On the universal relations for the formfactors in 1D quantum liquids.

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Abstract

For various one-dimensional quantum liquids in the framework of the Luttinger model (bosonization) we establish the relations between the coefficients before the power-law asymptotics of the correlators (prefactors) and the formfactors of the corresponding local operators. The derivation of these relations in the framework of the bosonization procedure allows to substantiate the prediction for the formfactors corresponding to the low-lying particle-hole excitations. We present an explicit expressions for the particle-hole formfactors for various one-dimensional models. We also obtain the formulas for the summation over the particle-hole states corresponding to the power-law asymptotics of the correlators.

1. Introduction

Calculation of correlation functions in the 1D quantum liquids or spin systems remains important problem both from theoretical and experimental points of view. Although the predictions of the critical exponents corresponding to the power-law decay at large distances obtained with the help of the mapping to the Luttinger model (bosonization) [1],[2] or conformal field theory [3], [4] are available for a long time, the calculation of the constants before the asymptotics (prefactors) remains an open problem. Recently some progress was achieved in relating of the prefactors to the certain formfactors of local operators by means of direct calculations in the integrable model [5] and by means of the conformal field theory [6]. Moreover, for the XXZ - quantum spin model the behaviour of the formfactors for the low-lying particle-hole excitations was found [7], which agrees with the predictions of the recent papers [6].

In Ref.[6] the arguments on the particle-hole formfactors are based on the proportionality of the correlator to the certain correlator in the Luttinger liquid theory. Although these arguments could in principle lead to conclusions made in these papers (proportionality of the two correlators means the correspondence of the matrix elements and the momenta for the intermediate states) the detailed explanation of the results for the particle-hole formfactors is absent. Therefore their results on the particle-hole formfactors are not grounded enough. It is the main goal of the present paper to present the detailed derivation of the results for the particle-hole formfactors in the framework of the bosonization approach. Next, in Ref.[6] the scaling relations for the lowest formfactors are derived using the conformal field theory. However, clearly, it would be successive to obtain the results for both type of the formfactors within the same method. Thus it is desirable to derive the scaling relations for the lowest formfactors entirely using the bosonization technique without use of the conformal field theory. Thus the second goal of the present paper is to derive the scaling relations for the lowest formfactors in the framework of the bosonization approach. To achieve these goals we introduce the extended bosonization concept (see the end of Section 2). Next we present an explicit expressions fo the particle-hole formfactors for different operators for various one-dimensional models. This goal can also be achieved only in the framework of the bosonization technique. We also point out the summation formulas which can be used to calculate the sum over the intermediate states for the correlators in the framework of the approach of ref.[7] to the correlators in the integrable (XXZ- spin chain) model. Finally, we present the results for the particle-hole formfactors for the excitations corresponding to the left Fermi-point. The short version of the present paper was published in Ref.[8].

In Section 2 we fix the notations and briefly review the theory of Luttinger liquid and the bosonization procedure. In Section 3 we explain how to derive the scaling relations for the lowest formfactors using the bosonization approach. Finally in Section 4 we derive the particle-hole formfactors for the low-lying states and present an explicit expressions for the formfactors of various operators for different models. We also present the summation formulas relevant to the calculation of the correlators.

2. Bosonization.

Consider the effective low -energy Hamiltonian which build up from the fermionic operators ($a_k,\ c_k,\ k = 2\pi n/L,\ n \in Z$, L- is the length of the chain)

$$a^+(x) = \frac{1}{\sqrt{L}} \sum_k e^{-ikx} a_k^+, \qquad a_k^+ = \frac{1}{\sqrt{L}} \int_0^L dx e^{ikx} a^+(x),$$

corresponding to the excitations around the right and the left Fermi- points and consists of the kinetic energy term and the interaction term $H = T + V$ with the coupling constant λ:

$$H = \sum_k k(a_k^+ a_k - c_k^+ c_k) + 2\pi\lambda/L \sum_{k,k',q} a_k^+ a_{k+q} c_{k'}^+ c_{k'-q}. \tag{1}$$

Defining the operators [9]

$$\rho_1(p) = \sum_k a_{k+p}^+ a_k, \qquad \rho_2(p) = \sum_k c_{k+p}^+ c_k,$$

where $|k|, |k+p| < \Lambda$, where Λ is some cut-off energy, which for the states with the filled

Dirac sea have the following commutational relations

$$[\rho_1(-p);\rho_1(p')] = \frac{pL}{2\pi}\delta_{p,p'} \qquad [\rho_2(p);\rho_2(-p')] = \frac{pL}{2\pi}\delta_{p,p'},$$

one can represent the Hamiltonian in the following form:

$$H = \frac{2\pi}{L}\sum_{p>0}\left(\rho_1(p)\rho_1(-p) + \rho_2(-p)\rho_2(p)\right) + \lambda\sum_{p>0}\frac{2\pi}{L}\left(\rho_1(p)\rho_2(-p) + \rho_1(-p)\rho_2(p)\right).$$

To evaluate the correlators in the system of finite length and make the connection with the conformal field theory predictions, one can proceed as follows. First one can define the lattice fields $n_{1,2}(x)$ with the help of the Fourier transform as

$$\rho_{1,2}(p) = \int_0^L dx e^{ipx} n_{1,2}(x), \qquad n_{1,2}(x) = \frac{1}{L}\sum_p e^{-ipx}\rho_{1,2}(p)$$

