LOCAL UNIQUENESS OF THE CIRCULAR INTEGRAL INVARIANT

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ABSTRACT. This article is concerned with the representation of curves by means of integral invariants. In contrast to the classical differential invariants they have the advantage of being less sensitive with respect to noise. The integral invariant most common in use is the circular integral invariant. A major drawback of this curve descriptor, however, is the absence of any uniqueness result for this representation. This article serves as a contribution towards closing this gap by showing that the circular integral invariant is continuously invertible in a neighbourhood of the circle. The proof is an application of Riesz–Schauder theory and the inverse function theorem in a Banach manifold setting.

1. INTRODUCTION

In many applications one faces the challenge to model objects, or parts of objects, in a mathematical framework. As an example, one important task is to extract an object from a given data set and manipulate it in a post-processing step in order to obtain further information. Typical applications include medical imaging, object tracking in a sequence of images, but also object recognition, where the post-processing step consists of the comparison of the extracted object with a database of reference objects. Similarly, such a comparison can be necessary in medical imaging in order to distinguish between healthy and diseased organs. To that end, however, one has to be able to decide whether two given objects are similar or not. This requires a representation of the objects that makes the application of standard similarity measures possible.

Finding a suitable representation of the object of interest, depending on the type of application, is crucial as a first step. For simplicity, it is often assumed that the object is a simply connected bounded domain, allowing for the identification of the domain with its boundary. From a mathematical point of view, this assumption reduces the complexity of the representation. In addition, there exists a larger

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number of descriptions of boundaries than of domains, and, consequently, more mathematical tools to analyze the geometry of the underlying objects.

In 2D a common approach is to encode the contour of an object by the curvature function of its boundary curve. This approach has, for instance, been used in [7], where the authors set up a shape space of planar curves, where the shapes are implicitly encoded by the curvature function. The main advantage of using a differential invariant — the most prominent representative being the curvature — to represent an object is the well investigated mathematical framework of this type of invariants (see [1, 9, 11]).

Since all kinds of differential invariants are based on derivatives, they suffer from the shortcoming of being sensitive with respect to small perturbations. To bypass this shortcoming Manay et al. [10] proposed to use integral invariants instead of their differential counterparts (see also [5, 6, 14]). Integral invariants have similar invariance properties as differential invariants, but have proven to be considerably more robust with respect to noise. Their theory, however, is not that well investigated as opposed to the theory of their differential counterparts.

Beside the classical approach of differential invariants and the novel approach of integral invariants, there exist several other concepts for encoding an object. For instance, in [3] the authors use the zero level set of a harmonic function, which is uniquely determined by prescribing two functions on the boundary of an annulus, to encode the boundary of a 2D object (see [4] for a generalization to compact surfaces in 3D). A similar encoding of the object by a function is given in the article of Sharon and Mumford [13]. Here, the authors first map the 2D object, which is supposed to be a smooth and simply closed curve, to the interior of the unit disc in the complex plane via the Riemann mapping theorem. This conformal mapping is composed with a second one, generated out of the exterior of the original object, and the composition is restricted to the boundary of the unit disc. Thus, the final mapping, which the authors call the fingerprint of the object, is a diffeomorphism from the unit circle onto itself.

One of the challenges in object encoding is the question of uniqueness of the encoding. More precisely, in many applications, e.g. object matching, the correspondence between the object and its encoding should be one-to-one. Thus, a thorough investigation of the operator that maps an object to its encoding is needed. In case of the encoding by a harmonic or conformal mapping — if possible — uniqueness is well known. Also for the encoding of an arc length parameterized curve by its curvature function, it is known that one obtains a one-to-one correspondence between the curve and its encoding (up to rigid body motions). One even has a complete characterization of the set of functions that arise as curvature functions of a class of sufficiently regular curves (see [2]). For integral invariants the situation is different; the cone area invariant, first introduced in [5], is an injective mapping independent of the space dimension, but its application is limited to star-shaped objects. In contrast, for the circular integral invariant, which is the integral invariant most common in use, there exists no proof for the uniqueness conjecture so far.

This article is a contribution towards this goal: We prove a local uniqueness result for the circular integral invariant. In fact, we show the even stronger result that there exists a neighborhood of the circle where the circular integral invariant is a homeomorphism onto its image. The proof of this theorem is based on the inverse function theorem on Banach manifolds and an application of Riesz–Schauder theory.

2. Setting

Let Emb be the space of all continuous embeddings from S^1 to $\mathbb{R}^2$. Then every curve $\gamma \in \mathrm{Emb}$ has a unique interior, denoted by $\mathrm{Int}(\gamma)$. Following [5, 10], this allows us to introduce the circular integral invariant:

Definition 2.1. For given $r > 0$ we define the *circular integral invariant*

$$I_r[\gamma]\colon S^1 \to \mathbb{R}$$

of a curve $\gamma \in \mathrm{Emb}$ as

$$I_r[\gamma](\varphi) := \mathrm{area}\big(B_r\big(\gamma(\varphi)\big) \cap \mathrm{Int}(\gamma)\big)\,,$$

where $B_r(p)$ denotes the ball of radius r centered at $p \in \mathbb{R}^2$.

