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Leibniz algebras on symplectic plane and cohomological vector fields

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Abstract

By using help of algebraic operad theory, Leibniz algebra theory and symplectic-Poisson geometry are connected. We introduce the notion of cohomological vector field defined on nongraded symplectic plane. It will be proved that the cohomological vector fields induce the finite dimensional Leibniz algebras by the derived bracket construction. This proposition is a Leibniz analogue of the cohomological field theory in the category of Lie algebras. The basic properties of the cohomological fields will be studied, in particular, we discuss a factorization problem with the cohomological fields and introduce the notion of double-algebra in the category of Leibniz algebras.

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1 Introduction

Leibniz algebras are noncommutative version of Lie algebras, that is, the bracket products are non commutative and satisfy the Leibniz identity instead of the Jacobi identity. The Leibniz algebras are introduced by J-L. Loday, motivated by the study of algebraic K-theory. Hence they are sometimes called the Loday algebras. Such algebras naturally arise in differential geometry and physics, for instance, in Nambu-mechanics as Leibniz algebroids, in topological field theory as Courant algebroids, in Poisson geometry and in representation theory of Lie algebras. Therefore it is interesting work to study the geometric aspect of Leibniz algebra.

In the present paper, we will prove that every finite dimensional Leibniz algebra can be represented on (nongraded-)symplectic plane as a special vector field, which is a *new type of cohomological vector field* and characterizes the Leibniz algebra structure on the symplectic plane.

In first, we rapidly recall a classical theory of cohomological vector field (cf. Kosmann-Schwarzbach [10], Roytenberg [14]). We consider a cohomology complex over a finite dimensional Lie algebra $\mathfrak{g}$,

$$\cdots \xrightarrow{d} \wedge^n \mathfrak{g}^* \xrightarrow{d} \wedge^{n+1}\mathfrak{g}^* \xrightarrow{d} \cdots,$$

where $\mathfrak{g}^*$ is the dual space of $\mathfrak{g}$. The total space $\bigwedge \mathfrak{g}^* = \sum_{n\geq 0} \wedge^n \mathfrak{g}^*$ is called a (linear) super plane[1]. Because the differential d is a derivation on the plane, it is regarded as a tangent vector field. Given a local coordinate, we have the following expression of the field,

$$d = \frac{1}{2} C^k_{ij} x^i \wedge x^j \frac{\partial}{\partial x^k},$$

where $x^{\cdot}$ is an odd variable and C^k_{ij} is the structure constant of the Lie algebra. This is a typical example of cohomological vector fields. In the topological field theory of AKSZ-type, $\bigwedge \mathfrak{g}^*$ is replaced with the extended plane $\bigwedge(\mathfrak{g}\oplus\mathfrak{g}^*)$, which is a canonical plane equipped with an even Poisson bracket defined by

$$\{p_1 + q^1, p_2 + q^2\}_+ := < p_1, q^2 > + < p_2, q^1 >, \tag{1}$$

where $p_i \in \mathfrak{g}$ and $q^i \in \mathfrak{g}^*$ for any $i \in \{1, 2\}$. The vector field d is also replaced by a Hamiltonian vector field, $d_\theta := \{\theta, -\}$, on the canonical plane, where θ is a Hamiltonian polynomial in $\bigwedge(\mathfrak{g}\oplus\mathfrak{g}^*)$ satisfying $\{\theta, \theta\} = 0$. This vector field d_θ deduces a Lie bracket on the double space $\mathfrak{g}\oplus\mathfrak{g}^*$ by the following method,

$$[-_1, -_2]_\theta := (\pm)\{d_\theta(-_1), -_2\},$$

which is called a derived bracket (Kosmann-Schwarzbach [8, 9]). One can easily check that the Lie bracket $[,]_\theta$ satisfies the invariance condition,

$$\{[-_1, -_2]_\theta, -_3\} = \{-_1, [-_2, -_3]_\theta\}.$$

It is known that the Drinfeld doubles defined over $\mathfrak{g}\oplus\mathfrak{g}^*$ are typical solutions of the cohomological condition $d_\theta d_\theta = 0$, or equivalently, $\{\theta, \theta\} = 0$.

Suppose that $\mathfrak{g}$ is a Leibniz algebra. It is known that the operad of Leibniz algebras is anti-cyclic (Getzler-Kapranov [5]), which implies that the invariant 2-forms in the category of Leibniz algebras are **anti-symmetric**. In [4], F. Chapoton proved

[1]The super plane is regarded as a sheaf of the super polynomial algebra over a point.

that the invariance conditions on Leibniz algebras are defined by the following two relations,

$$
\begin{aligned}
(-_1, [-_2, -_3])_- &= -([-_2, -_1], -_3)_-, \\
(-_1, [-_2, -_3])_- &= ([-_1, -_3] + [-_3, -_1], -_2)_-,
\end{aligned}
$$

where $(,)_-$ is an anti-symmetric pairing and $[,]$ is a Leibniz bracket. The double space $\mathfrak{g} \oplus \mathfrak{g}^*$ has a natural Poisson bracket,

$$\{p_1 + q^1, p_2 + q^2\}_- := < p_1, q^2 > - < p_2, q^1 >, \tag{2}$$

where the canonical variables have no degree (or even degree). The double space with the Poisson bracket becomes a linear symplectic plane. We notice that if $\mathfrak{g}$ is a Leibniz algebra, the suitable Poisson bracket on $\mathfrak{g}$ is the latter (2), not the former (1). That is, the Leibniz bracket on $\mathfrak{g}$ and the symplectic structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ are compatible.

The aim of this note is to construct a Lie-Leibniz analogue of the cohomological field theory above. This paper is organized as follows.

Section 2: We recall some basic properties of Leibniz algebras, in particular, the anti-invariant 2-form on $\mathfrak{g} \oplus \mathfrak{g}^*$ will be studied.
Section 3.1: The super space $\bigwedge(\mathfrak{g} \oplus \mathfrak{g}^*)$ is itself a complex of Lie algebra cohomology theory, on the other hand, the Leibniz complex of Loday-Pirashvili in [12] defined over $\mathfrak{g} \oplus \mathfrak{g}^*$ is a noncommutative space. Hence there is a difference between the symplectic plane and the complex. We will introduce a mapping, ψ, which connects the symplectic plane to the Leibniz complex (Definition 3.1 below).
Section 3.2: The notion of cohomological vector field, which is denoted by $\mathcal{L}$, will be defined over the linear symplectic plane (Definition 3.5 below). The cohomological condition of $\mathcal{L}$ is naturally defined by using the mapping ψ. That is, the target of $\mathcal{L}$ by ψ is a solution of a Maurer-Cartan equation,

$$\lfloor \psi_{\mathcal{L}}, \psi_{\mathcal{L}} \rfloor = 0,$$

where $\lfloor , \rfloor$ is a natural graded Lie bracket defined on the Leibniz complex (Definition 2.6 below). We will show that $\mathcal{L}$ induces a Leibniz bracket via the derived bracket construction

$$[l_1, l_2] = \{\mathcal{L}(l_1), l_2\}_-.$$

Here l_1, l_2 are linear functions on the symplectic plane and $\{,\}$ is the canonical Poisson bracket.

The Leibniz bracket above does not satisfy the invariance conditions in general.

This is a proper property that $\mathcal{L}$ has and is an appearance of the noncommutativity of Leibniz algebra. So we add an assumption that $\mathcal{L}$ satisfies

$$\{\{\mathcal{L}(l_1), l_2\}_-, l_3\}_- + \{\{\mathcal{L}(l_3), l_1\}_-, l_2\}_- + \{\{\mathcal{L}(l_2), l_3\}_-, l_1\}_- = 0,$$

which is called an **anti-cyclic** condition. Such a cohomological field is called a **Leibniz vector field**. We will show that given a Leibniz vector field, the double space $\mathfrak{g}\oplus\mathfrak{g}^*$ becomes a Leibniz algebra satisfying the invariance conditions (Theorem 3.8 below).