This fields have a physical meaning of the local number of the fermions above the Fermi level at the right and the left Fermi points. In terms of this fields the Hamiltonian has the following form:

$$H = 2\pi\sum_x\left(\frac{1}{2}(n_1^2(x) + n_2^2(x)) + \lambda n_1(x)n_2(x)\right)$$

Considering the average distribution of the number of extra particles we obtain the ground state energy in the form

$$\Delta E = \frac{2\pi}{L}\left(\frac{1}{2}\left((\Delta N_1)^2 + (\Delta N_2)^2\right) + \lambda\Delta N_1\Delta N_2\right)$$

where $\Delta N_{1,2}$ - are the numbers of additional particles at the two Fermi points. One can also rewrite the ground state energy in the sector with the total number of particles and the momentum $\Delta N = \Delta N_1 + \Delta N_2,\;\; \Delta Q = \Delta N_1 - \Delta N_2$ in such a way that the total Hamiltonian takes the following form (this form was first proposed in ref.[2]):

$$H = u(\lambda)\sum_p |p| b_p^+ b_p + \frac{\pi}{2L}u(\lambda)\left[\xi(\Delta N)^2 + (1/\xi)(\Delta Q)^2\right], \tag{2}$$

where the parameters $u(\lambda) = (1-\lambda^2)^{1/2}$ and $\xi = ((1+\lambda)/(1-\lambda))^{1/2}$. Next one establishes the commutational relations for the fields $n_{1,2}(x)$:

$$[n_1(x);n_1(y)] = -\frac{i}{2\pi}\delta'(x-y) \qquad [n_2(x);n_2(y)] = \frac{i}{2\pi}\delta'(x-y)$$

Then introducing the new variables $\tilde{n}_{1,2}(x) = \sqrt{2\pi}\, n_{1,2}(x)$, we have the following density of the Hamiltonian

$$H = \frac{1}{2}\left(\tilde{n}_1(x)\tilde{n}_1(x) + \tilde{n}_2(x)\tilde{n}_2(x)\right) + \lambda\tilde{n}_1(x)\tilde{n}_2(x). \tag{3}$$

We also have the following commutational relations $[\tilde{n}_1(x);\tilde{n}_1(y)] = -i\delta'(x-y)$. We then have the following conjugated field and the momenta:

$$\pi(x) = -\frac{1}{\sqrt{2}}(\tilde{n}_1(x)-\tilde{n}_2(x)); \qquad \partial_x\phi(x) = \frac{1}{\sqrt{2}}(\tilde{n}_1(x)+\tilde{n}_2(x))$$

In terms of these variables the Hamiltonian takes the following form:

$$H = \frac{1}{2}u(\lambda)\left[(1/\xi)\pi^2(x)+\xi(\partial\phi(x))^2\right] = \frac{1}{2}u(\lambda)\left[\hat{\pi}^2(x)+(\partial\hat{\phi}(x))^2\right], \tag{4}$$

where

$$\pi(x) = \sqrt{\xi}\;\hat{\pi}(x), \quad \phi(x) = (1/\sqrt{\xi})\hat{\phi}(x). \tag{5}$$

The last equation (5) is nothing else as the canonical transformation, which is equivalent to the Bogoliubov transformation for the original operators $\rho_{1,2}(p)$. Next to establish the expressions for Fermions one should use the commutational relations $[a^+(x);\rho_1(p)] = -e^{ipx}a^+(x)$ and the same for $c^+(x)$. Note that these last relations were obtained using the expression with original fermions: $\rho_1(p) = \int dy e^{ipy}a^+(y)a(y)$. In this way we obtain the following expressions for fermionic operators:

$$a^+(x) = K_1^+\frac{1}{\sqrt{2\pi\alpha}}\exp\left(\frac{2\pi}{L}\sum_{p\neq 0}\frac{\rho_1(p)}{p}e^{-ipx}e^{-\alpha|p|/2}\right) = K_1^+\frac{1}{\sqrt{2\pi\alpha}}\exp\left(-i2\pi\tilde{N}_1(x)\right), \tag{6}$$

$$c^+(x) = K_2^+\frac{1}{\sqrt{2\pi\alpha}}\exp\left(-\frac{2\pi}{L}\sum_{p\neq 0}\frac{\rho_2(p)}{p}e^{-ipx}e^{-\alpha|p|/2}\right) = K_2^+\frac{1}{\sqrt{2\pi\alpha}}\exp\left(i2\pi\tilde{N}_2(x)\right),$$

where the fields $\tilde{N}_{1,2}(x)$ are the analogs of the particle numbers with positions left to the point x and $K_{1,2}^+$ are the Klein factors - the operators which commute with the operators $\rho_{1,2}(p)$ and create the single particle at the right (left) Fermi -points when acting on the ground state. The parameter $\alpha \to 0$ is introduced to perform the ultraviolet cutoff. The operators (6) have the correct anticommutational relations.

It is assumed that there is an equality between the matrix elements of the specific 1D model for the low-energy states and the matrix elements of the corresponding operators over the corresponding eigenstates in the Luttinger liquid theory. The mapping between the two models is characterized by a single parameter ξ which should be the same for both models in a sense of the equation (2) i.e. the constant ξ in the effective Luttinger model is taken from the expression (2) for the original model. Thus the critical exponents in the asymptotics of the correlators for the 1D models are expressed in terms of the single parameter ξ i.e. exhibit the universal behaviour. The constant ξ determines the asymptotic behaviour of the correlators in the original model according to the prescriptions for the Luttinger model. This hypothesis is confirmed by the fact that the expression (2) is valid for the original model with the corresponding true speed of sound $v = u(\lambda)$. This was proved both for the continuous Bose or Fermi liquids and for the XXZ- spin chain [2].

