

PROJECTIVE HULLS AND CHARACTERIZATIONS OF MEROMORPHIC FUNCTIONS

JOHN T. ANDERSON, JOSEPH A. CIMA, NORM LEVENBERG,
AND THOMAS J. RANSFORD

ABSTRACT. We give conditions characterizing holomorphic and meromorphic functions in the unit disk of the complex plane in terms of certain weak forms of the maximum principle. Our work is directly inspired by recent results of John Wermer, and by the theory of the projective hull of a compact subset of complex projective space developed by Reese Harvey and Blaine Lawson.

1. INTRODUCTION

For a compact subset K of complex Euclidean space \mathbb{C}^n , we let $C(K)$ be the Banach space of continuous complex-valued functions on K with norm

$$\|f\|_K = \max\{|f(\zeta)| : \zeta \in K\}.$$

The classical maximum principle states that if Ω is a bounded connected open subset of \mathbb{C}^n , and f is holomorphic and non-constant in Ω and continuous on $\bar{\Omega}$, then for all $z \in \Omega$

$$|f(z)| < \|f\|_{\partial\Omega}.$$

Our goal in this paper is to explore converses to this basic property of holomorphic functions, i.e., to establish characterizations of holomorphic and meromorphic functions in terms of the existence of certain weak forms of the maximum principle.

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One such result, established by Walter Rudin in 1953 [10], goes as follows: let \mathbb{D} be the open unit disk in the complex plane, and let \mathbb{T} be its boundary. Let A be an algebra of continuous complex-valued functions on the closed unit disk $\overline{\mathbb{D}}$ such that

- i. for each $f \in A$ and $z \in \mathbb{D}$, we have $|f(z)| \leq \|f\|_{\mathbb{T}}$, and
- ii. A contains the disk algebra $A(\mathbb{D})$ of all functions holomorphic in \mathbb{D} and continuous in $\overline{\mathbb{D}}$,

then $A = A(\mathbb{D})$. In fact, this is a consequence of more general theorems that Rudin proved in [10] concerning maximum modulus algebras on plane sets.

Rudin ([11], Theorem 12.12) later relaxed the requirement that the set of functions in question be an algebra. He showed that if V is a vector subspace of the space $C(\overline{\mathbb{D}})$ of continuous complex-valued functions on $\overline{\mathbb{D}}$ such that

- i. for each $f \in V$ and $z \in \mathbb{D}$, we have $|f(z)| \leq \|f\|_{\mathbb{T}}$;
- ii. V contains the constant function 1; and
- iii. for each $f \in V$, the function $g(z) = zf(z)$ also belongs to V ,

then $V = A(\mathbb{D})$.

Note that the hypothesis (i) in both of these results is a weak form of the maximum principle: we do not demand that the maximum of $|f|$ is attained *only* on \mathbb{T} .

Rudin's later theorem can be rephrased as a statement about modules over the disk algebra: choose $\phi \in C(\overline{\mathbb{D}})$, and consider the space

$$(1.1) \quad \mathfrak{M} = \{a + b\phi : a, b \in A(\mathbb{D})\}.$$

If for each $f \in \mathfrak{M}$ and $z \in \mathbb{D}$,

$$(1.2) \quad |f(z)| \leq \|f\|_{\mathbb{T}}$$

then $\mathfrak{M} = A(\mathbb{D})$, i.e., $\phi \in A(\mathbb{D})$. These two formulations are easily seen to be equivalent: first, assuming that (1.2) holds, then \mathfrak{M} is a vector subspace of $C(\overline{\mathbb{D}})$ satisfying the hypotheses (i), (ii) and (iii) of Rudin's result, and so $\mathfrak{M} = A(\mathbb{D})$. Conversely, given any vector space V satisfying (i), (ii) and (iii), choose any $\phi \in V$. Then by assumptions (ii) and (iii), V contains all functions of the form $a + b\phi$ where a, b are polynomials. Since polynomials are dense in $A(\mathbb{D})$, assumption (i) implies that (1.2) holds for the space \mathfrak{M} defined in (1.1), and so ϕ is holomorphic in \mathbb{D} . Since $\phi \in V$ was arbitrary, $V = A(\mathbb{D})$.

