There are many interesting examples of normal ideals [25, 29]. For instance if $\mathcal{C}$ is the clutter of all subsets of $Y = \{y_1, \ldots, y_n\}$ of a fixed size $k \geq 1$, then I is normal. If $\mathcal{C}$ is the clutter of bases of a matroid, then I is also normal. There is a combinatorial description of the normality of ideals generated by square-free monomials of degree 2, i.e., of ideals such that $\mathcal{C}$ is a graph (see [20, 21]). In particular if $\mathcal{C}$ is a complete or bipartite graph, then I is normal. The edge ideal I is also normal if $\mathcal{C}$ is any odd cycle or any unicyclic graph.

Let $\mathcal{B} \subset \mathbb{Z}^{n+1}$. The *polyhedral cone* generated by $\mathcal{B}$ is denoted by $\mathbb{R}_+\mathcal{B}$. A polyhedral cone containing no lines is called *pointed*. The set $\mathcal{B}$ is called a *Hilbert basis* if $\mathbb{N}\mathcal{B} = \mathbb{R}_+\mathcal{B} \cap \mathbb{Z}^{n+1}$, where $\mathbb{N}\mathcal{B}$ is the semigroup generated by $\mathcal{B}$.

We come to the main result of this section, an upper bound for the minimum distance of $C_X(d)$ valid for certain normal edge ideals of uniform clutters.

Theorem 5.1. *Let $\mathcal{C}$ be a uniform clutter whose incidence matrix has rank n and let $I \subset R$ be its edge ideal. If I is normal and $\mathbb{T}$ is a projective torus in $\mathbb{P}^{n-1}$, then:*

(a) *The degree of $S/I(X)$ is equal to $|X| = (q-1)^{n-1}$.*
(b) $\delta_d \leq \delta_d'$, *where δ_d' is the minimum distance of the linear code $C_{\mathbb{T}}(d)$.*
(c) $\operatorname{reg}(S/I(X)) \leq \operatorname{reg}(S'/I(\mathbb{T})) = (q-2)(n-1)$, *where $S' = K[t_1, \ldots, t_n]$.*

Proof. (a) The ideal I is normal. Then by [7, Theorem 3.15] the set $\mathcal{B} = \{(v_i, 1)\}_{i=1}^s$ is a Hilbert basis. Therefore, using [17, Theorem 3.5], we obtain that $(q-1)^{n-1}$ divides $|X|$. On the other hand there is an epimorphism of multiplicative groups

$$\theta\colon \mathbb{T}^* \to X\,; \qquad (x_1, \ldots, x_n) \stackrel{\theta}{\longmapsto} [(x^{v_1}, \ldots, x^{v_s})],$$

where $\mathbb{T}^* = (K^*)^n$ is an affine algebraic torus. The kernel of θ contains the diagonal subgroup

$$\mathcal{D}^* = \{(\lambda, \ldots, \lambda) \,|\, \lambda \in K^*\}.$$

Thus $|X|$ divides $(q-1)^{n-1}$. Putting altogether, we get $|X| = (q-1)^{n-1}$.

(b) The set $\mathcal{B} = \{(v_i, 1)\}_{i=1}^s$ is a Hilbert basis (see the proof of part (a)). Hence using a result of [9], after permutation of the $(v_i, 1)$'s, we may assume that $\mathcal{B}' = \{(v_1, 1), \ldots, (v_n, 1)\}$ is a Hilbert basis and a linearly independent set. Then, it is seen that the group $\mathbb{Z}^{n+1}/\mathbb{Z}\mathcal{B}'$ is torsion-free. We set $L'=\mathbb{Z}\{v_i-v_1\}_{i=2}^n$. It is not hard to see that there is an isomorphism of groups

$$\tau\colon T(\mathbb{Z}^n/L')\rightarrow T(\mathbb{Z}^{n+1}/\mathbb{Z}\mathcal{B}')$$

given by $\tau(\overline{a})=\overline{(a,0)}$, where $T(M)$ denotes the torsion subgroup of an abelian group M, i.e., $T(M)$, is the set of all m in M such that $pm=0$ for some $0\neq p\in\mathbb{Z}$. From this isomorphism we conclude that $T(\mathbb{Z}^n/L')=0$, i.e., $\mathbb{Z}^n/L'$ is also torsion-free.