We propose that the correspondence between the two models is not limited only by the correspondence between the critical exponents. There is an exact mapping between the operators and the eigenstates for the original and the Luttinger models. The matrix elements and the correlators in the original model are *equal* to the matrix elements and the correlators in the Luttinger model provided the operators are replaced by the corresponding operators in the Luttinger model and the eigenstates are replaced by the corresponding eigenstates. This correspondence for the low-lying states is valid if at the large distances the correlators are effectively described by the small energies and the contributions of the large energies are parametrized by the non-universal constants entering the definition of the operators in the Luttinger model. Thus for example the correlators at large distances can be calculated exactly in the framework of the effective low-energy Luttinger model. One can call these ideas by the extended bosonization concept, which means the exact mapping between the original and the Luttinger liquid models.

3. Lowest formfactors.

Let us start with the derivation of the scaling relations for the lowest formfactors for the XXZ- quantum spin chain:

$$H = \sum_{i=1}^{L} \left(\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right),$$

where the sites $L+1$ and 1 are coincide. First, let us establish the relation for the formfactors of $\sigma_i^{\pm}$- operators. The relations for the other operators (σ_i^z) as well as for the formfactors for the other 1D systems can be obtained in a similar way. Calculation of finite - size corrections to the energy of the ground state for the XXZ- spin chain (see for example [10]) leads to the expression (2) and allows one to obtain the parameter ξ which leads to the predictions of critical indices according to the conformal field theory. The calculation gives the value $\xi = 2(\pi - \eta)/\pi$, where the parameter η is connected with the anisotropy parameter of the XXZ - chain as $\Delta = \cos(\eta)$. Using the Jordan-Wigner transformation $\sigma_x^+ = a_x^+ \exp(i\pi N(x))$, where a_x^+ stands for the "original" lattice fermionic operator, $N(x) = \sum_{l=1}^{x-1} n_l$ $(n_l = a_l^+ a_l)$, and performing the obvious substitutions $N(x) \to x/2 + N_1(x) + N_2(x)$ and $a_x^+ \to e^{-ip_F x} a^+(x) + e^{ip_F x} c^+(x)$, $\quad p_F = \pi/2$, we obtain after the canonical transformation (5) the expression for the leading term in the asympotics of the correlator for the XXZ -chain:

$$\langle \sigma_x^+ \sigma_0^- \rangle = \langle a_x^+ e^{i\pi N(x)} a_0 \rangle \sim (-1)^x \langle 0 | e^{-i\pi\sqrt{\xi}(\tilde{N}_1(x) - \tilde{N}_2(x))} e^{i\pi\sqrt{\xi}(\tilde{N}_1(0) - \tilde{N}_2(0))} | 0 \rangle, \qquad (7)$$

where $\tilde{N}_{1,2}(x)$ - are corresponds to the free fields $\hat{\pi}(x)$, $\hat{\phi}(x)$, obtained after the transformation (5). To these operators correspond the new operators $\rho_{1,2}(p)$ and the new fermionic operators (quasiparticles). Averaging the product of exponents in bosonic operators for the expression (7) and using the properties of $\rho_{1,2}(p)$, $\langle\rho_1(-p)\rho_1(p)\rangle = \frac{pL}{2\pi}\theta(p)$ and $\langle\rho_2(p)\rho_2(-p)\rangle = \frac{pL}{2\pi}\theta(p)$, we get for the correlation function $G(x) = \langle 0|\sigma^+_{i+x}\sigma^-_i|0\rangle$ the following sum in the exponent:

$$C\,\exp\left(\frac{\xi}{4}\sum_{n=1}^{\infty}\frac{1}{n}e^{in(2\pi x/L)} + h.c.\right),$$

where C - is some constant. Then using the formula $\sum_{n=1}^{\infty}\frac{1}{n}z^n = -\ln(1-z)$ and substituting the value $\xi = 2(\pi-\eta)/\pi$ we obtain the following expression for the XXZ - chain:

$$G(x) = C_0\frac{(-1)^x}{\left(L\sin(\frac{\pi x}{L})\right)^{\alpha}}, \quad \alpha = \frac{\xi}{2} = \frac{\pi-\eta}{\pi} \quad (x >> 1). \tag{8}$$

Thus, although bosonization, which deals with the low-energy effective theory, is not able to predict the constant before the asymptotics, the critical exponent and the functional form are predicted in accordance with conformal field theory.

From the relation (7) it follows the following equation (the local operators of the XXZ- chain should be represented as the local operators of the Luttinger model):

$$\sigma_0^- = C'K_1 e^{i\pi\sqrt{\xi}(\tilde{N}_1(0)-\tilde{N}_2(0))} + \ldots. \tag{9}$$

where C' is some constant and the dots stand for the different (subleading) operators. Eq.(9) should be understood in a sense of the correspondence between the XXZ and the Luttinger liquid models: the formfactors for the corresponding states should be equal to each other. In particular for the ground states of M ($|t\rangle$) and $M-1$ ($|\lambda\rangle$) particles (up-spins) for the XXZ spin chain we obtain:

$$\langle\lambda|\sigma_0^-|t\rangle = C'\langle 0|e^{i\pi\sqrt{\xi}(\tilde{N}_1(0)-\tilde{N}_2(0))}|0\rangle. \tag{10}$$