Note that \mathfrak{M} is a module over the disk algebra generated by the constant function 1 and ϕ . In fact, the same result holds for any subset \mathfrak{M}' of $C(\overline{\mathbb{D}})$ that forms a module over the disk algebra (with the usual multiplication of functions) and contains 1: if we assume (1.2) holds

for all $f \in \mathfrak{M}'$ then $\mathfrak{M}' = A(\mathbb{D})$. To see this, choose any ϕ in \mathfrak{M}' , form the submodule $\mathfrak{M} \subset \mathfrak{M}'$ as in (1.1), and conclude that $\phi \in A(\mathbb{D})$. The same remark obtains for the result we discuss next.

Recently, John Wermer ([16], Theorem 2) gave the following generalization of Rudin's Theorem:

Theorem 1.1. *Let \mathfrak{M} be as in (1.1), and assume that for each $z \in \mathbb{D}$ there exists a constant C_z so that*

$$(1.3) \quad |f(z)| \leq C_z \|f\|_{\mathbb{T}}$$

for all $f \in \mathfrak{M}$. If the restriction of ϕ to \mathbb{T} is real-analytic, then ϕ is holomorphic in \mathbb{D} .

In section 2 we will strengthen Wermer's result (see Theorem 2.1, Corollary 2.3, and Remark 2.4) to give a characterization of functions meromorphic in the disk, while also relaxing the requirement that ϕ be real-analytic on the circle.

Wermer's hypothesis (1.3) was inspired by the concept of the *projective hull* \hat{K} of a compact subset K of projective space \mathbb{P}^n introduced by Harvey and Lawson in [6]. For a compact subset K of an affine coordinate chart \mathbb{C}^n the intersection of \hat{K} with \mathbb{C}^n is the set of points $z \in \mathbb{C}^n$ such that there exists a constant C_z with

$$(1.4) \quad |p(z)| \leq C_z^{\deg p} \|p\|_K$$

for all polynomials p (see [6], section 6). For compact subsets K of \mathbb{C}^n , the projective hull is thus a generalization of the polynomially convex hull \hat{K}_P , defined as the set of points $z \in \mathbb{C}^n$ for which (1.4) holds with $C_z = 1$.

Harvey and Lawson were in turn motivated by the desire to find an appropriate generalization for curves in projective space of a celebrated theorem of Wermer [15]: if $\gamma \subset \mathbb{C}^n$ is a closed real-analytic curve, and $\hat{\gamma}_P \neq \gamma$, then $\hat{\gamma}_P \setminus \gamma$ is a one-dimensional complex-analytic variety. This result was later extended by various authors to smooth and even merely rectifiable curves. However, the analogue of Wermer's theorem for projective hulls may fail spectacularly if γ is smooth but not real-analytic: Diederich and Fornaess [2] constructed a C^∞ curve $\gamma \subset \mathbb{C}^2$ such that γ is not pluripolar, and (see section 3) this is equivalent to the statement that $\hat{\gamma} = \mathbb{P}^2$.

Wermer also considered characterizations of functions meromorphic in the punctured disk $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ in terms of projective hulls of graphs. If $\phi \in C(\mathbb{D}^* \cup \mathbb{T})$ we set

$$\Sigma := \{(z, \phi(z)) : z \in \mathbb{D}^*\}$$

and

$$\gamma := \{(z, \phi(z)) : z \in \mathbb{T}\}.$$

Wermer showed ([16], Proposition 3)

Theorem 1.2. *If $\phi \in C(\mathbb{D}^* \cup \mathbb{T})$ is meromorphic in \mathbb{D} , then $\Sigma \subset \hat{\gamma}$.*

Note that the condition $\Sigma \subset \hat{\gamma}$ is equivalent to the statement that for each $z \in \mathbb{D}^*$ there exists a constant C_z such that

$$(1.5) \quad |p(z, \phi(z))| \leq C_z^{\deg p} \|p\|_\gamma$$

for all polynomials p in two variables, while the hypothesis (1.3) in Theorem 1.1 is equivalent to the existence of a constant C_z such that

$$|p(z, \phi(z))| \leq C_z \|p\|_\gamma$$

for all polynomials $p(z, w)$ of degree at most one in w and of arbitrary degree in z .

In section 3 we will consider the question of a converse to Theorem 1.2. The existence (noted above) of smooth curves with projective hull equal to \mathbb{P}^2 makes it necessary to place some restriction on the curve γ . Wermer in [16] posed the question: *Suppose $\phi \in C(\mathbb{D}^* \cup \mathbb{T})$ and ϕ is real-analytic on \mathbb{T} . If $\Sigma \subset \hat{\gamma}$, is ϕ in fact meromorphic in \mathbb{D} ?* Under a much stronger assumption on ϕ (namely, that ϕ is in fact real-analytic on all of \mathbb{C}^2), Wermer showed that the answer is yes (in this case the conclusion is that ϕ is in fact an entire function).