Section 4: We consider a direct decomposition of $\mathfrak{g}\oplus\mathfrak{g}^*$ by two Lagrangian subspaces.

$$\mathfrak{g}\oplus\mathfrak{g}^* \cong D \oplus D_*,$$

where D and D_* are Lagrangian subspaces on the symplectic plane. Given such a pair (D, D_*), a linear canonical transformation is deduced. We will study how the Leibniz field is transformed by the canonical transformation given by the Lagrangian decomposition (Proposition 4.8 below). As an application of the study, we consider a Maurer-Cartan type equation,

$$[\mathcal{H}, \mathcal{H}]_{\mathcal{L}} = [[\mathcal{L}, \mathcal{H}], \mathcal{H}] = 0. \qquad (LYBE)$$

which is called a Leibniz Yang-Baxter equation (LYBE). Here $\mathcal{H}$ is a Hamiltonian vector field on the symplectic plane. The solutions of LYBE are regarded as Leibniz analogues of classical triangular r-matrices. We will give some examples of r-matrices.

Section 5: We shortly discuss Nijenhuis-, complex-structures defined on Leibniz algebras. The Nijenhuis-, complex-structures on Courant algebroids[2] are interesting geometric objects and have been studied by several authors, in terms of Poisson geometry, by Carinena-Grabowski-Marmo [3] [6] and by Kosmann-Schwarzbach [11]. We will study pure Leibniz version of Nijenhuis-, complex-structures.

Summary of Lie-Leibniz analogues:

operad	Lie	Leibniz
invariance	cyclic	anti-cyclic
target space	graded symplectic plane	symplectic plane
structure	θ	$\psi_{\mathcal{L}}$
cohomological vector field	d_θ	$\mathcal{L}$
integrability condition	CYBE	LYBE

[2]The Courant algebroid is a vector bundle in which the space of sections is a Leibniz algebra satisfying some additional properties (see [10] [14]).

2 Preliminaries

Assumptions. In this note, we assume that the characteristic of ground field is 0 and that the Leibniz algebras are finite dimensional.

2.1 Leibniz algebras and invariant 2-forms

A (left-)Leibniz algebra ([12, 13]) is by definition a vector space $\mathfrak{g}$ equipped with a binary bracket product $[,]$ satisfying the Leibniz identity

$$[p_1,[p_2,p_3]] = [[p_1,p_2],p_3] + [p_2,[p_1,p_3]],$$

where $p_i \in \mathfrak{g}$ for each $i \in \{1,2,3\}$. The Lie algebras are, of course, Leibniz algebras whose brackets are skewsymmetric.

We recall the notion of invariant 2-form in the category of Leibniz algebras.

Definition 2.1 (Chapoton [4])**.** *An invariant 2-form on a Leibniz algebra is by definition a nondegenerate skewsymmetric 2-form* $(,)_-$ *satisfying*

$$(-_1,[-_2,-_3])_- = -([-_2,-_1],-_3)_-, \tag{3}$$

$$(-_1,[-_2,-_3])_- = ([-_1,-_3]+[-_3,-_1],-_2)_-, \tag{4}$$

where the bracket is the Leibniz bracket.

The first application of the conditions (3) and (4) is as follows.

Let $\mathfrak{g}$ be a Leibniz algebra and let $\mathfrak{g}^*$ the dual space of $\mathfrak{g}$. Define the two-side actions $\mathfrak{g} \curvearrowright \mathfrak{g}^* \curvearrowleft \mathfrak{g}$ by

$$\begin{aligned} < p_1,[p_2,q] > &:= - < [p_2,p_1],q >, \\ < p_1,[q,p_2] > &:= < [p_1,p_2]+[p_2,p_1],q >, \end{aligned}$$

where $p_1,p_2 \in \mathfrak{g}$ and $q \in \mathfrak{g}^*$.

Proposition 2.2. *Then the semi-direct product* $\mathfrak{g} \ltimes \mathfrak{g}^*$ *becomes a Leibniz algebra.*

Proof. (Sketch) An arbitrary Leibniz bracket satisfies $[[x,y],z] = -[[y,x],z]$, which is used in the following proof. We prove that the triple (q,p_2,p_3) satisfies the Leibniz identity.

$$\begin{aligned} < p_1,[q,[p_2,p_3]] > &= < [p_1,[p_2,p_3]]+[[p_2,p_3],p_1],q >, \\ < p_1,[[q,p_2],p_3] > &= < [p_2,[p_1,p_3]]+[p_2,[p_3,p_1]],q >, \\ < p_1,[p_2,[q,p_3]] > &= - < [[p_2,p_1],p_3]+[p_3,[p_2,p_1]],q >, \end{aligned}$$

where $[[p_1, p_3] + [p_3, p_1], p_2] = 0$ is used. By the Leibniz identity on $\mathfrak{g}$, we obtain

$$< p_1, [q, [p_2, p_3]] - [[q, p_2], p_3] - [p_2, [q, p_3]] >= 0,$$

which gives the Leibniz identity of the triple. For other triples, (p_1, q, p_3) and (p_1, p_2, q), one can prove the Leibniz identity in a similar manner. $\square$

If $\mathfrak{g}$ is Lie, then the right adjoint action $\mathfrak{g}^* \curvearrowleft \mathfrak{g}$ is trivial and then the Leibniz bracket on $\mathfrak{g} \ltimes \mathfrak{g}^*$ has the same form as a *demisemi-direct* product in Kinyon-Weinstein [7] (see also [16]).

Proposition 2.3. *The double space $\mathfrak{g} \oplus \mathfrak{g}^*$ has the natural anti-symmetric pairing,*

$$(p_1 + q^1, p_2 + q^2)_- :=< p_1, q^2 > - < p_2, q^1 >,$$

which is invariant on the Leibniz algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$.

Example 2.4. *We consider a space $\mathcal{E}_V := \mathfrak{gl}(V) \oplus V$, which has a Leibniz bracket*[3]

$$[f_1 + v^1, f_2 + v^2] := [f_1, f_2] + f_1(v^2),$$

where $f_i \in \mathfrak{gl}(V)$, $v^i \in V$ and where $[f_1, f_2]$ is the Lie bracket on $\mathfrak{gl}(V)$ and $f_1(v^2)$ is the natural action $\mathfrak{gl}(V) \curvearrowright V$. We consider the semi-direct product algebra $\mathcal{E}_V \ltimes \mathcal{E}_{V^*}$. Here $\mathcal{E}_V^* = \mathcal{E}_{V^*}$. The Leibniz bracket on $\mathcal{E}_V \ltimes \mathcal{E}_{V^*}$ has the following form,

$$\Big[(f_1 + v^1) \oplus (g^1_* + v_1),\ (f_2 + v^2) \oplus (g^2_* + v_2)\Big] =$$
$$= \Big([f_1, f_2] + f_1(v^2)\Big) \oplus \Big([f_1, g^2_*] + v_1 \otimes v^2 - f^1_*(v_2) + f^2_*(v_1)\Big),$$

where $f^i_*, g^j_* \in \mathfrak{gl}(V^*)$ and $v_i \in V^*$, and where f^i_* is the dual element of f_i, that is, $f^i_* = (f_i)^*$, g^j_* also keeps the same manner.

We will study above example again in Section 4.

We consider an extension of Leibniz algebras,

$$0 \longrightarrow \mathfrak{g}^* \longrightarrow E \longrightarrow \mathfrak{g} \longrightarrow 0, \tag{5}$$

where $E \cong \mathfrak{g} \oplus \mathfrak{g}^*$ as a linear space. An isomorphism class of the extensions bijectively corresponds to a Leibniz cohomology class of 2-cochain $\phi : \mathfrak{g}^{\otimes 2} \to \mathfrak{g}^*$ (see [12]). We add a condition that the Leibniz algebra E satisfies the invariance conditions above. Then, the cochain should satisfy the invariance conditions,

$$(p_1, \phi(p_2, p_3))_- = -(\phi(p_2, p_1), p_3)_-, \tag{6}$$
$$(p_1, \phi(p_2, p_3))_- = (\phi(p_1, p_3) + \phi(p_3, p_1), p_2)_-, \tag{7}$$

[3]This Leibniz algebra $\mathcal{E}_V$ is called an omni-Lie algebra ([16]), which is regarded as a toy-model of Courant algebroid.

and the splitting map, $E \leftarrow \mathfrak{g} : \mathbf{s}$, should preserve the pairing, namely,

$$(p_1, \mathbf{s}(p_2))_- = (p_2, \mathbf{s}(p_1))_-. \tag{8}$$

We will discuss the identities (6), (7) and (8) in Section 4.