The circular integral invariant behaves well under several group actions:

- I_r is invariant with respect to Euclidean motions: For $A \in \mathrm{SE}[2]$ we have
$$I_r[A \circ \gamma] = I_r[\gamma]\,.$$
- I_r is equivariant with respect to reparametrizations: For every homeomorphism $\Phi\colon S^1 \to S^1$ we have
$$I_r[\gamma \circ \Phi] = I_r[\gamma] \circ \Phi\,.$$
- For every scalar $t > 0$ we have
$$I_r[\gamma] = t^2 I_{r/t}[\gamma]\,.$$

The observations above suggest to consider the integral invariant on the space $\mathcal{C}$ of all curves modulo Euclidean motions and reparametrizations. Moreover, we assume as an additional smoothness property that the considered curves are of class C^1. Then it makes sense to use the following representation of $\mathcal{C}$, as it avoids working with equivalence classes of curves.

Definition 2.2. Denote by $\mathcal{C} \subset \mathrm{Emb}$ the space of all curves $\gamma \in C^1(S^1, \mathbb{R}^2)$ satisfying the following conditions:

- γ has constant speed, i.e., there exists a constant $c_\gamma > 0$ such that $\|\dot\gamma(\varphi)\| = c_\gamma$ for all $\varphi \in S^1$.
- $\gamma(0) = (1, 0)$ and $\dot\gamma(0) = (0, c_\gamma)$, where we identify the circle S^1 with the interval $[0, 2\pi)$.
- γ is an embedding, that is, $\gamma(\varphi) \neq \gamma(\psi)$ for all $\varphi \neq \psi$.

In the proof of our main theorem we will apply the inverse function theorem on the space $\mathcal{C}$. Therefore we need the following result from differential geometry concerning the manifold structure of $\mathcal{C}$:

Theorem 2.3. *The space $\mathcal{C}$ is a smooth submanifold of the Banach space of all C^1-curves from S^1 to $\mathbb{R}^2$. Its tangent space $T_\gamma\mathcal{C}$ at a curve $\gamma \in \mathcal{C}$ consists of all C^1-curves σ with*

$$\langle \dot\sigma(\varphi), \dot\gamma(\varphi)\rangle = C \text{ for some } C \in \mathbb{R}, \quad \sigma(0) = (0, 0) \text{ and } \langle \dot\sigma(0), \gamma(0)\rangle = 0\,.$$

Proof. The proof of the submanifold result is similar to [12, Thm. 2.2]. In our case the situation is less complicated, as we only deal with C^1-curves instead of Sobolev curves. The constant speed parameterization yields the condition

$$2C = \partial_\varepsilon|_0 \langle \dot\gamma(\varphi) + \varepsilon\dot\sigma(\varphi), \dot\gamma(\varphi) + \varepsilon\dot\sigma(\varphi)\rangle = 2\langle \dot\sigma(\varphi), \dot\gamma(\varphi)\rangle \; .$$

The remaining constraints follow directly from the initial conditions. □

If we assume a stronger smoothness condition for the curve γ, we obtain the following characterization of the tangent space $T_\gamma\mathcal{C}$.

Lemma 2.4. *The tangent space of $\mathcal{C}$ at a curve $\gamma \in \mathcal{C} \cap C^2(S^1, \mathbb{R}^2)$ with curvature function*

$$\kappa_\gamma(\varphi) := \frac{\langle \dot\gamma(\varphi)^\perp, \ddot\gamma(\varphi)\rangle}{c_\gamma^3}$$

consists of all C^1-curves $\sigma(\varphi) = a(\varphi)\dot\gamma(\varphi)^\perp + b(\varphi)\dot\gamma(\varphi)$ satisfying:

- $\dot b(\varphi) = \dot b(0) - a(\varphi)\kappa_\gamma(\varphi)c_\gamma^{-1/2}$.
- $a(0) = b(0) = 0$.
- $\dot a(0) = 0$.

Proof. Theorem 2.3 and the fact that $\langle \dot\gamma(\varphi), \ddot\gamma(\varphi)\rangle = 0$ imply that there exists a constant $C \in \mathbb{R}$ such that

$$\begin{aligned} C &= \langle \dot\sigma(\varphi), \dot\gamma(\varphi)\rangle = \langle \dot\gamma(\varphi), \dot a(\varphi)\dot\gamma(\varphi)^\perp + a(\varphi)\ddot\gamma(\varphi)^\perp + \dot b(\varphi)\dot\gamma(\varphi) + b(\varphi)\ddot\gamma(\varphi)\rangle \\ &= a(\varphi)\langle \dot\gamma(\varphi), \ddot\gamma(\varphi)^\perp\rangle + \dot b(\varphi)c_\gamma^2 = -a(\varphi)c_\gamma^{3/2}\kappa_\gamma(\varphi) + \dot b(\varphi)c_\gamma^2 \; . \end{aligned}$$

Using the initial conditions for σ, we obtain the initial conditions for a and b and the value of $C = \dot b(0)$. □

We are now able to formulate the main result of this article:

Theorem 2.5. *The circular integral invariant $I_r \colon \mathcal{C} \to C^1(S^1, \mathbb{R})$ is locally continuously invertible at the circle of radius $R > r/2$.*

Remark 1. The condition on r to be smaller than $2R$ is necessary, because otherwise the circular integral invariant in each point φ is constant equal to $R^2\pi$, the area of the circle, and the same holds for any sufficiently small deformation of the circle which preserves the area.