This equation is valid if we assume that the eigenstate $\langle\lambda|$ corresponds to the state $\langle -1|_{(1)}$ i.e. the state with the absence of particle at the first branch of the Luttinger model. Calculating the average at the right-hand side of the equation (10) we obtain $C' = C(L/2\pi\alpha)^{\xi/4}$, where $C = \langle\lambda|\sigma_0^-|t\rangle$ is the value of the lowest formfactor. Next calculating the correlator $G(x)$ in the framework of the Luttinger model using the equation (9) we obtain the following result:

$$G(x) = (C')^2\langle 0|e^{-i\pi\sqrt{\xi}(\tilde{N}_1(x)-\tilde{N}_2(x))}e^{i\pi\sqrt{\xi}(\tilde{N}_1(0)-\tilde{N}_2(0))}|0\rangle = \frac{C^2}{(2\sin(\pi x/L)^{\xi/2}}. \tag{11}$$

Note that the dependence of $G(x)$ on the parameter α is cancelled due to the dependence of the constant $C' = C(L/2\pi\alpha)^{\xi/4}$ on α which indicates the independence of the scaling relations on the details of physics at high momenta which is model-dependent. Comparing the equation (11) with the equation (8) we obtain the desired relation between the lowest formfactor C and the prefactor C_0:

$$C^2 = \left(\frac{2}{L}\right)^{\xi/2} C_0. \tag{12}$$

For the case of the XX- spin chain ($\xi = 1$) the relation between the formfactor and the prefactor (12) coincides with the relation obtained in Ref.[11] using the completely different method. Thus we see that while the prefactors are not the universal quantities the relations between them and the corresponding formfactors are universal.

Let us consider the density-density correlator for the XXZ- spin chain, namely $\Pi(x) = \langle \sigma_x^z \sigma_0^z \rangle$. Substituting the expressions for the original lattice Jordan-Wigner fermions into the density operator $n_x = a_x^+ a_x$, we obtain the general expression for the density in terms of the Luttinger liquid operators:

$$n_x = \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{\xi}} \partial_x \hat{\phi}(x) + C_1' e^{-i2p_F x} K_1^+ K_2 e^{-i2\pi(1/\sqrt{\xi})\left(\tilde{N}_1(x)+\tilde{N}_2(x)\right)} + h.c. + \ldots, \tag{13}$$

where $p_F = \pi/2$, the operators $\tilde{N}_{1,2}(x)$ correspond to the new fields $\hat{\pi}(x), \hat{\phi}(x)$ and the dots stand for the operators, corresponding to the higher order terms in the expansion of the correlator. Consider the matrix element $\langle t'|n_0|t\rangle = C_1$, where $|t\rangle$ - is the ground state of the XXZ- spin chain and $|t'\rangle$ is the eigenstate which is obtained from the ground state by adding one particle at the right Fermi- point and removing one particle from the left Fermi- point. Taking the corresponding matrix element for both sides of the equation (13) we obtain:

$$\langle t'|n_0|t\rangle = C_1 = C_1' \langle 0| e^{-i2\pi(1/\sqrt{\xi})\left(\tilde{N}_1(0)+\tilde{N}_2(0)\right)} |0\rangle. \tag{14}$$

Calculating the average at the right- hand side side of Eq.(14) we obtain the constant $C_1' = C_1 (L/2\pi\alpha)^{1/\xi}$. Calculating the correlator $\Pi(x)$ in the framework of the Luttinger liquid theory using the equation (13), we obtain the expression for the first two terms in the expansion in the form:

$$\Pi(x) = -\frac{1}{2\xi(L\sin(\pi x/L))^2} + e^{i2p_F x} \frac{C_1^2}{(2\sin(\pi x/L))^{2/\xi}} + h.c.. \tag{15}$$

Again the parameter α drops out of this equation. Comparing this equation with the general expression for the correlator:

$$\Pi(x) = -\frac{1}{2\xi(L\sin(\pi x/L))^2} + \frac{C_{10}\cos(2p_F x)}{(L\sin(\pi x/L))^{2/\xi}} + \ldots, \tag{16}$$

we obtain the scaling relation:

$$C_1^2 = \frac{1}{2} \left(\frac{2}{L}\right)^{2/\xi} C_{10}. \tag{17}$$

The equation (17) was proved explicitly by direct computations for the XXZ spin chain in Ref.[5].

The examples presented above allow one to conclude the following. First, it is clear that in the same way the similar relations can be obtained for the other models of the 1D quantum liquids, in particular for the correlators of the continuous Bose- or Fermi-liquids. Second, the scaling relations can be easily generalized to the case of the lowest formfactors corresponding to the higher terms in the asymptotics of the correlators. The corresponding states are the states obtained from the ground state by moving an arbitrary number (m) of particles from the left to the right Fermi- points, which corresponds to the operators containing an additional powers of the operator $(a^+(x)c(x))$: $(a^+c)^m$. It is clear that in all the cases the results have the same form as the equations (12), (17) with the corresponding critical exponent $\alpha(m)$. An explicit expressions for the correlators and the corresponding scaling relations for the 1D continuous Bose- and Fermi- liquids are presented in Ref.[6]. For completeness we present these results in the Appendix A. Here we would like to stress once more that these results of Ref.[6] can be equally well obtained in the framework of the bosonization approach.