In the final of this subsection, we introduce Leibniz analogues of Cartan 3-forms:

Definition 2.5. *Let $\mathfrak{g}$ be a Leibniz algebra with an invariant 2-form $(\cdot,\cdot)_-$. We call a 3-form $(p_1, p_2, p_3)_L := ([p_1, p_2], p_3)_-$ a Leibniz 3-form.*

We will recall the Leibniz 3-form in Section 3.

2.2 Leibniz cohomology complex

In this subsection we suppose that V is an arbitrary vector space.

In general, a complex of Leibniz cohomology theory ([12]) is defined as a collection of the multilinear maps on a vector space V,

$$C^{n-1}(V) := \mathrm{Hom}(V^{\otimes n}, V),$$

where $C^{-2}(V) := \mathbb{K}$ and $C^{-1}(V) := V$. We consider the total space which consists of the non-negative degree parts,

$$C^*(V) := C^0(V) \oplus C^1(V) \oplus \cdots .$$

Take the non-negative cochains $M \in C^{m-1}(V)$ and $N \in C^{n-1}(V)$. A composition of M and N is defined by, for any $v_i \in V$,

$$M \bar{\circ} N := \sum_{\bar{\sigma},\sigma} (\pm) M(v_{\bar{\sigma}(1)}, ..., v_{\bar{\sigma}(k)}, N(v_{\sigma(1)}, ..., v_{\sigma(n)}), v_{\sigma(n)+1}, ..., v_{m+n-1}),$$

where $\bar{\sigma}$, σ are permutations satisfying

$$\bar{\sigma}(1) < \cdots < \bar{\sigma}(k) < \sigma(n) \quad \text{and} \quad \sigma(1) < \cdots < \sigma(n),$$

and where $\bar{\sigma}$ depends on σ. The sign $(\pm)$ is determined by the same rule as the case of Lie (i.e. Koszul sign convention). For example, if $M \in C^3(V)$, $N \in C^1(V)$,

$$\begin{aligned} M \bar{\circ} N = {} & M(N(v_1, v_2), v_3, v_4, v_5) + M(v_2, N(v_1, v_3), v_4, v_5) - M(v_1, N(v_2, v_3), v_4, v_5) \\ & + M(v_2, v_3, N(v_1, v_4), v_5) - M(v_1, v_3, N(v_2, v_4), v_5) + M(v_1, v_2, N(v_3, v_4), v_5) \\ & + M(v_2, v_3, v_4, N(v_1, v_5)) - M(v_1, v_3, v_4, N(v_2, v_5)) + M(v_1, v_2, v_4, N(v_3, v_5)) \\ & - M(v_1, v_2, v_3, N(v_4, v_5)). \end{aligned}$$

It is known that a graded Lie bracket is defined on $C^*(V)$.

Definition 2.6 (cf. [2] [1] or [15]). *The graded commutator of $M \in C^{m-1}(V)$ and $N \in C^{n-1}(V)$ with respect to $\bar{\circ}$,*

$$\lfloor M, N \rfloor := M \bar{\circ} N - (-1)^{(m-1)(n-1)} N \bar{\circ} M,$$

becomes a graded Lie bracket on the complex.

The negative degree parts $C^{-1}(V)$ and $C^{-2}(V)$ are inconsistent with the Lie algebra structure on $C^*(V)$.

The binary operations in $C^1(V)$

$$\mu : V^{\otimes 2} \to V, \quad v_1 \otimes v_2 \mapsto [v_1, v_2]$$

are Leibniz brackets if and only if there are Maurer-Cartan elements in the Lie algebra $C^*(V)$, that is, $\lfloor \mu, \mu \rfloor = 0$. Given a such μ, a differential is defined by the usual manner,

$$d_\mu := \lfloor \mu, - \rfloor,$$

which is the differential of the Leibniz cohomology theory. For example, if $\beta \in C^0(V)$,

$$d_\mu \beta = \mu(\beta \otimes 1) + \mu(1 \otimes \beta) - \beta \circ \mu.$$

For the negative degree cochains, the differential is defined "by hand" as follows.

$$d_\mu \mathbb{K} := 0, \tag{9}$$

$$d_\mu V := ad_V := [V, -]. \tag{10}$$

2.3 Linear symplectic planes

A symplectic plane is by definition a polynomial algebra $\mathbb{K}[p, q]$ over 2n variables,

$$(p, q) := (p_1, ..., p_n, q^1, ..., q^n),$$

equipped with the canonical Poisson bracket

$$\begin{aligned} \{p_i, p_j\} &= 0, \\ \{p_i, q^j\} &= \delta_i^j = -\{q^j, p_i\}, \\ \{q_i, q_j\} &= 0. \end{aligned}$$

We assume that the (canonical) coordinate transformations on the symplectic plane preserve the degree of polynomials, or equivalently, linear. Hence our symplectic plane is linear[4]. The canonical variables (p, q) generate a double vector space $V \oplus V^*$, where V is generated by (p_i) and V^* is generated by (q^j). The canonical duality of V and V^* is compatible with the Poisson bracket, that is, $< p_i, q^j >= \{p_i, q^j\}$.

The space $V \oplus V^*$ is regarded as the set of linear functions on the plane. In the following, we will denote elements in $V \oplus V^*$ by $l, l', l_1, l_2, ...$

The set of quadratic polynomials on the symplectic plane becomes a Lie algebra, which is known as $\mathfrak{sp}(2n)$. Since the polynomial degree of Poisson bracket is -2, $\mathfrak{sp}(2n)$ acts on the double space by the Poisson bracket, $A \cdot l := \{A, l\}$, where $A \in \mathfrak{sp}(2n)$, $l \in V \oplus V^*$.

We recall some basic concepts defined on symplectic plane.

Lagrangian subspaces. The subspaces of $V \oplus V^*$, D, are called the Lagrangian subspaces, if they are maximally isotropic with respect to the Poisson bracket, that is, $\dim D = \dim V$ and $\{l_1, l_2\} = 0$ for any $l_1, l_2 \in D$. It is obvious that V and V^* are Lagrangians.

Polynomial bidegree. Since our symplectic plane is linear, it naturally has the system of bidegree. The bidegree of the canonical variables are by definition $|p_i| := (1, 0)$ and $|q^i| := (0, 1)$, respectively, for each i. The basis of vector fields also has the bidegree,

$$\begin{aligned} |\frac{\partial}{\partial p_i}| &:= (-1, 0), \\ |\frac{\partial}{\partial q^i}| &:= (0, -1). \end{aligned}$$

If an object has the bidegree (m, n), then its total degree is by definition $m+n$. We notice that the bidegree of the Poisson bracket is $(-1, -1)$ and that the bidegree of Lie bracket (commutator) defined on the tangent vector fields is $(0, 0)$. The total degree of the bidegree is equal to the polynomial degree.

Remark 2.7. *Our symplectic plane has no degree except the polynomial degree, however it is possible to invest the degrees in the canonical coordinate (p, q), for instance, $|p| := 2$ and $|q| := 2$. Then the linear symplectic plane is considered to be a graded cotangent bundle $T^*[4]V[2]$. In this connection, the classical super plane $\bigwedge(V \oplus V^*)$ is equivalent with $T^*[2]V[1]$.*

[4]The assumption of linearity is natural, because the super symplectic plane $\bigwedge(\mathfrak{g} \oplus \mathfrak{g}^*)$ is also linear (see Remark 2.7).

2.4 Schouten-Nijenhuis calculus

Let χ be the space of vector fields on the symplectic plane. We consider the graded commutative algebra,

$$\bigwedge \chi := \mathbb{K}[p,q] \oplus \chi \oplus \wedge^2\chi \oplus \cdots .$$

The Lie bracket on χ can be extended on $\bigwedge \chi$ as a graded Poisson bracket of degree -1, which is called a Schouten-Nijenhuis bracket[5], or SN-bracket shortly. Here the elementary brackets are defined by $[X,f] := X(f)$ and $[f,g] := 0$ for any $f,g \in \mathbb{K}[p,q]$ and $X \in \chi$.

The Poisson bracket structure on the symplectic plane is a bivector field,

$$\pi := \sum_i \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i},$$

which is an element in $\wedge^2\chi$. The Poisson bracket of $f,g \in \mathbb{K}[p,q]$ is given by

$$\{f,g\} := [[\pi,f],g]$$

where $[,]$ is the SN-bracket.