3. Variation of the Circular Integral Invariant

For the application of the inverse function theorem we need the variation of $I_r[\gamma]$, which we derive in the following. In order to make the notation less cumbersome, we omit the argument φ in $\gamma(\varphi)$, $\sigma(\varphi)$ and similar expressions if the argument is clear from the context.

Lemma 3.1. *Let $r > 0$ and $\gamma \in \mathcal{C}$ a curve satisfying:*

- *For each $\varphi \in S^1$ the circle $B_r\big(\gamma(\varphi)\big)$ intersects the curve γ in exactly two points, denoted by $\gamma(p) = \gamma\big(p(\varphi)\big)$ and $\gamma(m) = \gamma\big(m(\varphi)\big)$. Here $m(\varphi)$ denotes the previous intersection parameter and $p(\varphi)$ the next one (see Figure 1).*
- *For each $\varphi \in S^1$ we have $\langle \dot\gamma(p), \gamma(p) - \gamma\rangle \neq 0$ and $\langle \dot\gamma(m), \gamma(m) - \gamma\rangle \neq 0$.*

Then the first variation in direction $\sigma \in T_\gamma\mathcal{C}$ of $I_r[\gamma]$ is given by

$$
\begin{aligned}
2I_r'[\gamma](\sigma) = 2\int_m^p &\langle \sigma(\psi), \dot\gamma(\psi)^\perp\rangle d\psi \\
&- \langle \sigma, \gamma(p)^\perp - \gamma(m)^\perp\rangle + \langle \gamma(p)-\gamma, \sigma(p)^\perp\rangle - \langle \gamma(m)-\gamma, \sigma(m)^\perp\rangle \\
&+ \langle \gamma(p)-\gamma, \dot\gamma(p)^\perp\rangle \frac{\langle \sigma-\sigma(p), \gamma(p)-\gamma\rangle}{\langle \dot\gamma(p), \gamma(p)-\gamma\rangle} \\
&- \langle \gamma(m)-\gamma, \dot\gamma(m)^\perp\rangle \frac{\langle \sigma-\sigma(m), \gamma(m)-\gamma\rangle}{\langle \dot\gamma(m), \gamma(m)-\gamma\rangle} \\
&- \frac{r^2}{\sqrt{r^4 - \langle \gamma(p)-\gamma, \gamma(m)-\gamma\rangle^2}} \Bigg(\frac{\langle \sigma-\sigma(p), \gamma(p)-\gamma\rangle}{\langle \dot\gamma(p), \gamma(p)-\gamma\rangle} \langle \dot\gamma(p), \gamma(m)-\gamma\rangle \\
&\qquad + \langle \sigma(p)-\sigma, \gamma(m)-\gamma\rangle + \langle \gamma(p)-\gamma, \sigma(m)-\sigma\rangle \\
&\qquad + \frac{\langle \sigma-\sigma(m), \gamma(m)-\gamma\rangle}{\langle \dot\gamma(m), \gamma(m)-\gamma\rangle} \langle \gamma(p)-\gamma, \dot\gamma(m)\rangle \Bigg) .
\end{aligned}
$$

Proof. Using the first assumption we can rewrite $I_r[\gamma]$ as

$$
(1) \quad I_r[\gamma](\varphi) = \frac{1}{2}\int_m^p \langle \gamma(\psi)-\gamma, \dot\gamma(\psi)^\perp\rangle d\psi + \frac{r^2}{2}\arccos\left(r^{-2}\langle \gamma(p)-\gamma, \gamma(m)-\gamma\rangle\right) .
$$

This formula can be easily deduced from Figure 1.

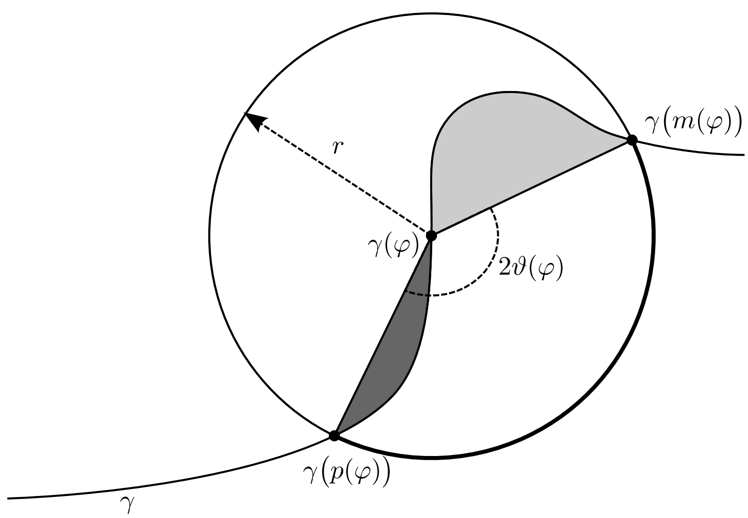

FIGURE 1. Sketch of the derivation of the analytical formula for the circular integral invariant assuming two points of intersection.