Consider the algebraic toric set parameterized by $y^{v_1},\ldots,y^{v_n}$:

$$X_1=\{[(x^{v_1},\ldots,x^{v_n})]\,|\,x_i\in K^* \text{ for all } i\}\subset\mathbb{P}^{n-1}.$$

We claim that $I(X_1)=(\{t_i^{q-1}-t_n^{q-1}\}_{i=1}^{n-1})$. We set $\mathcal{A}'=\{v_1,\ldots,v_n\}$. Notice that the set $\mathcal{A}'$ is also linearly independent. Since $I_{\mathcal{A}'}=(0)$ and $\mathbb{Z}^n/L'$ is torsion-free, by Theorem 3.8, we obtain

$$\begin{aligned}(\{t_i^{q-1}-t_n^{q-1}\}_{i=1}^{n-1}) &= (\{t_i^{q-1}-t_n^{q-1}\}_{i=1}^{n-1})\colon(t_1\cdots t_n)^\infty)\\ &= (I_{\mathcal{A}'}+(\{t_i^{q-1}-t_n^{q-1}\}_{i=1}^{n-1}))\colon(t_1\cdots t_n)^\infty)\\ &\overset{3.8}{=} I(X_1).\end{aligned}$$

Let $\mathbb{T}$ be a projective torus in $\mathbb{P}^{n-1}$. By Proposition 2.5, we have $I(\mathbb{T})=I(X_1)$. Consequently $X_1=\mathbb{T}$ because X_1 and $\mathbb{T}$ are projective varieties. Let δ_d' be the minimum distance of $C_{X_1}(d)$. Next we show that $\delta_d\leq\delta_d'$. There is a well defined epimorphism

$$\overline{\theta}_1\colon X\rightarrow X_1,\qquad [(x^{v_1},\ldots,x^{v_s})]\mapsto[(x^{v_1},\ldots,x^{v_n})]$$

induced by the projection map $[(\alpha_1,\ldots,\alpha_s)]\mapsto[(\alpha_1,\ldots,\alpha_n)]$. By part (a) one has $|X|=|X_1|=(q-1)^{n-1}$. Therefore the map $\overline{\theta}_1$ is an isomorphism of multiplicative groups. For any homogeneous polynomial F, we denote its zero set by $Z_X(F)=\{[P]\in X\,|\,F(P)=0\}$. Let $S'=K[t_1,\ldots,t_n]=\oplus_{d=0}^\infty S_d'$ and let $F_1\in S_d'$ be a polynomial such that $\mathrm{ev}_d(F_1)\neq 0$ and with $|Z_{X_1}|$ as large as possible, i.e., we choose F_1 so that $\delta_d'=|X_1|-|Z_{X_1}(F_1)|$. We can regard the polynomial $F_1=F_1(t_1,\ldots,t_n)$ as an element of S and denote it by F. The map $\overline{\theta}_1$ induces a bijective map

$$\overline{\theta}_1\colon Z_X(F)\mapsto Z_{X_1}(F_1),\qquad [P]\mapsto[\overline{\theta}_1(P)].$$

Therefore we have the inequality

$$\max\{|Z_X(F)|\colon F\in S_d;\,\mathrm{ev}_d(F)\neq 0\}\quad\geq\quad\max\{|Z_{X_1}(F_1)|\colon F_1\in S_d';\,\mathrm{ev}_d(F_1)\neq 0\}.$$

Consequently $\delta_d\leq\delta_d'$.