3 Leibniz structures

3.1 Anti-cyclic calculus

We recall the Leibniz cohomology theory in Section 2.2. The Leibniz complex is, on the symplectic plane, defined by $C^n(V \oplus V^*)$, $n \geq -2$. We introduce a natural map, ψ, from the space of vector fields to the Leibniz complex.

Definition 3.1 (Higher derived brackets)**.** *If X is a vector field of the polynomial degree n, then*

$$\psi_X := \{...\{\{X(l_1), l_2\}, l_3\}, ..., l_{n+1}\},$$

becomes a cochain in $C^n(V \oplus V^*)$.

A linear map on $V \oplus V^*$ is identified with a vector field of the polynomial degree 0. Hence, when $n = 0$, $\psi_X := X$ naturally.

Since the Poisson bracket is nondegenerate, ψ is injective. If f is a polynomial of degree n, we define ψ_f by $\psi_f := \psi_{\mathcal{H}_f}$. Here $\mathcal{H}_f := \{f, -\}$ is the Hamiltonian vector field of the polynomial degree $n-2$.

The following lemma is the key of this note.

[5] The Schouten-Nijenhuis bracket is the free Poisson bracket over χ.

Lemma 3.2 (cf. Remark 3.12 below)**.** *Let X be a vector field of the polynomial degree $+1$ and let $\mathcal{H}$ be a Hamiltonian vector field of degree 0. Then $\psi[X, \mathcal{H}] = \lfloor \psi_X, \psi_{\mathcal{H}} \rfloor$, namely,*

$$\{[X, \mathcal{H}](l_1), l_2\} = \{X\mathcal{H}(l_1), l_2\} + \{X(l_1), \mathcal{H}(l_2)\} - \mathcal{H}\{X(l_1), l_2\}.$$

Proof. By $-\{\mathcal{H}X(l_1), l_2\} = -\mathcal{H}\{X(l_1), l_2\} + \{X(l_1), \mathcal{H}(l_2)\}$. □

We introduce the notion of **anti-cyclic vector field**, which is considered to be a generalization of Hamiltonian vector fields.

Definition 3.3 (Anti-cyclic fields)**.** *A vector field X which has the polynomial degree n is called anti-cyclic, if it satisfies,*

$$\sum_{\tau} sgn(\tau)\{...\{\{X(l_{\tau(1)}), l_{\tau(2)}\}, l_{\tau(3)}\}..., l_{\tau(n+2)}\} = 0,$$

where τ are cyclic permutations in S_{n+2}. In particular, when $n = 0$, the condition is the same as $\{X(l_1), l_2\} = -\{l_1, X(l_2)\}$.

One can easily prove that if X is a Hamiltonian vector field of even-degree, it satisfies the anti-cyclic condition. If X is an anti-cyclic Hamiltonian vector field of odd-degree, then it is trivial; for instance, when $n = 3$,

$$\begin{aligned}&\{\{X(l_1), l_2\}, l_3\} = \{X\{l_1, l_2\}, l_3\} - \{\{l_1, X(l_2)\}, l_3\} = \\ &\qquad \{\{X(l_2), l_1\}, l_3\} = \{X(l_2), \{l_1, l_3\}\} - \{l_1, \{X(l_2), l_3\}\} = \{\{X(l_2), l_3\}, l_1\},\end{aligned}$$

which gives

$$\sum_{\tau} sgn(\tau)\{\{X(l_{\tau(1)}), l_{\tau(2)}\}, l_{\tau(3)}\} = 3\{\{X(l_1), l_2\}, l_3\} = 0,$$

which implies $X = 0$.

Lemma 3.4. *Let X be an anti-cyclic vector field of odd-degree and let $\mathcal{H}$ be a Hamiltonian vector field of degree 0. Then $[X, \mathcal{H}]$ is also anti-cyclic.*

Proof. We show a case that the degree of $X = 1$. We recall that the bidegree of the Lie derivation is $(0, 0)$. Hence $[X, \mathcal{H}]$ also has the polynomial degree $+1$. We denote $\{\{-_1, -_2\}, -_3\}$ by simply $\{-_1, -_2, -_3\}$.

$$\begin{aligned}&\{[X, \mathcal{H}](l_1), l_2, l_3\} + cyclic = \{X\mathcal{H}(l_1), l_2, l_3\} - \{\mathcal{H}X(l_1), l_2, l_3\} + cyclic \\ &\qquad = \{X\mathcal{H}(l_1), l_2, l_3\} + \{X(l_1), \mathcal{H}(l_2), l_3\} + \{X(l_1), l_2, H(l_3)\} + cyclic \\ &\qquad\qquad = \{X\mathcal{H}(l_1), l_2, l_3\} + \{X(l_3), \mathcal{H}(l_1), l_2\} + \{X(l_2), l_3, H(l_1)\} + cyclic.\end{aligned}$$

From the assumption that X is anti-cyclic, we obtain the desired condition. □

We will use this lemma in Section 4.

3.2 Structure fields

Definition 3.5. *We call a vector field $\mathcal{L}$ on the symplectic plane a **cohomological vector field**, if $\mathcal{L}$ has the polynomial degree $+1$ and satisfies*

$$\lfloor \psi_{\mathcal{L}}, \psi_{\mathcal{L}} \rfloor = 0. \tag{11}$$

Since $\psi_{\mathcal{L}}(l_1, l_2) = \{\mathcal{L}(l_1), l_2\}$, if $\mathcal{L}$ is a cohomological vector field, then the derived bracket, $[l_1, l_2]_{\mathcal{L}} := \{\mathcal{L}(l_1), l_2\}$, becomes a Leibniz bracket on $V \oplus V^*$.

Claim 3.6. *The condition (11) is equivalent with*

$$\{\mathcal{L}(l_1), \mathcal{L}(l_2)\} - \mathcal{L}\{\mathcal{L}(l_1), l_2\} = 0, \tag{11$'$}$$

where $l_i \in V \oplus V^*$.

Proof. The defining equation (11) is equal to

$$\begin{aligned}
&[l_1, [l_2, l_3]_{\mathcal{L}}]_{\mathcal{L}} - [[l_1, l_2]_{\mathcal{L}}, l_3]_{\mathcal{L}} - [l_2, [l_1, l_3]_{\mathcal{L}}]_{\mathcal{L}} = \\
&\qquad = \{\mathcal{L}(l_1), \{\mathcal{L}(l_2), l_3\}\} - \{\mathcal{L}\{\mathcal{L}(l_1), l_2\}, l_3\} - \{\mathcal{L}(l_2), \{\mathcal{L}(l_1), l_3\}\} = \\
&\qquad\qquad = \{\{\mathcal{L}(l_1), \mathcal{L}(l_2)\}, l_3\} - \{\mathcal{L}\{\mathcal{L}(l_1), l_2\}, l_3\} = 0,
\end{aligned}$$

which gives the identity of the claim. □

Since the polynomial degree of $\mathcal{L}$ is $+1$, $\mathcal{L}$ is regarded as a mapping from $V \oplus V^*$ to $\mathfrak{sp}(2n)$. Hence the induced Leibniz bracket is coherent with the $\mathfrak{sp}(2n)$-action.

Definition 3.7. *A cohomological vector field is called **Leibniz**, if it is anti-cyclic, i.e.,*

$$\{\{\mathcal{L}(l_1), l_2\}, l_3\} + \{\{\mathcal{L}(l_3), l_1\}, l_2\} + \{\{\mathcal{L}(l_2), l_3\}, l_1\} = 0, \tag{12}$$

for each $l_i \in V \oplus V^*$.