Let $\gamma_\varepsilon(\varphi) := \gamma(\varphi) + \varepsilon\sigma(\varphi)$ with $\sigma \in T_\gamma\mathcal{C}$. Denote the intersection parameters of the curve γ_ε with the circle of radius r centered at $\gamma_\varepsilon(\varphi)$ by $m_\varepsilon = m_\varepsilon(\varphi)$ and $p_\varepsilon = p_\varepsilon(\varphi)$. The governing equations for m_ε and p_ε are (see Figure 1)

$$
\langle \gamma_\varepsilon(p_\varepsilon) - \gamma_\varepsilon, \gamma_\varepsilon(p_\varepsilon) - \gamma_\varepsilon\rangle = r^2, \qquad \langle \gamma_\varepsilon(m_\varepsilon) - \gamma_\varepsilon, \gamma_\varepsilon(m_\varepsilon) - \gamma_\varepsilon\rangle = r^2 .
$$

Taking the derivative with respect to ε yields

$$\begin{aligned} 0 &= \partial_\varepsilon|_0 \langle \gamma_\varepsilon(p_\varepsilon) - \gamma_\varepsilon, \gamma_\varepsilon(p_\varepsilon) - \gamma_\varepsilon \rangle \\ &= 2\langle \partial_\varepsilon|_0 (\gamma(p_\varepsilon) + \varepsilon\sigma(p_\varepsilon) - \gamma - \varepsilon\sigma), \gamma(p) - \gamma \rangle \\ &= 2\langle \dot\gamma(p)\partial_\varepsilon|_0 p_\varepsilon + \sigma(p) - \sigma, \gamma(p) - \gamma \rangle \end{aligned}$$

and

$$\begin{aligned} 0 &= \partial_\varepsilon|_0 \langle \gamma_\varepsilon(m_\varepsilon) - \gamma_\varepsilon, \gamma_\varepsilon(m_\varepsilon) - \gamma_\varepsilon \rangle \\ &= 2\langle \partial_\varepsilon|_0 (\gamma(m_\varepsilon) + \varepsilon\sigma(m_\varepsilon) - \gamma - \varepsilon\sigma), \gamma(m) - \gamma \rangle \\ &= 2\langle \dot\gamma(m)\partial_\varepsilon|_0 m_\varepsilon + \sigma(m) - \sigma, \gamma(m) - \gamma \rangle \ . \end{aligned}$$

Using the second assumption of the lemma we can divide by $\langle \dot\gamma(p), \gamma(p) - \gamma \rangle$ and $\langle \dot\gamma(m), \gamma(m) - \gamma \rangle$, respectively, and obtain the following expressions for the variation of the intersection parameters

$$\partial_\varepsilon|_0 p_\varepsilon = \frac{\langle \sigma - \sigma(p), \gamma(p) - \gamma \rangle}{\langle \dot\gamma(p), \gamma(p) - \gamma \rangle} \ , \qquad \partial_\varepsilon|_0 m_\varepsilon = \frac{\langle \sigma - \sigma(m), \gamma(m) - \gamma \rangle}{\langle \dot\gamma(m), \gamma(m) - \gamma \rangle} \ .$$

To calculate the first variation of the circular integral invariant, we treat the two terms of formula (1) separately.

For the first term we obtain

$$\begin{aligned} &\partial_\varepsilon|_0 \int_{m_\varepsilon}^{p_\varepsilon} \langle \gamma_\varepsilon(\psi) - \gamma_\varepsilon, \dot\gamma_\varepsilon(\psi)^\perp \rangle d\psi \\ &= \int_m^p \langle \sigma(\psi) - \sigma, \dot\gamma(\psi)^\perp \rangle + \langle \gamma(\psi) - \gamma, \dot\sigma(\psi)^\perp \rangle d\psi \\ &\qquad + \langle \gamma(p) - \gamma, \dot\gamma(p)^\perp \rangle \partial_\varepsilon|_0 p_\varepsilon - \langle \gamma(m) - \gamma, \dot\gamma(m)^\perp \rangle \partial_\varepsilon|_0 m_\varepsilon \\ &= \int_m^p \langle \sigma(\psi) - \sigma, \dot\gamma(\psi)^\perp \rangle - \int_m^p \langle \dot\gamma(\psi), \sigma(\psi)^\perp \rangle d\psi \\ &\qquad + \langle \gamma(p) - \gamma, \sigma(p)^\perp \rangle - \langle \gamma(m) - \gamma, \sigma(m)^\perp \rangle \\ &\qquad + \langle \gamma(p) - \gamma, \dot\gamma(p)^\perp \rangle \partial_\varepsilon|_0 p_\varepsilon - \langle \gamma(m) - \gamma, \dot\gamma(m)^\perp \rangle \partial_\varepsilon|_0 m_\varepsilon \\ &= \int_m^p 2\langle \sigma(\psi), \dot\gamma(\psi)^\perp \rangle d\psi - \langle \sigma, \gamma(p)^\perp - \gamma(m)^\perp \rangle \\ &\qquad + \langle \gamma(p) - \gamma, \sigma(p)^\perp \rangle - \langle \gamma(m) - \gamma, \sigma(m)^\perp \rangle \\ &\qquad + \langle \gamma(p) - \gamma, \dot\gamma(p)^\perp \rangle \partial_\varepsilon|_0 p_\varepsilon - \langle \gamma(m) - \gamma, \dot\gamma(m)^\perp \rangle \partial_\varepsilon|_0 m_\varepsilon \ . \end{aligned}$$