4. Particle-hole formfactors.

Let us consider the calculation of the low-energy particle-hole formfactors in the framework of the bosonization approach. As an example consider the formfactor of the operator σ_0^- for the XXZ spin chain. Suppose we have the eigenstate $\langle\lambda(p_i, q_i)|$ obtained from the ground state by creating the holes with the momenta q_i and the particles with the momenta p_i ($i = 1, \ldots n$) located in the vicinity of the right Fermi- point. The formfactor corresponding to this state has the following representation in terms of Luttinger liquid matrix element:

$$\langle\lambda(p_i, q_i)|\sigma_0^-|t\rangle = C'\langle p_i, q_i|e^{i\pi\sqrt{\xi}(\tilde{N}_1(0)-\tilde{N}_2(0))}|0\rangle, \tag{18}$$

where at the right-hand side $p_i > 0$ and $q_i \leq 0$ are the positions of the particles and the holes at the first branch of the Luttinger liquid (i.e. correspond to the operators $a^+(x)$, $a(x)$) and the constant C' was introduced in Eq.(9). Calculating the average at the right-hand side of the equation (18) we obtain the following result:

$$\langle\lambda(p_i, q_i)|\sigma_0^-|t\rangle = C\langle p_i, q_i|e^{a\frac{2\pi}{L}\sum_{p>0}\frac{\rho_1(p)}{p}}|0\rangle, \qquad a = -\sqrt{\xi}/2, \tag{19}$$

where C is the value of the lowest formfactor $C = \langle\lambda|\sigma_0^-|t\rangle$. The average at the right-hand side of the equation (19) was calculated in Ref.[12] (see Appendix B):

$$\langle p_i, q_i|e^{a\frac{2\pi}{L}\sum_{p>0}\frac{\rho_1(p)}{p}}|0\rangle = F_a(p_i, q_i) = \det_{ij}\left(\frac{1}{p_i - q_j}\right)\prod_{i=1}^{n} f^+(p_i)\prod_{i=1}^{n} f^-(q_i), \tag{20}$$

where

$$f^{+}(p) = \frac{\Gamma(p+a)}{\Gamma(p)\Gamma(a)}, \qquad f^{-}(q) = \frac{\Gamma(1-q-a)}{\Gamma(1-q)\Gamma(1-a)}.$$

In the equation (20) p_i and q_i are assumed to be integers (corresponding to the momenta $2\pi p_i/L$ and $2\pi q_i/L$) and $p_i > 0$, $q_i \leq 0$.

Thus we have calculated the particle-hole formfactor in the form $\langle \lambda(p_i, q_i)|\sigma_0^-|t\rangle = CF(p_i, q_i)$, and have shown that the constant C is in fact the lowest formfactor, which is not clear within the approach of Ref.[6]. The same analysis can be performed for the formfactor $\langle t'(p_i, q_i)|n_0|t\rangle = C_1 F_a(p_i, q_i)$, where the constant a in Eq.(20) is now equals $a = 1/\sqrt{\xi}$, and the formfactors for the continuous Bose- and Fermi- liquids. Note that the dependence on the particles and holes momenta (20) can be easily obtained from the expressions for the formfactors in the case of the XX- spin chain in the form of the Cauchy determinant in Ref.[11] ($\xi = 1$, $a = -1/2$). We considered the formfactors corresponding to the particles and the holes located near the right Fermi- point. The same analysis can be performed for the particles and the holes located near the left Fermi- point (see below). Clearly, the total formfactor is a product of these two terms. Note that the corresponding particle-hole formfactors for the continuous Bose- liquid are given by the same formulas with the same value of the parameter a.

To sum up the formfactor series for the correlators in the framework of the approach of Ref.[7] (where the dependence on the particle and hole positions was found for the formfactors of the XXZ- spin chain) it is useful to have the formula for the sum:

$$\sum_{n} \sum_{p_i > 0, q_i \leq 0} |F_a(p_i, q_i)|^2 e^{i(p-q)2\pi x/L} = \frac{1}{(1 - e^{i2\pi x/L})^{a^2}}, \tag{21}$$

where $p = \sum_{i=1}^{n} p_i$, $q = \sum_{i=1}^{n} q_i$ and p_i, q_i are integers. The result (21) can be easily obtained with the help of the following correlator:

$$G_a(x) = \langle 0|e^{-a\frac{2\pi}{L}\sum_{p<0}\frac{\rho_1(p)}{p}e^{-ipx}} e^{a\frac{2\pi}{L}\sum_{p>0}\frac{\rho_1(p)}{p}}|0\rangle = \frac{1}{(1 - e^{i2\pi x/L})^{a^2}},$$

which can be calculated either with the help of inserting of the complete set of intermediate states or by means of the standard formulas used in the bosonization approach. Thus the two sides of the equation (21) should be equal to each other.

Finally, it would be interesting to verify the sum rule for the function $F(p_i, q_i)$ which can be obtained from the equation (21) by means of expanding it into the Fourier series:

$$\sum_{n} \sum_{p_i, q_i, p-q=m} |F_a(p_i, q_i)|^2 = \frac{\Gamma(a^2 + m)}{\Gamma(m+1)\Gamma(a^2)},$$

where $p = \sum_{i=1}^{n} p_i$, $q = \sum_{i=1}^{n} q_i$.