The anti-cyclic condition (12) is the same as

$$(l_1, l_2, l_3)_{\mathcal{L}} + (l_3, l_1, l_2)_{\mathcal{L}} + (l_2, l_3, l_1)_{\mathcal{L}} = 0,$$

where $(\cdot, \cdot, \cdot)_{\mathcal{L}}$ is the Leibniz 3-form defined in Section 2. We should remark that the cohomological condition (11) is independent from the anti-cyclic condition. In the after of Example 3.11, we give an example of $\mathcal{L}$ which is cohomological but not anti-cyclic

Theorem 3.8. *Let $\mathcal{L}$ be a Leibniz vector field on the symplectic plane. Then the Leibniz algebra $(V \oplus V^*, [,]_{\mathcal{L}})$ satisfies the invariance conditions (3) and (4) with respect to the Poisson bracket.*

Proof. The left-invariance condition is followed from the Jacobi identity of the Poisson bracket:

$$\begin{aligned}\{l_2, \{\mathcal{L}(l_1), l_3\}\} &= \{\{l_2, \mathcal{L}(l_1)\}, l_3\} + \{\mathcal{L}(l_1), \{l_2, l_3\}\} \\ &= -\{\{\mathcal{L}(l_1), l_2\}, l_3\}\end{aligned}$$

and the right-invariance one is deduced by the anti-cyclic condition:

$$\begin{aligned}&\{\{\mathcal{L}(l_1), l_2\}, l_3\} + \{\{\mathcal{L}(l_3), l_1\}, l_2\} + \{\{\mathcal{L}(l_2), l_3\}, l_1\} = \\ &\quad = \{\{\mathcal{L}(l_1), l_2\}, l_3\} - \{l_2, \{\mathcal{L}(l_3), l_1\}\} + \{\mathcal{L}(l_2), \{l_3, l_1\}\} - \{l_3, \{\mathcal{L}(l_2), l_1\}\} = \\ &\qquad = \{\{\mathcal{L}(l_1), l_2\}, l_3\} - \{l_2, \{\mathcal{L}(l_3), l_1\}\} + \{\{\mathcal{L}(l_2), l_1\}, l_3\} = 0.\end{aligned}$$

□

In the following, we recall basic properties that the Leibniz fields satisfy. The bracket $\{\mathcal{L}(f_1), f_2\}$ can be extended on the symplectic plane for any $f_1, f_2 \in \mathbb{K}[p, q]$. However the Leibniz identity is broken in general.

Proposition 3.9. *For any $l \in V \oplus V^*$, $f \in \mathbb{K}[p, q]$, define the bracket $[l, f]_{\mathcal{L}}$ by the same manner as above. Then it becomes a left-Leibniz representation of $V \oplus V^* \curvearrowright \mathbb{K}[p, q]$.*

Proof. By $\{-, f\} = \{-, q_i\}\frac{\partial f}{\partial q_i} + \{-, p_i\}\frac{\partial f}{\partial p_i}$. □

We consider the skewsymmetry brackets of the Leibniz brackets,

$$[l_1, l_2]_- := \{\mathcal{L}(l_1), l_2\} - \{\mathcal{L}(l_2), l_1\}.$$

Proposition 3.10. *The bracket $[l_1, l_2]_-$ has the structure tensor in $\wedge^2\chi$, that is, there exists $\Lambda \in \wedge^2\chi$ satisfying $[l_1, l_2]_- = [[\Lambda, l_1], l_2]$, where $[,]$ is the SN-bracket in Section 2.*

Proof. Since $\mathcal{L}\{l_1, l_2\} = 0$,

$$[l_1, l_2]_- = \{\mathcal{L}(l_1), l_2\} + \{l_1, \mathcal{L}(l_2)\} - \mathcal{L}\{l_1, l_2\},$$

which gives $[l_1, l_2]_- = \{\{[\pi, \mathcal{L}], l_1\}, l_2\}$. Thus we obtain $\Lambda = [\pi, \mathcal{L}]$. □

Let $\omega = \sum dp_i \wedge dq^i$ be the symplectic 2-form on the plane. Given a Leibniz vector field, a 1-form is defined to be the inner-derivation, $\theta_{\mathcal{L}} := i_{\mathcal{L}}\omega$, which is called a **Leibniz 1-form**. It is possible to define the skewsymmetry bracket above by using the differential of $\theta_{\mathcal{L}}$,

$$[l_1, l_2]_- = d\theta_{\mathcal{L}}(\mathcal{H}_{l_1}, \mathcal{H}_{l_2}),$$

where $\mathcal{H}_{l_i} := \{l_i, -\}$ is the Hamiltonian vector field for each $i \in \{1, 2\}$.

Now, we recall a typical example of Leibniz vector fields.

Example 3.11. *Let $\mathfrak{g}(= V)$ be a Leibniz algebra with the Leibniz bracket*

$$[p_i, p_j] = C_{ij}^k p_k,$$

where C_{ij}^k is the structure constant. Define a vector field by

$$\mathcal{L} := -C_{ij}^k p_k q^j \frac{\partial}{\partial p_i} - C_{ij}^k q^i q^j \frac{\partial}{\partial q^k}. \tag{13}$$

This is the Leibniz vector field which corresponds to the semi-direct product algebra $\mathfrak{g} \ltimes \mathfrak{g}^*$. The bidegree of $\mathcal{L}$ is equal to $(0, 1)$. The Leibniz 1-form has the form,

$$\theta_{\mathcal{L}} = -C_{ij}^k p_k q^j dq^i + C_{ij}^k q^i q^j dp_k.$$

The first term of (13) satisfies the cohomological condition (11), however (12) false.

Remark 3.12 (Recall Lemma 3.2)**.** *The Leibniz coboundary of a 1-cochain $\mathcal{H}$ is computed by the Lie bracket $[\mathcal{L}, \mathcal{H}]$, because ψ is mono. The good relation in Lemma 3.2 is broken on the level of higher cochains, however the elementary differentials (9) and (10) are systematically derived to be the natural derivations $\mathcal{L}(\mathbb{K}) = 0$ and $ad_l = \{\mathcal{L}(l), -\}$. From these observation, one can state that the canonical calculus and Leibniz cohomology theory are, on the level of elementary cochains, coherent each other.*

4 Canonical doubles and r-matrices

4.1 Lagrangian factorization

We consider the Lagrangian decompositions of the Leibniz algebra $(V \oplus V^*, \mathcal{L})$:

$$D \oplus D_* \cong V \oplus V^*,$$

where D and D_* are both Lagrangian subspaces of $V \oplus V^*$. Since $\{D, D\} = 0$ and $\{D_*, D_*\} = 0$, D_* is identified with the dual space of D via the Poisson bracket.

Definition 4.1. *The pair (D, D_*) is called a canonical pair, if D and D_* are both Leibniz subalgebras of $V \oplus V^*$.*

Let $(\bar{p},\bar{q})$ be a second canonical coordinate associated with the decomposition $D\oplus D_{*}\cong V\oplus V^{*}$, where $\bar{p}\in D$ and $\bar{q}\in D_{*}$. We should remark that the coordinate transformation $(p,q)\to(\bar{p},\bar{q})$ is a canonical transformation which preserves the Poisson bracket. Because D and D_{*} are subalgebras, the Leibniz brackets $[\bar{p}_i,\bar{p}_j]$ and $[\bar{q}^i,\bar{q}^j]$ have been defined. By the projections $V\oplus V^{*}\rightrightarrows D,D_{*}$, the bracket $[\bar{p}_i,\bar{q}^k]$ (resp. $[\bar{q}^k,\bar{p}_i]$) is decomposed into a D-valued bracket and a D_{*}-valued one

$$[\bar{p}_i,\bar{q}^k]=[\bar{p}_i,\bar{q}^k]_{D_*}\oplus[\bar{p}_i,\bar{q}^k]_{D},$$

where $[\bar{p}_i,\bar{q}^k]_{D_*}\in D$ and $[\bar{p}_i,\bar{q}^k]_{D}\in D_{*}$. By the invariance conditions, we obtain

$$\{\bar{p}_j,[\bar{p}_i,\bar{q}^k]\}=\{\bar{p}_j,[\bar{p}_i,\bar{q}^k]_D\}=-\{[\bar{p}_i,\bar{p}_j],\bar{q}^k\},$$

which implies that $[\bar{p}_i,\bar{q}^k]_D$ is the canonical left-action $D\curvearrowright D_{*}$. In the same way, one can prove that $[\bar{q}^k,\bar{p}_i]_D$ is the right-action $D_{*}\curvearrowleft D$. Hence $D\ltimes D_{*}$ becomes a subalgebra of $V\oplus V^{*}$. Dually, $D_{*}\ltimes D$ is also so. In this way, the Leibniz product on $V\oplus V^{*}$ is factorized into the two semi-direct products. The structure field $\mathcal{L}$ is also factorized into the sum of two Leibniz vector fields, $\mathcal{L}=\overline{\mathcal{L}}+\overline{\mathcal{L}}_{*}$, where $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}_{*}$ correspond to $D\ltimes D_{*}$ and $D_{*}\ltimes D$ respectively.