For the second term we obtain

$$\begin{aligned} &\partial\varepsilon|_0 \frac{r^2}{2} \arccos\left(\frac{\langle \gamma_\varepsilon(p_\varepsilon) - \gamma_\varepsilon, \gamma_\varepsilon(m_\varepsilon) - \gamma_\varepsilon \rangle}{r^2} \right) \\ &= -\frac{\langle \dot\gamma(p)\partial_\varepsilon|_0 p_\varepsilon + \sigma(p) - \sigma, \gamma(m) - \gamma \rangle + \langle \gamma(p) - \gamma, \dot\gamma(m)\partial_\varepsilon|_0 m_\varepsilon + \sigma(m) - \sigma \rangle}{2r^{-2}\sqrt{r^4 - \langle \gamma(p) - \gamma, \gamma(m) - \gamma \rangle^2}} \ . \end{aligned}$$

Using the formulas for the intersection parameters, we obtain the desired result. □

In the special case where γ equals the unit circle the lemma above reduces to:

Lemma 3.2. *Let $\gamma \in \mathcal{C}$ be the constant speed parameterized unit circle, that is,*

$$\gamma(\varphi) = \big(\cos(\varphi), \sin(\varphi)\big) \ ,$$

and let $r < 2$. Then the first variation of $I_r[\gamma]$ in direction $\sigma \in T_\gamma\mathcal{C}$ with

$$\sigma(\varphi) = a(\varphi)\dot\gamma(\varphi)^\perp + b(\varphi)\dot\gamma(\varphi)$$

is given by

$$I_r'[\gamma](\sigma)(\varphi) = \int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)\, d\psi - 2\sin(\vartheta)a(\varphi) = \big(\chi_{[-\vartheta,\vartheta]} * a\big)(\varphi) - 2\sin(\vartheta)a(\varphi)$$

with

$$\vartheta := \arccos\left(1 - \frac{r^2}{2}\right) .$$

The proof of this lemma is postponed to the appendix.

4. Proof of the Main Theorem

Proof. Without loss of generality we may assume that γ is the unit circle. Define for given $\sigma \in T_\gamma\mathcal{C}$ the function $A\sigma\colon S^1 \to \mathbb{R}$ by

$$A\sigma(\varphi) := \langle \sigma(\varphi), \dot\gamma(\varphi)^\perp\rangle .$$

Because γ is a C^∞-curve, it follows that $A\sigma$ is C^1. Using Lemma 2.4 it follows that $A\sigma(0) = 0$ and $\partial_\varphi(A\sigma)(0) = 0$. Denoting by $C^1_0(S^1,\mathbb{R})$ the space of all C^1-functions a on the circle satisfying $a(0) = \dot a(0) = 0$, it follows that A is a bounded linear mapping from $T_\gamma\mathcal{C}$ to $C^1_0(S^1,\mathbb{R})$.

In addition, it follows from Lemma 2.4 that A is boundedly invertible with A^{-1} given by

$$A^{-1}a = a\dot\gamma^\perp + b\dot\gamma$$

with

$$b(\varphi) = \dot b(0)\varphi - \int_0^\varphi a(\tau)\, d\tau \quad \text{and} \quad \dot b(0) = \frac{1}{2\pi}\int_0^{2\pi} a(\tau)\, d\tau .$$

The expression for $\dot b(0)$ is due to the periodicity of b, which implies that

$$0 = b(0) = b(2\pi) = 2\pi\dot b(0) - \int_0^{2\pi} a(\tau)\, d\tau .$$

Therefore A is in fact an isomorphism between $T_\gamma\mathcal{C}$ and $C^1_0(S^1,\mathbb{R})$.

According to Lemma 3.2, the derivative $I_r'[\gamma]$ in direction $\sigma = a\dot\gamma^\perp + b\dot\gamma$ can be written as

$$I_r'[\gamma](\sigma) = \chi_{[-\vartheta,\vartheta]} * a - 2\sin(\vartheta)a .$$

Thus $I_r'[\gamma]$ can be decomposed into

$$I_r'[\gamma] = B \circ \imath \circ A ,$$

where the operator $B\colon C^1(S^1,\mathbb{R}) \to C^1(S^1,\mathbb{R})$ is given by

$$Ba = \chi_{[-\vartheta,\vartheta]} * a - 2\sin(\vartheta)a$$

and $\imath$ is the embedding from $C^1_0(S^1,\mathbb{R})$ into $C^1(S^1,\mathbb{R})$. Lemma 5.1 (see Appendix) implies that the mapping $\sigma \mapsto \chi_{[-\vartheta,\vartheta]} * a$ is compact and thus B is a compact perturbation of the identity. Therefore the Riesz–Schauder theory (see [15, Chap. X.5]) implies that B has a closed range.