(c) We continue to use the notation and definitions used in the proof of part (b). Since $X_1=\mathbb{T}$, it suffices to show that $H_{X_1}(d)\leq H_X(d)$ for $d\geq 1$. Using that $I(X)$ and $I(X_1)$ are vanishing ideals generated by homogeneous polynomials, it is not hard to show that $S'\cap I(X)=I(X_1)$. Thus, we have a graded monomorphism

$$0\rightarrow K[t_1,\ldots,t_n]/I(X_1)\rightarrow K[t_1,\ldots,t_s]/I(X),\qquad \overline{F}_1\mapsto\overline{F}_1.$$

Hence $H_{X_1}(d)\leq H_X(d)$ and consequently $\mathrm{reg}(S/I(X))\leq\mathrm{reg}(S'/I(\mathbb{T}))=(q-2)(n-1)$. □

There is a nice recent formula for δ_d':

Theorem 5.2. [19, Theorem 3.4] *If $\mathbb{T}$ is a projective torus in $\mathbb{P}^{n-1}$ and $d\geq 1$, then the minimum distance of $C_{\mathbb{T}}(d)$ is given by*

$$\delta_d'=\begin{cases}(q-1)^{n-(k+2)}(q-1-\ell) & \text{if}\quad d\leq(q-2)(n-1)-1,\\ 1 & \text{if}\quad d\geq(q-2)(n-1),\end{cases}$$

where k and ℓ are the unique integers such that $k\geq 0$, $1\leq\ell\leq q-2$ and $d=k(q-2)+\ell$.

Remark 5.3. (i) When d is greater than or equal to the index of regularity of $S/I(X)$, by Proposition 2.3, one has that $\delta_d = 1$. Thus, for $d \geq \text{reg}(S/I(X))$ our codes are useless from a practical point of view. For some other values of the parameters however, the bound δ'_d does not prevent our codes from having a large (although not optimal) minimum distance. In Example 5.4 we provide specific values of the parameters, for which our codes have a large minimum distance relative to $|X|$.

(ii) Let $\mathcal{C}$ be a unicyclic connected graph with n vertices and with a unique cycle of odd length. Then, $X = \mathbb{T}$ is a projective torus in $\mathbb{P}^{n-1}$ by Corollary 3.10. Thus, the minimum distance δ_d of $C_X(d)$ is equal to δ'_d by Theorem 5.2. This means that the bound of Theorem 5.1(b) is optimal.

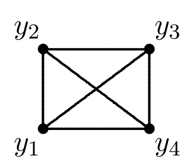
(iii) The problem of computing the minimum distance of a linear code is NP-hard [28]. It might not be easy to compute the minimum distance of $C_X(d)$ for graphs with large number of edges and vertices. However, for a complete graph with 4 vertices it is not hard to compute the minimum distance and to compare the bound δ'_d with the Singleton bound, see Example 5.5.

Example 5.4. Let $\mathcal{C}$ be a cycle of length 3, let X be the algebraic toric set parameterized by y_1y_2, y_2y_3, y_1y_3 and let $C_X(d)$ be the parameterized code of order d over the field $K = \mathbb{F}_{11}$. Using *Macaulay2*, together with Remark 5.3(ii), we obtain its basic parameters:

d	1	2	3	4	5	6	7	8	9	10	11	12	13
$\|X\|$	100	100	100	100	100	100	100	100	100	100	100	100	100
$\dim C_X(d)$	3	6	10	15	21	28	36	45	55	64	72	79	85
δ_d	90	80	70	60	50	40	30	20	10	9	8	7	6

d	14	15	16	17	18
$\|X\|$	100	100	100	100	100
$\dim C_X(d)$	90	94	97	99	100
δ_d	5	4	3	2	1

Example 5.5. Let $\mathcal{C}$ be the following complete graph on four vertices and let X be the algebraic toric set parameterized by all y_iy_j such that $\{y_i, y_j\}$ is an edge of $\mathcal{C}$.



Let $C_X(d)$ be the parameterized code of order d over the field $K = \mathbb{F}_3$, let b_d (resp. δ'_d) be the Singleton bound (resp. the bound of Theorem 5.1), and let δ_d be the minimum distance of $C_X(d)$. Using *Macaulay2*, we obtain:

d	1	2	3
b_d	3	1	1
δ'_d	4	2	1
δ_d	2	1	1

If $C_X(d)$ is the parameterized code of order d over the field $K = \mathbb{F}_4$, then we get:

d	1	2	3	4	5	6
b_d	22	9	1	1	1	1
δ'_d	18	9	6	3	2	1
δ_d	12	3	1	1	1	1

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