Definition 4.2. *If (D,D_{*}) is a canonical pair, the total Leibniz algebra $D\oplus D_{*}$ is called a **canonical double** of the pair, or shortly, double, which is denoted by $D\bowtie D_{*}$. The total structure $\overline{\mathcal{L}}+\overline{\mathcal{L}}_{*}$ is also called a **canonical double** of the pair $(\overline{\mathcal{L}},\overline{\mathcal{L}}_{*})$.*

The following proposition is in general not correct in the category of Lie algebras.

Proposition 4.3. *Let $\mathfrak{g}$ be a semi-simple Lie algebra. Then $\mathfrak{g}^{*}$ has a Lie algebra structure which is isomorphic to the one of $\mathfrak{g}$. The two Lie algebra structures on $\mathfrak{g}$ and $\mathfrak{g}^{*}$ are compatible in the category of Leibniz algebras, that is, they are canonically unified into the double algebra $\mathfrak{g}\bowtie\mathfrak{g}^{*}$.*

Proof. Since $\mathfrak{g}$ is semi-simple, there exists a linear isomorphism $K:\mathfrak{g}\to\mathfrak{g}^{*}$, which is defined by

$$\{p_1,K(p_2)\}=<p_1,p_2>,$$

where $<,>$ is the Killing 2-form. The Lie bracket on $\mathfrak{g}^{*}$ is isomorphically defined by

$$[K(p_1),K(p_2)]:=K[p_1,p_2].$$

Thus, two Leibniz algebras $\mathfrak{g}\ltimes\mathfrak{g}^{*}$ and $\mathfrak{g}^{*}\ltimes\mathfrak{g}$ are defined, where the left-actions $\mathfrak{g}\curvearrowright\mathfrak{g}^{*}$ and $\mathfrak{g}^{*}\curvearrowright\mathfrak{g}$ are respectively given by

$$[p_1,K(p_2)]=K[p_1,p_2], \tag{14}$$
$$[K(p_1),p_2]=[p_1,p_2].$$

We prove that the algebra structures on $\mathfrak{g} \ltimes \mathfrak{g}^*$ and $\mathfrak{g}^* \ltimes \mathfrak{g}$ are compatible, that is, they are unified into the double $\mathfrak{g} \bowtie \mathfrak{g}^*$. It suffices to show that any bracket defined on $\mathfrak{g} \oplus \mathfrak{g}^*$ is Leibniz. This is an easy exercise. For instance,

$$\begin{aligned}
&[p_1, [K(p_2), p_3]] = [p_1, [p_2, p_3]], \\
&[[p_1, K(p_2)], p_3] = [K[p_1, p_2], p_3] = [[p_1, p_2], p_3], \\
&[K(p_2), [p_1, p_3]] = [p_2, [p_1, p_3]],
\end{aligned}$$

which gives the Leibniz identity for the triple $(p_1, K(p_2), p_3)$. In this way, for any triple, the Leibniz identity holds. □

The proposition above holds for any Lie algebra which has an invariant nondegenerate symmetric 2-form, because the mapping $K : \mathfrak{g} \to \mathfrak{g}^*$ above is not necessarily Killing form.

We recall another example of factorization.

Example 4.4. *Recall the Leibniz algebra $\mathcal{E}_V \ltimes \mathcal{E}_{V^*}$ in Example 2.4. We notice that $\mathfrak{gl}(V) \ltimes_{Lie} V^*$, which is the semi-direct product in the category of Lie algebras, is a Lie subalgebra of $\mathcal{E}_V \ltimes \mathcal{E}_{V^*}$ with the bracket $[f_1 + v_1, f_2 + v_2]$. It is obvious that $\mathfrak{gl}(V) \ltimes_{Lie} V^*$ and $\mathfrak{gl}(V^*) \oplus V$ are Lagrangian subspaces. Thus, we obtain the following factorization,*

$$\mathcal{E}_V \ltimes \mathcal{E}_{V^*} \cong (\mathfrak{gl}(V) \ltimes_{Lie} V^*) \ltimes (\mathfrak{gl}(V^*) \oplus V).$$

Here, $\mathfrak{gl}(V^*) \oplus V$ is a trivial subalgebra of $\mathcal{E}_V \ltimes \mathcal{E}_{V^*}$.

If $\mathcal{L} + \mathcal{L}_*$ is a canonical double, the bidegrees of $\mathcal{L}$ and $\mathcal{L}_*$ are respectively $(0,1)$ and $(1,0)$. In general, the Leibniz vector field is decomposed into the 4-substructures of bidegrees $(0,1)$, $(-1,2)$, $(1,0)$ and $(2,-1)$,

$$\mathcal{L}^{total} = \mathcal{L} + \Phi + \mathcal{L}_* + \Phi_*, \tag{15}$$

where Φ and Φ_* are the substructures of $(-1,2)$ and $(2,-1)$ respectively. Under the local coordinate, $\mathcal{L}$ has the same form as (13) and

$$\begin{aligned}
\mathcal{L} &= -C^k_{ij} p_k q^j \frac{\partial}{\partial p_i} - C^k_{ij} q^i q^j \frac{\partial}{\partial q^k}, \\
\Phi &= \frac{1}{2} \phi_{ijk} q^i q^j \frac{\partial}{\partial p_k}, \\
\mathcal{L}_* &= C^{ij}_{*k} q^k p_j \frac{\partial}{\partial q^i} + C^{ij}_{*k} p_i p_j \frac{\partial}{\partial p_k}, \\
\Phi_* &= \frac{1}{2} \phi^{ijk}_* p_i p_j \frac{\partial}{\partial q^k},
\end{aligned}$$

where ϕ_{ijk} and ϕ_*^{ijk} are partially symmetric on the subscripts (i,j).

We recall the identities (6), (7) and (8) in Section 2. Eqs. (6) and (7) are equivalent with the anti-cyclic condition,

$$\{\{\Phi(l_1),l_2\},l_3\}+\{\{\Phi(l_3),l_1\},l_2\}+\{\{\Phi(l_2),l_3\},l_1\}=0 \qquad (6)'(7)'$$

and eq. (8) is the same as

$$\{\mathbf{s}(p_1),p_2\}+\{p_1,\mathbf{s}(p_2)\}=0. \qquad (8)'$$

Since $E\leftarrow\mathfrak{g}:\mathbf{s}$ is a splitting map, one can assume that $\mathbf{s}(p_j):=p_j+s_{jk}q^k$. By $(8)'$, the matrix s_{jk} is symmetric. Define a Hamiltonian by $h:=\frac{1}{2}s_{jk}q^jq^k$. Then we have $\{p_j,h\}=s_{jk}q^k$. Thus, $\mathbf{s}$ is regarded as a Hamiltonian flow,

$$\mathbf{s}=1+\{-,h\}+\frac{1}{2!}\{\{-,h\},h\}+\cdots.$$

This is consistent as a geometric representation of $\mathbf{s}$, because $\{\{p,h\},h\}=0$ for any $p\in\mathfrak{g}$.

4.2 LYBE

Let $\mathfrak{g}$ be a Leibniz algebra and let $\mathfrak{g}\ltimes\mathfrak{g}^*$ the semi-direct product algebra. We call a matrix operator $r:\mathfrak{g}^*\to\mathfrak{g}$ an r-matrix, if r is a solution of the identity,

$$[r(q^1),r(q^2)]=r[r(q^1),q^2]+r[q^1,r(q^2)], \qquad (16)$$

for any $q^1,q^2\in\mathfrak{g}^*$. We call (16) a **Leibniz-Yang-Baxter equation** over $\mathfrak{g}$, or LYBE for short. It is easy to check that the operator r is a solution of LYBE if and only if the graph of r, $\{(r(q),q)\}$, is a subalgebra of $\mathfrak{g}\ltimes\mathfrak{g}^*$. Hence, if r is an r-matrix, then $\mathfrak{g}^*$ becomes a Leibniz algebra. If an r-matrix satisfies

$$\{r(q^1),q^2\}+\{q^1,r(q^2)\}=0, \qquad (17)$$

then r is called an **anti-triangular** r-matrix. If $\mathfrak{g}$ is Lie, then (16) is reduced to

$$[r(q^1),r(q^2)]=r[r(q^1),q^2],$$

because the right-action $\mathfrak{g}^*\curvearrowleft\mathfrak{g}$ is trivial.