Now let us consider the particle-hole excitations in the vicinity of the left Fermi-point. For the case of the operator σ_0^- for the XXZ- spin chain the formfactor is given by the equation

$$\langle\lambda(p_i,q_i)|\sigma_0^-|t\rangle = C'\langle p_i,q_i|e^{i\pi\sqrt{\xi}(\tilde{N}_1(0)-\tilde{N}_2(0))}|0\rangle, \tag{22}$$

where now the momenta $p_i < 0$, $q_i \geq 0$ are the positions of the particles and the holes at the second branch of the Luttinger liquid, i.e. the momenta with respect to the left Fermi-point. Calculating the average at the right-hand side of Eq.(22) we obtain:

$$\langle\lambda(p_i,q_i)|\sigma_0^-|t\rangle = C\langle p_i,q_i|e^{c\frac{2\pi}{L}\sum_{p<0}\frac{\rho_2(p)}{p}}|0\rangle, \qquad c=\sqrt{\xi}/2, \tag{23}$$

where C is again the value of the lowest formfactor $C=\langle\lambda|\sigma_0^-|t\rangle$. The average at the right-hand side of Eq.(23) is calculated in the Appendix B:

$$\langle p_i,q_i|e^{c\frac{2\pi}{L}\sum_{p<0}\frac{\rho_2(p)}{p}}|0\rangle = F_c(p_i,q_i) = \det_{ij}\left(\frac{1}{p_i-q_j}\right)\prod_{i=1}^{n} f^+(p_i)\prod_{i=1}^{n} f^-(q_i), \tag{24}$$

where now

$$f^+(p)=\frac{\Gamma(-p-c)}{\Gamma(-p)\Gamma(1-c)}, \qquad f^-(q)=\frac{\Gamma(1+q+c)}{\Gamma(1+q)\Gamma(c)}.$$

In the equation (24) p_i and q_i are assumed to be integers (corresponding to the momenta $2\pi p_i/L$ and $2\pi q_i/L$) and $p_i<0$, $q_i\geq 0$. The same analysis can be performed for the formfactor $\langle t'(p_i,q_i)|n_0|t\rangle = C_1F_c(p_i,q_i)$, where the particles and the holes are located near the left Fermi-point and the constant c in Eq.(24) is now equals $c=-1/\sqrt{\xi}$, and the formfactors for the continuous Bose- and Fermi- liquids. For example for the continuous Bose- liquid we obtain the same expression as for the XXZ- spin chain:

$$\langle\lambda(p_i,q_i)_{1(2)}|\phi(0)|t\rangle = CF_{a(c)}(p_i,q_i),$$

with $a=-\sqrt{\xi}/2$, $c=\sqrt{\xi}/2$, where the field $\phi(x)$ corresponds to the Bose- particles and C is the value of the lowest formfactor $\langle\lambda|\phi(0)|t\rangle$. The formfactor $\langle t'(p_i,q_i)|\rho(0)|t\rangle$ of the density operator $\rho(x)$ for the Bose- and Fermi- liquids also coincides with the corresponding particle-hole formfactor $\langle t'(p_i,q_i)|n_0|t\rangle$ for the XXZ- spin chain.

To sum up the formfactor series for the correlators it is useful to have the formula for the sum:

$$\sum_n\sum_{p_i<0,q_i\geq 0}|F_c(p_i,q_i)|^2 e^{i(p-q)2\pi x/L} = \frac{1}{(1-e^{-i2\pi x/L})^{c^2}}, \tag{25}$$

where $p=\sum_{i=1}^n p_i$, $q=\sum_{i=1}^n q_i$ and p_i, q_i are integers. The result (25) can be easily obtained with the help of the following correlator:

$$G_c(x)=\langle 0|e^{-c\frac{2\pi}{L}\sum_{p>0}\frac{\rho_2(p)}{p}e^{-ipx}}e^{c\frac{2\pi}{L}\sum_{p<0}\frac{\rho_2(p)}{p}}|0\rangle = \frac{1}{(1-e^{-i2\pi x/L})^{c^2}},$$

which can be calculated again either with the help of inserting of the complete set of intermediate states or by means of the standard formulas used in the bosonization approach. Thus the two sides of the equation (25) should be equal to each other.

Let us mention that if one have the formulas (21), (25) and the form of the particle-hole formfactors then one can prove the scaling relations for the lowest formfactors. Alternatively, if one have the formulas (21), (25) and the scaling relations for the lowest formfactors, then assuming the formulas (19), (23) one can prove that the constant C in this formulas is in fact the lowest formfactor.

Finally let us calculate the particle-hole formfactors for the continuous Fermi-liquid. We have the following expression for the Fermi-operator:

$$\psi(0) = C' K_1 e^{i\pi\sqrt{\xi}(\tilde{N}_1(0)-\tilde{N}_2(0))+i\pi(1/\sqrt{\xi})(\tilde{N}_1(0)+\tilde{N}_2(0))}+$$

$$C' K_2 e^{i\pi\sqrt{\xi}(\tilde{N}_1(0)-\tilde{N}_2(0))-i\pi(1/\sqrt{\xi})(\tilde{N}_1(0)+\tilde{N}_2(0))}+\ldots,$$

where the dots stand for the different powers of the Klein factors. The constant $C' = C(L/2\pi\alpha)^{(1/4)(\xi+1/\xi)}$ is related to the lowest formfactors

$$C = \langle\lambda_1|\psi(0)|t\rangle = \langle\lambda_2|\psi(0)|t\rangle,$$

where $\langle\lambda_1|$ ($\langle\lambda_2|$) is the eigenstate corresponding to the absense of particle at the right(left) Fermi-point. We then have the following expressions for the particle-hole formfactors corresponding to the right(left) Fermi-point:

$$\langle\lambda_1(p_i,q_i)_1|\psi(0)|t\rangle = CF_a(p_i,q_i), \qquad a = -\frac{1}{2}(\sqrt{\xi}+1/\sqrt{\xi}),$$

$$\langle\lambda_1(p_i,q_i)_2|\psi(0)|t\rangle = CF_c(p_i,q_i), \qquad c = \frac{1}{2}(\sqrt{\xi}-1/\sqrt{\xi}),$$

$$\langle\lambda_2(p_i,q_i)_1|\psi(0)|t\rangle = CF_a(p_i,q_i), \qquad a = -\frac{1}{2}(\sqrt{\xi}-1/\sqrt{\xi}),$$

$$\langle\lambda_2(p_i,q_i)_2|\psi(0)|t\rangle = CF_c(p_i,q_i), \qquad c = \frac{1}{2}(\sqrt{\xi}+1/\sqrt{\xi}).$$