Next we compute the kernel of B. To that end we consider the mapping in the Fourier basis. A short calculation shows that in this basis the operator B is the diagonal operator that maps a sequence of (complex) Fourier coefficients $(c_k)_{k\in\mathbb{Z}}$ to the sequence $(d_k c_k)_{k\in\mathbb{Z}}$, where

$$d_k = \begin{cases} 2\big(1-\sin(\vartheta)\big) & \text{if } k=0\,,\\ 0 & \text{if } k=\pm 1\,,\\ \Big[2\frac{\sin(k\vartheta)}{k} - 2\sin(\vartheta)\Big] & \text{else.}\end{cases}$$

Because $\sin(\vartheta)\neq 1$ and $\sin(k\vartheta)\neq k\sin(\vartheta)$ whenever $k\in\mathbb{Z}\setminus\{-1,0,1\}$ (see Lemma 5.2 in the Appendix), it follows that the kernel of B consists of the functions a of the form $a(\varphi) = c_{-1}\exp(-i\varphi) + c_1\exp(i\varphi)$ for some c_{-1}, $c_1\in\mathbb{C}$.

In the next step we show that the kernel of $I_r'[\gamma] = B\circ\imath\circ A$ is trivial. Therefore assume that $a = c_{-1}\exp(-i\cdot) + c_1\exp(i\cdot)\in C_0^1(S^1,\mathbb{R})\cap\operatorname{Ker} B$. Because $a(0)=0$, it follows that $c_{-1}+c_1 = 0$; because $\dot a(0) = 0$, it follows that $-c_{-1}+c_1 = 0$. Together, this shows that $c_{-1} = c_1 = 0$, implying that the intersection of $\operatorname{Ker} B$ with $C_0^1(S^1,\mathbb{R})$ is trivial. Since A is an isomorphism this proves the injectivity of $I_r'[\gamma]$.

We have thus shown that $I_r'[\gamma] = B\circ\imath\circ A$ is strongly closed and injective. Moreover the range of $I_r'[\gamma]$ has a finite codimension (its codimension equals the dimension of the kernel of B, which is 2), showing that $\operatorname{Ran} I_r'[\gamma]$ is complemented in $C^1(S^1,\mathbb{R})$. Therefore, the inverse function theorem (see [8, Prop. 2.3]) implies that I_r is continuously invertible at γ, which concludes the proof. □

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5. Appendix

Lemma 5.1. *The mapping $a\mapsto K_\vartheta a := \chi_{[-\vartheta,\vartheta]} * a$ is compact as a mapping from $C^1(S^1,\mathbb{R})$ to $C^1(S^1,\mathbb{R})$.*

Proof. Denoting by B_1 the unit ball in $C^1(S^1,\mathbb{R})$, we have to show that the image of B_1 under K_ϑ is precompact in $C^1(S^1,\mathbb{R})$. Applying the Arzelà–Ascoli Theorem (see [15, Chap. III.3]), we have to show that $K_\vartheta(B_1)$ is bounded and the derivatives of the functions in $K_\vartheta(B_1)$ are equicontinuous. The boundedness of $K_\vartheta(B_1)$ is obvious, K_ϑ being a bounded linear mapping (of norm 2ϑ). Now assume that $a\in B_1$. Then $\partial_\varphi(K_\vartheta a)(\varphi) = a(\varphi+\vartheta) - a(\varphi-\vartheta)$. Because $\|\dot a\|_\infty\le 1$, it follows that $\partial_\varphi(K_\vartheta a)$ is Lipschitz continuous with Lipschitz constant at most 2. Hence $K_\vartheta(B_1)$ is a precompact set. □

Lemma 5.2. *Let $0<\vartheta<\pi$ and $k\in\mathbb{Z}\setminus\{-1,0,1\}$. Then $\sin(k\vartheta)\neq k\sin(\vartheta)$.*

Proof. Assume first that $0<\vartheta\le\pi/2$. We show that in this case the equation $\sin(s) = s\sin(\vartheta)/\vartheta$ has the only solutions $s=0$ and $s=\pm\vartheta$. First note that the strict concavity of the sine function on the interval $[0,\pi]$ implies that on this interval we only have two solutions, namely 0 and ϑ. Moreover, the concavity of the sine implies that $\sin(\vartheta)/\vartheta \ge \sin(\pi/2)/(\pi/2) = 2/\pi$, and therefore $\pi\sin(\vartheta)/\vartheta\ge 2$. This, however, implies that the equation $\sin(s) = s\sin(\vartheta)/\vartheta$ cannot have any solutions for $s>\pi$, as the right hand side is strictly larger than 2. The fact that $-\vartheta$ is the only negative solution follows by symmetry. In particular, setting $s = k\vartheta$, this proves the assertion in the case $0<\vartheta\le\pi/2$.

Now assume that $\pi/2 < \vartheta < \pi$ and let $\psi := \pi - \vartheta$. Then

$$\sin(\vartheta) = \sin(\pi - \psi) = \sin(\psi)\ .$$

Now, if k is odd, then

$$\sin(k\vartheta) = \sin(k\pi - k\psi) = \sin(\pi - k\psi) = \sin(k\psi)\ .$$

Thus $k\sin(\vartheta) = \sin(k\vartheta)$, if and only if $k\sin(\psi) = \sin(k\psi)$. Because $0 < \psi < \pi/2$, the first part of the proof can be applied, showing that $k\sin(\vartheta) \neq \sin(k\vartheta)$ unless $k = \pm 1$.