The operator $r:\mathfrak{g}^*\to\mathfrak{g}$ can be extended on $\mathfrak{g}\ltimes\mathfrak{g}^*$ by the natural manner,

$$\hat{r}(p,q):=(r(q),0).$$

One can easily check that r is a solution of LYBE if and only if $\hat{r}$ satisfies

$$[\hat{r}(l_1),\hat{r}(l_2)]=\hat{r}[\hat{r}(l_1),l_2]+\hat{r}[l_1,\hat{r}(l_2)], \qquad (LYBE)$$

for any $l_1,l_2\in\mathfrak{g}\ltimes\mathfrak{g}^*$. This operator identity is also called an **LYBE**. We sometimes identify r with $\hat{r}$.

Lemma 4.5. *The anti-triangular r-matrices are represented as the Hamiltonian vector fields as follows,*

$$r \equiv \hat{r} = \{\frac{1}{2}r^{ij}p_i p_j, -\}.$$

Proof. Define a matrix r^{ij} by $r(q^i) = r^{ij}p_j$. Eq. (17) implies that r^{ij} is symmetric. Hence the polynomial $\frac{1}{2}r^{ij}p_i p_j$ can be defined without lose of information. Thus we obtain the representation of r. $\square$

The bidegree of $r(=\hat{r})$ is $(1, -1)$.

Proposition 4.6. *Let $\mathcal{L}$ be a Leibniz vector field and let $\mathcal{H}$ be a Hamiltonian vector field of bidegree $(1, -1)$. Then $\mathcal{H}$ is a solution of LYBE if and only if*

$$\frac{1}{2}[\mathcal{H}, \mathcal{H}]_{\mathcal{L}} = 0. \tag{18}$$

where $[\mathcal{H}, \mathcal{H}]_{\mathcal{L}} := [[\mathcal{L}, \mathcal{H}], \mathcal{H}]$.

Proof. Since the bidegree of $\mathcal{H}$ is $(1, -1)$, for any linear functions, $\mathcal{H}\mathcal{H}(l) = 0$. From the assumptions,

$$\begin{aligned}
\{[[\mathcal{L}, \mathcal{H}], \mathcal{H}](l_1), l_2\} &= \{(-2\mathcal{H}\mathcal{L}\mathcal{H} + \mathcal{H}\mathcal{H}\mathcal{L})(l_1), l_2\}, \\
-2\{\mathcal{H}\mathcal{L}\mathcal{H}(l_1), l_2\} &= -2\mathcal{H}\{\mathcal{L}\mathcal{H}(l_1), l_2\} + 2\{\mathcal{L}\mathcal{H}(l_1), \mathcal{H}(l_2)\}, \\
\{\mathcal{H}\mathcal{H}\mathcal{L}(l_1), l_2\} &= -2\mathcal{H}\{\mathcal{L}(l_1), \mathcal{H}(l_2)\}.
\end{aligned}$$

Therefore we have $\{[[\mathcal{L}, \mathcal{H}], \mathcal{H}](l_1), l_2\}/2 =$

$$\begin{aligned}
&= \{\mathcal{L}\mathcal{H}(l_1), \mathcal{H}(l_2)\} - \mathcal{H}\{\mathcal{L}\mathcal{H}(l_1), l_2\} - \mathcal{H}\{\mathcal{L}(l_1), \mathcal{H}(l_2)\} \\
&= [\mathcal{H}(l_1), \mathcal{H}(l_2)] - \mathcal{H}[\mathcal{H}(l_1), l_2] - \mathcal{H}[l_1, \mathcal{H}(l_2)].
\end{aligned}$$

$\square$

Lemma 4.7. *Under the same assumptions,*

$$\frac{1}{3!}[[[\mathcal{L}, \mathcal{H}], \mathcal{H}], \mathcal{H}] = \mathcal{H}\{\mathcal{L}\mathcal{H}, \mathcal{H}\}$$

and $[[[[\mathcal{L}, \mathcal{H}], \mathcal{H}], \mathcal{H}], \mathcal{H}] = 0$ automatically.

Proof. Replace $\mathcal{L}$ in Proposition 4.6 into $[\mathcal{L}, \mathcal{H}]$. Then we obtain

$$\frac{1}{2}\{[[[\mathcal{L}, \mathcal{H}], \mathcal{H}], \mathcal{H}](l_1), l_2\} = 3\mathcal{H}\{\mathcal{L}\mathcal{H}(l_1), \mathcal{H}(l_2)\},$$

which is the identity of the lemma. $\square$

We study how the Leibniz vector field $\mathcal{L}$ is transformed by a canonical transformation generated by $\mathcal{H}$. The push-forward by the canonical transformation is defined by the exponential operation,

$$exp(X_{\mathcal{H}}) := 1 + [-,\mathcal{H}] + \frac{1}{2!}[[-,\mathcal{H}],\mathcal{H}] + \cdots ,$$

where $X_{\mathcal{H}} := [-,\mathcal{H}]$. The main result of this section is as follows.

Proposition 4.8. *If $\mathcal{L}$ is a Leibniz vector field and if $\mathcal{H}$ is a Hamiltonian vector field of bidegree $(1,-1)$, then $exp(X_{\mathcal{H}})(\mathcal{L})$ is again a Leibniz vector field.*

Proof. From Lemma 4.7,

$$exp(X_{\mathcal{H}})(\mathcal{L}) = \mathcal{L} + [\mathcal{L},\mathcal{H}] + \frac{1}{2}[\mathcal{H},\mathcal{H}]_{\mathcal{L}} + \frac{1}{6}[\mathcal{H},\mathcal{H},\mathcal{H}]_{\mathcal{L}},$$

where $[\mathcal{H},\mathcal{H},\mathcal{H}]_{\mathcal{L}} := [[[\mathcal{L},\mathcal{H}],\mathcal{H}],\mathcal{H}]$. From Lemma 3.4, we realize that $exp(X_{\mathcal{H}})(\mathcal{L})$ is anti-cyclic. We should prove that $exp(X_{\mathcal{H}})(\mathcal{L})$ satisfies the cohomological condition. Apply ψ, which is defined in Section 4.1, on $exp(X_{\mathcal{H}})(\mathcal{L})$. Then we have

$$exp(X_{\psi_{\mathcal{H}}})(\psi_{\mathcal{L}}) = \psi_{\mathcal{L}} + \lfloor\psi_{\mathcal{L}},\psi_{\mathcal{H}}\rfloor + \frac{1}{2}\lfloor\psi_{\mathcal{H}},\psi_{\mathcal{H}}\rfloor_{\psi_{\mathcal{L}}} + \frac{1}{6}\lfloor\psi_{\mathcal{H}},\psi_{\mathcal{H}},\psi_{\mathcal{H}}\rfloor_{\psi_{\mathcal{L}}},$$

where $X_{\psi_{\mathcal{H}}} = \lfloor -,\psi_{\mathcal{H}}\rfloor$. Because $exp(X_{\psi_{\mathcal{H}}})$ preserves the graded Lie bracket $\lfloor,\rfloor$, we obtain

$$exp(X_{\psi_{\mathcal{H}}})\lfloor\psi_{\mathcal{L}},\psi_{\mathcal{L}}\rfloor = \lfloor exp(X_{\psi_{\mathcal{H}}})(\psi_{\mathcal{L}}), exp(X_{\psi_{\mathcal{H}}})(\psi_{\mathcal{L}})\rfloor.$$

Since $\lfloor\psi_{\mathcal{L}},\psi_{\mathcal{L}}\rfloor = 0$, we have

$$\lfloor exp(X_{\psi_{\mathcal{H}}})(\psi_{\mathcal{L}}), exp(X_{\psi_{\mathcal{H}}})(\psi_{\mathcal{L}})\rfloor = 0.$$

The proof of the proposition is completed. □

The Leibniz field $exp(X_{\mathcal{H}})(\mathcal{L})$ is also decomposed into the 4-subfields, like (15). The homogeneous components of $exp(X_{\mathcal{H}})(\mathcal{L})$ have the following form,

$$\begin{aligned}
&\mathcal{L} + [\Phi,\mathcal{H}],\\
&\Phi,\\
&\mathcal{L}_* + [\mathcal{L},\mathcal{H}] + \frac{1}{2}[[\Phi,\mathcal{H}],\mathcal{H}],\\
&\Phi_* + [\mathcal{L}_*,\mathcal{H}] + \frac{1}{2}[\mathcal{H},\mathcal{H}]_{\mathcal{L}} + \frac{1}{3!}[[[\Phi,\mathcal{H}],\mathcal{H}],\mathcal{H}],
\end{aligned}$$

where the bidegrees are respectively $(0,1)$, $(-1,2)$, $(1,0)$ and $(2,-1)$. For example, if the bidegree of $\mathcal{L}$ is $(0,1)$, then $[\mathcal{H},\mathcal{H},\mathcal{H}]_{\mathcal{L}} = 0$ automatically. Therefore, in that case,

$$exp(X_{\mathcal{H}})(\mathcal{L}) = \mathcal{L} + [\mathcal{L},\mathcal{H}] + \frac{1}{2}[\mathcal{H},\mathcal{H}]_{\mathcal{L}},$$

and if $\mathcal{H}$ is a solution of LYBE, then $\mathcal{L}+[\mathcal{L},\mathcal{H}]$ becomes a canonical double.