For the formfactors corresponding to the eigenstates with the particles and the holes located at both sides of the Fermi-interval we have the product of these two functions $F_a(p_i,q_i)$ and $F_c(p_i,q_i)$.

In conclusion, for various one-dimensional quantum liquids in the framework of the Luttinger model (bosonization) we established the relations between the prefactors of the correlators and the formfactors of the corresponding local operators. The physical reason for existing of these relations is that in 1D only the low-lying particle-hole excitations contribute to the power-law asymptotics of the correlators. In fact, it turns out that in the 1D models there is the separation of energies. The contributions of large energies are all included in the non-universal constants in front of the operators of the effective low-energy Luttinger model, while the contributions of small energies are described by the operators of the Luttinger liquid. Since the sums in the equations (21), (25) are convergent, at large distances the correlator is determined by the small energies so it can be calculated exactly in the framework of the effective Luttinger liquid theory. The derivation of the scaling relations in the framework of the bosonization procedure allows one to substantiate the prediction for the formfactors corresponding to the low-lying particle-hole excitations. Let us stress once more that the relations of the type (19) can be obtained only in the framework of the bosonization procedure, since only within this approach one can see that the constant C in Eq.(19) is in fact the lowest formfactor. Second, using the so called extended bosonization concept introduced in the present paper (exact mapping of the original model to the Luttinger liquid model) we derived the expressions for the particle-hole formfactors. Let us note that these results could be obtained also from the correspondence of the two correlators of Eq.(7). We obtained an explicit expressions for the particle-hole formfactors both for the XXZ- spin chain and the continuous Bose- and Fermi- liquids including the particle-hole formfactors with the excitations corresponding to the left Fermi- point. We also obtained the formulas for the summation over the particle-hole states corresponding to the power-law asymptotics of the correlators.

Appendix A.

Here we present without derivation the scaling relations for the formfactors for the continuous Bose- and Fermi- liquids and the XXZ- spin chain including the relations for the higher order terms in the asymptotics of the correlators. The results for the Bose-liquid have exactly the same form as the results for the XXZ- spin chain. We denote by $\phi(x)$ ($\psi(x)$) the fields corresponding to the Bose (Fermi) particles and by $\rho(x)$ the density operator. The general expressions for the expansions of the Bose- field and the density operators are:

$$\phi(x) = \sum_m C'_m e^{-i2p_F mx} e^{i\pi\sqrt{\xi}(\tilde{N}_1(x)-\tilde{N}_2(x))} e^{-i2\pi m(1/\sqrt{\xi})(\tilde{N}_1(x)+\tilde{N}_2(x))},$$

$$\psi(x) = \sum_m B'_m e^{i(2m+1)p_F x} e^{i\pi\sqrt{\xi}(\tilde{N}_1(x)-\tilde{N}_2(x))} e^{i\pi(2m+1)(1/\sqrt{\xi})(\tilde{N}_1(x)+\tilde{N}_2(x))},$$

$$\rho(x) = \frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\xi}}\partial_x\hat{\phi}(x) + \sum_{m\neq 0} A'_m e^{-i2p_F mx} e^{-i2\pi m(1/\sqrt{\xi})(\tilde{N}_1(x)+\tilde{N}_2(x))},$$

where $p_F = \pi\rho_0$ is the Fermi- momentum (here we omit the Klein factors, for the correlators one should take the averages of the terms, corresponding to the same harmonics (same m)). Exactly the same expressions hold for the operator σ_x^- and σ_x^z for the XXZ- spin chain ($p_F = \pi/2$) The general expressions for the equal-time correlators have the form:

$$G_B(x) = \langle\phi^+(x)\phi(0)\rangle = \sum_{m\geq 0} C_m \frac{\cos(2p_F mx)}{(L\sin(\pi x/L))^{\xi/2+m^2(2/\xi)}},$$

$$G_F(x) = \langle \psi^+(x)\psi(0)\rangle = \sum_{m\geq 0} B_m \frac{\sin((2m+1)p_F x)}{(L\sin(\pi x/L))^{\xi/2+(2m+1)^2/2\xi}},$$

$$\Pi(x) = \langle \rho(x)\rho(0)\rangle = \rho_0^2 - \frac{1}{2\xi(L\sin(\pi x/L))^2} + \sum_{m\geq 1} A_m \frac{\cos(2p_F m x)}{(L\sin(\pi x/L))^{(2/\xi)m^2}}.$$