On the other hand, if k is even, we have

$$\sin(k\vartheta) = \sin(k\pi - k\psi) = \sin(-k\psi) = -\sin(k\psi)\ .$$

Thus, $k\sin(\vartheta) = \sin(k\vartheta)$, if and only if $k\sin(\psi) = -\sin(k\psi)$. Now note that the equation $-\sin(s) = s\sin(\psi)/\psi$ has only the trivial solution $s = 0$, because $\sin(\psi)/\psi \geq 2/\pi$ and the left hand side is negative for $0 < s < \pi$. As a consequence, the equation $k\sin(\psi) = -\sin(k\psi)$ only holds for $k = 0$, which concludes the proof. □

5.1. **Proof of Lemma 3.2.**

Proof. Let $\gamma \in \mathcal{C}$ be the constant speed parameterized unit circle. Then

$$\gamma(\varphi) = \big(\cos(\varphi), \sin(\varphi)\big)\,, \qquad \dot\gamma(\varphi) = \big(-\sin(\varphi), \cos(\varphi)\big)\,,$$
$$\gamma(\varphi)^\perp = \big(\sin(\varphi), -\cos(\varphi)\big)\,, \qquad \dot\gamma(\varphi)^\perp = \big(\cos(\varphi), \sin(\varphi)\big)\,.$$

Because γ is the unit circle, there exists $\vartheta \in S^1$ such that

$$p(\varphi) = \varphi + \vartheta\,, \qquad m(\varphi) = \varphi - \vartheta\ .$$

Obviously, all assumptions of Lemma 3.1 are satisfied. It remains to calculate all the terms that appear in the expression of the first variation in Lemma 3.1 for the special case of the unit circle. In particular, we obtain

$$\begin{aligned}
r^2 = \|\gamma(p) - \gamma\|^2 &= 2\big(1 - \cos(\vartheta)\big)\,,\\
\langle \gamma(p) - \gamma, \gamma(m) - \gamma\rangle &= -2\cos(\vartheta)\big(1 - \cos(\vartheta)\big)\,,\\
\langle \dot\gamma(p), \gamma(m) - \gamma\rangle &= \sin(\vartheta)\big(1 - 2\cos(\vartheta)\big)\,,\\
\langle \dot\gamma(p)^\perp, \gamma(p) - \gamma\rangle &= 1 - \cos(\vartheta)\,,\\
\langle \dot\gamma(p), \gamma(p) - \gamma\rangle &= \sin(\vartheta)\,,\\
\langle \dot\gamma, \gamma(p) - \gamma\rangle &= \sin(\vartheta)\,,\\
\langle \dot\gamma^\perp, \gamma(p) - \gamma\rangle &= \cos(\vartheta) - 1\,,\\
\langle \dot\gamma(p)^\perp, \gamma(m) - \gamma\rangle &= \cos^2(\vartheta) - \cos(\vartheta) - \sin^2(\vartheta)\,,\\
\langle \dot\gamma^\perp, \gamma(m) - \gamma\rangle &= \cos(\vartheta) - 1\,,\\
\langle \dot\gamma(m), \gamma(p) - \gamma\rangle &= -\sin(\vartheta)\big(1 - 2\cos(\vartheta)\big)\,,\\
\langle \dot\gamma(m)^\perp, \gamma(m) - \gamma\rangle &= 1 - \cos(\vartheta)\,,\\
\langle \dot\gamma(m), \gamma(m) - \gamma\rangle &= -\sin(\vartheta)\,,\\
\langle \dot\gamma, \gamma(m) - \gamma\rangle &= -\sin(\vartheta)\,,\\
\langle \dot\gamma(m)^\perp, \gamma(p) - \gamma\rangle &= \cos^2(\vartheta) - \cos(\vartheta) - \sin^2(\vartheta)\,,\\
\langle \dot\gamma, \gamma(p) - \gamma(m)\rangle &= 2\sin(\vartheta)\ .
\end{aligned}$$

In the following we treat the variation in direction $a\dot\gamma^\perp$ and $b\dot\gamma$ separately. Inserting the formulas above in the expression for $I_r'[\gamma](b\dot\gamma)$ yields