When the bidegree of $\mathcal{H}$ is $(-1,1)$, $exp(X_{\mathcal{H}})(\mathcal{L})$ can be computed by a similar manner. In the next section, we will study a more general modification of $\mathcal{L}$.

We recall some examples of r-matrices.

Example 4.9 (cf. Proposition 4.3). *Let $\mathfrak{g}$ be a semi-simple Lie algebra with the Killing form $K:\mathfrak{g}\to\mathfrak{g}^*$. Define an operator by $r:=K^{-1}$. Then, from (14), we obtain $r[r(q^1),q^2]=[r(q^1),r(q^2)]$. The matrix $r=(r^{ij})$ is symmetric, because the Killing pairing is so. Hence the Killing forms are anti-triangular r-matrices.*

It is well-known that the Hamiltonian polynomials associated with K^{-1},

$$h_c:=\frac{1}{2}r^{ij}p_ip_j$$

are natural Casimir functions defined on the semi-simple Lie algebras.

Example 4.10. *Recall again the Leibniz algebra $\mathcal{E}_V\ltimes\mathcal{E}_{V^*}$ in Example 2.4. Define $r:\mathcal{E}_{V^*}\to\mathcal{E}_V$ to be a special projection*

$$r:(f^{\cdot}_*+\mathbf{v}.)\mapsto -f..$$

From Example 2.4, on the graph of r, the Leibniz bracket has the following form

$$[-f_1\oplus(f^1_*+v_1),-f_2\oplus(f^2_*+v_2)]=[f_1,f_2]\oplus\Big(-[f_1,f^2_*]+f^1_*(v_2)-f^2_*(v_1)\Big). \quad (19)$$

Since $[f_1,f^2_*]$ is the dual of $[f_1,f_2]$, the graph of r is closed under the bracket. It is obvious that the projection satisfies the anti-triangularity.

We notice that the second term of (19),

$$-[f_1,f^2_*]+f^1_*(v_2)-f^2_*(v_1)$$

is the bracket of the semi-direct product Lie algebra $\mathfrak{gl}(V^*)\ltimes_{Lie}V^*$, because $-[f_1,f^2_*]=[f^1_*,f^2_*]$. Thus, we obtain a canonical double, $\mathcal{E}_V\bowtie(\mathfrak{gl}(V^*)\ltimes_{Lie}V^*)$.

5 Nijenhuis operators and complex structures

In this section, we study some basic properties of Nijenhuis operators in the category of Leibniz algebras (cf. [3] [6] [11]).

A **Nijenhuis operator** on a Leibniz algebra $\mathfrak{g}$ is by definition a 1-cochain $\mathcal{N}\in C^0(\mathfrak{g})$ whose torsion vanishes, i.e.,

$$\begin{aligned}\mathrm{Tor}_{\mathcal{N}}(p_1,p_2) &:= [\mathcal{N}p_1,\mathcal{N}p_2]-\mathcal{N}[\mathcal{N}p_1,p_2]-\mathcal{N}[p_1,\mathcal{N}p_2]+\mathcal{N}^2[p_1,p_2]\\ &= 0.\end{aligned}$$

Let μ be the Leibniz bracket, $\mu(p_1, p_2) = [p_1, p_2]$. Given a 1-cochain $\mathcal{N}$, by a direct computation we obtain

$$\frac{1}{2}\lfloor\lfloor\mu, \mathcal{N}\rfloor, \mathcal{N}\rfloor - \mathrm{Tor}_{\mathcal{N}} = \frac{1}{2}\lfloor\mu, \mathcal{N}^2\rfloor, \tag{20}$$

which implies that the differential of the torsion, $\lfloor\mu, \mathrm{Tor}_{\mathcal{N}}\rfloor$, is trivial if and only if $\lfloor\mu, \mathcal{N}\rfloor$ is a Leibniz bracket ([3] [6]). Since $\lfloor\mu, \lfloor\mu, N\rfloor\rfloor = 0$, if $\lfloor\mu, \mathcal{N}\rfloor$ is Leibniz, then $\mu + t\lfloor\mu, \mathcal{N}\rfloor$ is a 1-parameter deformation of the Leibniz bracket.

If $\mathrm{Tor}_{\mathcal{N}} = 0$ and $\mathcal{N}^2 = -1$, $\mathcal{N}$ is called a **complex structure**. When $\mathcal{N}^2 = -1$, the right-hand side of (20) is equal to $-\mu/2$. Therefore, if $\mathcal{N}$ is a complex structure, then

$$\lfloor\lfloor\mu, \mathcal{N}\rfloor, \mathcal{N}\rfloor = -\mu. \tag{21}$$

In our context, $\mathfrak{g} = V \oplus V^*$, the symplectic plane, and $\mu = \psi_{\mathcal{L}}$, the Leibniz vector field. We consider the cases that $\mathcal{N}$ is a Hamiltonian vector field of the polynomial degree 0. Then $\mathcal{N}$ is identified with an element in $\mathfrak{sp}(2n)$ as follows,

$$\begin{aligned} \mathcal{N} &:= \{h, -\}, \\ h &:= \frac{1}{2} r^{ij} p_i p_j + n^i_j p_i q^j + \frac{1}{2} o_{ij} q^i q^j, \end{aligned}$$

where r^{ij}, n^i_j and o_{ij} are structure constants.

One can easily verify the two propositions below.

Proposition 5.1. *If the Hamiltonian vector field $\mathcal{N}(= \psi_{\mathcal{N}})$ above is a Nijenhuis operator on the Leibniz algebra $(V \oplus V^*, \mathcal{L})$, then $[\mathcal{L}, \mathcal{N}]$ is also a Leibniz vector field and then $\mathcal{L} + t[\mathcal{L}, \mathcal{N}]$ is a 1-parameter deformation of $\mathcal{L}$.*

Proof. Since $\mathcal{N}$ is a Hamiltonian vector field, $[\mathcal{L}, \mathcal{N}]$ becomes an anti-cyclic field (cf. Lemma 3.4). Applying ψ, we have $\psi[\mathcal{L}, \mathcal{N}] = \lfloor\psi_{\mathcal{L}}, \psi_{\mathcal{N}}\rfloor$. From the assumptions of the proposition, $\lfloor\psi_{\mathcal{L}}, \psi_{\mathcal{N}}\rfloor$ becomes a Leibniz bracket. Hence $[\mathcal{L}, \mathcal{N}]$ is Leibniz. □

Since ψ is mono, we obtain

Proposition 5.2. *If the Hamiltonian vector field $\mathcal{N}$ above is a complex structure on the Leibniz algebra $(V \oplus V^*, \mathcal{L})$, then*

$$[[\mathcal{L}, \mathcal{N}], \mathcal{N}] = [\mathcal{N}, \mathcal{N}]_{\mathcal{L}} = -\mathcal{L}.$$

We recall an example of complex structures.

Example 5.3. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with the Killing form $K : \mathfrak{g} \to \mathfrak{g}^*$. Define a 1-cochain on $\mathfrak{g} \ltimes \mathfrak{g}^*$ by*

$$\mathcal{N} := K^{-1} - K.$$

Then $\mathcal{N}$ becomes a complex structure on the Leibniz algebra. The associated Hamiltonian of $\mathcal{N}$ is the harmonic oscillator,

$$h = \frac{1}{2}\sum r^{ij}p_ip_j + r_{ij}q^iq^j.$$

where r_{ij} is K and r^{ij} is K^{-1}.

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