The expressions for $G_B(x)$ and $\Pi(x)$ also hold for the XXZ- spin chain (the correlators $G(x)$ and $\Pi(x)$). Then the scaling relations for the lowest formfactors have the following form:

$$|\langle \lambda(m)|\phi(0)|t\rangle|^2 = \frac{(-1)^m C_m}{2-\delta_{0,m}} \left(\frac{2}{L}\right)^{\xi/2+m^2(2/\xi)},$$

$$|\langle \lambda(m)|\psi(0)|t\rangle|^2 = \frac{(-1)^m B_m}{2} \left(\frac{2}{L}\right)^{\xi/2+(2m+1)^2/2\xi},$$

$$|\langle t(m)|\rho(0)|t\rangle|^2 = \frac{A_m}{2} \left(\frac{2}{L}\right)^{(2/\xi)m^2},$$

where $|\lambda(m)\rangle$ is the eigenstate with the number of particles equal to $M-1$ and with m particles removed from the right and created at the left Fermi- point, and $|t\rangle$ is the ground state of M particles ($\rho_0 = M/L$). For fermions the formfactors $\langle \lambda_{1,2}|\psi(0)|t\rangle$ introduced above correspond to the formfactor $\langle \lambda(0)|\psi(0)|t\rangle$ from the last equations and the higher formfactors $\langle \lambda(m)|\psi(0)|t\rangle$ ($m > 0$) correspond to the states with the momentum $\pm(2m+1)p_F$. The two eigenstates with an opposite momentum correspond to the two different harmonics in the factor $\sin((2m+1)p_F x)$. For bosons the eigenstate $|\lambda(0)\rangle$ is not degenerate. For the case of the XXZ- spin chain the particles correspond to the up-spins and we have the same formulas as the formulas for the bosons ($\phi(0) \to \sigma_0^-$, $\rho(0) \to \sigma_0^z$). The factors $(-1)^m$ in the last equations appear in the process of the calculations of the corresponding averages for the correlators at $m \neq 0$.

Appendix B.

Let us calculate the matrix elements (20), (24) which correspond to the particle-hole formfactors. Let us begin with the matrix element (20) corresponding to the first branch of the Luttinger model [12]. We start with the matrix element for the operators in the coordinate space $\langle 0|a^+(x)a(y)e^{B_a}|0\rangle$, $B_a = a(2\pi/L)\sum_{p>0}\rho_1(p)/p$. Using the formula $Ae^B = e^B(\sum_n (1/n!)\,[A,B]_n)$, one can commute the exponent e^{B_a} to the left. Thus we obtain the following equation:

$$\langle 0|a^+(x)a(y)e^{B_a}|0\rangle = \left(\frac{1-e^{ix'}}{1-e^{iy'}}\right)^a \langle 0|a^+(x)a(y)|0\rangle = \frac{1}{L}\left(\frac{1-e^{ix'}}{1-e^{iy'}}\right)^a \frac{e^{iy'}}{e^{iy'}-e^{ix'}},$$

where $x' = 2\pi x/L$, $y' = 2\pi y/L$. Then for the derivative we obtain

$$(i\partial_x + i\partial_y)\langle 0|a^+(x)a(y)e^{B_a}|0\rangle = -a\frac{2\pi}{L^2}\left(1-e^{ix'}\right)^{a-1} e^{iy'}\left(1-e^{iy'}\right)^{-(a+1)}.$$

Calculating the Fourier transform for both sides of this equation we obtain for the matrix element $\langle 0|a_q^+ a_p e^{B_a}|0\rangle$ the expression (20) for $n=1$ with the factors

$$f^+(p) = a\int_0^{2\pi} \frac{dy}{2\pi} e^{-i(p-1)y}\left(1-e^{iy}\right)^{-(a+1)}, \qquad p>0,$$

$$f^-(q) = \int_0^{2\pi} \frac{dx}{2\pi} e^{iqx}\left(1-e^{ix}\right)^{a-1}, \qquad q\le 0,$$

where p and q are integers. Calculating the integrals and using the Wick's theorem we obtain the equation (20) with the functions $f^+(p)$, $f^-(q)$ presented in the text.

The derivation of the equation (24) is similar. We consider the matrix element $\langle 0|c^+(x)c(y)e^{B_c}|0\rangle$ with $B_c = c(2\pi/L)\sum_{p<0}\rho_2(p)/p$. Commuting e^{B_c} to the left we obtain:

$$\langle 0|c^+(x)c(y)e^{B_c}|0\rangle = \left(\frac{1-e^{-iy'}}{1-e^{-ix'}}\right)^c \langle 0|c^+(x)c(y)|0\rangle = \frac{1}{L}\left(\frac{1-e^{-iy'}}{1-e^{-ix'}}\right)^c \frac{e^{-iy'}}{e^{-iy'}-e^{-ix'}}.$$

Calculation of the derivatives gives

$$(i\partial_x + i\partial_y)\langle 0|c^+(x)c(y)e^{B_c}|0\rangle = -c\frac{2\pi}{L^2}\left(1-e^{-iy'}\right)^{c-1} e^{-iy'}\left(1-e^{-ix'}\right)^{-(c+1)}.$$

Calculating the Fourier transform we obtain for the matrix element $\langle 0|c_q^+ c_p e^{B_c}|0\rangle$ the expression (24) for $n=1$ with the factors

$$f^+(p) = \int_0^{2\pi} \frac{dy}{2\pi} e^{-i(p+1)y}\left(1-e^{-iy}\right)^{c-1}, \qquad p<0,$$

$$f^-(q) = c\int_0^{2\pi} \frac{dx}{2\pi} e^{iqx}\left(1-e^{-ix}\right)^{-(c+1)}, \qquad q\ge 0.$$

Calculating the integrals we obtain the equation (24) with the functions $f^+(p)$, $f^-(q)$ presented in the text.

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