$$\begin{aligned}
2I_r'[\gamma](b\dot\gamma) &= 2\int_m^p b(\psi)\langle\dot\gamma(\psi),\dot\gamma(\psi)^\perp\rangle d\psi \\
&\quad - \langle b\dot\gamma, \gamma(p)^\perp - \gamma(m)^\perp\rangle + \langle\gamma(p)-\gamma, b(p)\dot\gamma(p)^\perp\rangle \\
&\quad - \langle\gamma(m)-\gamma, b(m)\dot\gamma(m)^\perp\rangle + \langle\gamma(p)-\gamma,\dot\gamma(p)^\perp\rangle\frac{\langle b\dot\gamma - b(p)\dot\gamma(p),\gamma(p)-\gamma\rangle}{\langle\dot\gamma(p),\gamma(p)-\gamma\rangle} \\
&\quad - \langle\gamma(m)-\gamma,\dot\gamma(m)^\perp\rangle\frac{\langle b\dot\gamma - b(m)\dot\gamma(m),\gamma(m)-\gamma\rangle}{\langle\dot\gamma(m),\gamma(m)-\gamma\rangle} \\
&\quad - \frac{r^2}{\sqrt{r^4 - \langle\gamma(p)-\gamma,\gamma(m)-\gamma\rangle^2}}\Bigg(\frac{\langle b\dot\gamma - b(p)\dot\gamma(p),\gamma(p)-\gamma\rangle}{\langle\dot\gamma(p),\gamma(p)-\gamma\rangle}\langle\dot\gamma(p),\gamma(m)-\gamma\rangle \\
&\qquad + \langle b(p)\dot\gamma(p) - b\dot\gamma,\gamma(m)-\gamma\rangle + \langle\gamma(p)-\gamma, b(m)\dot\gamma(m) - b\dot\gamma\rangle \\
&\qquad + \frac{\langle b\dot\gamma - b(m)\dot\gamma(m),\gamma(m)-\gamma\rangle}{\langle\dot\gamma(m),\gamma(m)-\gamma\rangle}\langle\gamma(p)-\gamma,\dot\gamma(m)\rangle\Bigg) \\
&= 0 - 0 + \big(1-\cos(\vartheta)\big)\big(b(p) - b(m) + b - b(p) - b + b(m)\big) \\
&\quad - \Big(\big(1 - 2\cos(\vartheta)\big)\big(b - b(p) + b(p) - b(m) + b(m) - b\big) + (b-b)\Big) \\
&= 0\,.
\end{aligned}$$

For the variation in normal direction $a\dot\gamma^\perp$ we get

$$\begin{aligned}
2I_r'[\gamma](a\dot\gamma^\perp) &= 2\int_m^p a(\psi)d\psi - a\langle\dot\gamma^\perp,\gamma(p)^\perp - \gamma(m)^\perp\rangle \\
&\quad - a(p)\langle\gamma(p)-\gamma,\dot\gamma(p)\rangle + a(m)\langle\gamma(m)-\gamma,\dot\gamma(m)\rangle \\
&\quad + \langle\gamma(p)-\gamma,\dot\gamma(p)^\perp\rangle\frac{\langle a\dot\gamma^\perp - a(p)\dot\gamma(p)^\perp,\gamma(p)-\gamma\rangle}{\langle\dot\gamma(p),\gamma(p)-\gamma\rangle} \\
&\quad - \langle\gamma(m)-\gamma,\dot\gamma(m)^\perp\rangle\frac{\langle a\dot\gamma^\perp - a(m)\dot\gamma(m)^\perp,\gamma(m)-\gamma\rangle}{\langle\dot\gamma(m),\gamma(m)-\gamma\rangle} \\
&\quad - \frac{r^2}{\sqrt{r^4 - \langle\gamma(p)-\gamma,\gamma(m)-\gamma\rangle^2}}\Bigg(\frac{\langle a\dot\gamma^\perp - a(p)\dot\gamma(p)^\perp,\gamma(p)-\gamma\rangle}{\langle\dot\gamma(p),\gamma(p)-\gamma\rangle}\langle\dot\gamma(p),\gamma(m)-\gamma\rangle \\
&\qquad + \langle a(p)\dot\gamma(p)^\perp - a\dot\gamma^\perp,\gamma(m)-\gamma\rangle + \langle\gamma(p)-\gamma, a(m)\dot\gamma(m)^\perp - a\dot\gamma^\perp\rangle \\
&\qquad + \frac{\langle a\dot\gamma^\perp - a(m)\dot\gamma(m)^\perp,\gamma(m)-\gamma\rangle}{\langle\dot\gamma(m),\gamma(m)-\gamma\rangle}\langle\gamma(p)-\gamma,\dot\gamma(m)\rangle\Bigg)
\end{aligned}$$

Inserting the expressions for γ equal to the unit circle that have been calculated previously, we obtain

$$
\begin{aligned}
&2I_r'[\gamma](a\dot\gamma^\perp)\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi - \sin(\vartheta)\big(a(p)+2a+a(m)\big) - \frac{(\cos(\vartheta)-1)^2}{\sin(\vartheta)}\big(a(p)+2a+a(m)\big)\\
&\qquad - \frac{1}{\sin(\vartheta)}\Big(\big(a+a(p)\big)\big(\cos(\vartheta)-1\big)\big(1-2\cos(\vartheta)\big) - 2a(\cos(\vartheta)-1)\\
&\qquad\quad + \big(a(p)+a(m)\big)\big(\cos^2(\vartheta)-\cos(\vartheta)-\sin^2(\vartheta)\big)\\
&\qquad\quad + \big(a+a(m)\big)\big(\cos(\vartheta)-1\big)\big(1-2\cos(\vartheta)\big)\Big)\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi + \frac{\cos(\vartheta)-1}{\sin(\vartheta)}\big(2a(p)+4a+2a(m)-2a(p)+4\cos(\vartheta)a-2a(m)\big)\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi + 4a\frac{(\cos(\vartheta)-1)(\cos(\vartheta)+1)}{\sin(\vartheta)}\\
&= 2\int_{\varphi-\vartheta}^{\varphi+\vartheta} a(\psi)d\psi - 4a\sin(\vartheta)\ .
\end{aligned}
$$

Therefore,

$$
I_r'[\gamma](a\dot\gamma^\perp + b\dot\gamma) = \chi_{[-\vartheta,\vartheta]} * a - 2\sin(\vartheta)a\ . \qquad \square
$$

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