We will show in section 3 (Theorem 3.2) that if $\phi \in C(\mathbb{D}^* \cup \mathbb{T})$ and γ is pluripolar, then using a deep result of Shcherbina it is easy to conclude that $\Sigma \subset \hat{\gamma}$ implies that ϕ is holomorphic in \mathbb{D}^* . In order to show that ϕ has at most a pole at the origin, we will make a stronger assumption on the projective hull (Theorem 3.5): we assume that $\Sigma = \hat{\gamma} \cap (\mathbb{D}^* \times \mathbb{C})$. We do not know if this stronger assumption is necessary.

Finally, we note that Josip Globevnik has recently given in [4] a characterization of meromorphic functions that bears some resemblance to the Wermer/Rudin results but postulates a condition on winding numbers: $\phi \in C(\mathbb{T})$ extends meromorphically to \mathbb{D} iff there exists $N \in \mathbb{N} \cup \{0\}$ such that for all f of the form $a + b\phi$ with $a, b \in A(\mathbb{D})$ the change in the argument of f around \mathbb{T} is bounded by $-2\pi N$. We are indebted to Dmitry Khavinson for pointing out Globevnik's work to us.

2. MODULES OVER THE DISK ALGEBRA

We will denote by H^p the Hardy space consisting of functions holomorphic in the disk and satisfying (if $0 < p < \infty$)

$$\|f\|_p^p := \sup_{r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

while H^∞ will denote the space of functions bounded and holomorphic in the disk, with $\|f\|_\infty = \sup\{|f(z)| : |z| < 1\}$. For $\zeta \in \mathbb{T}$ and $\alpha > 1$ we let $\Gamma_\alpha(\zeta)$ be the nontangential approach region

$$\Gamma_\alpha(\zeta) := \{z \in \mathbb{D} : |z - \zeta| < \alpha(1 - |z|)\}.$$

Then for a function $f \in H^p$, $0 < p \leq \infty$, the nontangential limit $f^*(\zeta) = \lim_{z \in \Gamma_\alpha(\zeta) \rightarrow \zeta} f(z)$ exists for almost every $\zeta \in \mathbb{T}$ and every $\alpha > 1$. Moreover $f^* \in L^p(\mathbb{T})$ with respect to normalized Lebesgue measure $d\theta/2\pi$ on the circle and $\|f\|_p = \|f^*\|_{L^p(\mathbb{T})}$, for $0 < p \leq \infty$.

We first give a condition equivalent to the existence of the inequality (1.3) at a single point, assuming only that $\phi \in C(\mathbb{T})$.

Theorem 2.1. *If $\phi \in C(\mathbb{T})$, $z \in \mathbb{D}$, and $\lambda \in \mathbb{C}$, then the following are equivalent:*

- (1) *There exists a constant C such that $|a(z) + b(z)\lambda| \leq C\|a + b\phi\|_{\mathbb{T}}$ for all $a, b \in A(\mathbb{D})$;*
- (2) *There exist bounded holomorphic functions f, g in \mathbb{D} such that $h = f/g$ satisfies $h(z) = \lambda$ and $h^* = \phi$ a.e. on \mathbb{T} .*

Proof. First, suppose that (1) holds. Since composition with a fixed element of the holomorphic automorphism group of the disk preserves $A(\mathbb{D})$ and is a norm-preserving map of $C(\mathbb{T})$ to itself, we may assume that $z = 0$. Let

$$\mathfrak{M} = \{a + b\phi : a, b \in A(\mathbb{D})\}.$$

Restricting functions in \mathfrak{M} to the circle, we may consider \mathfrak{M} as a subspace of $C(\mathbb{T})$. By (1) the linear functional given by $a + b\phi \mapsto a(0) + b(0)\lambda$ is bounded on \mathfrak{M} , and therefore by the Hahn-Banach theorem extends to a continuous linear functional on $C(\mathbb{T})$. By the Riesz representation theorem there exists a complex measure μ supported on \mathbb{T} representing this linear functional, i.e.,

$$(2.1) \quad \int_{\mathbb{T}} (a + b\phi) d\mu = a(0) + b(0)\lambda$$

for all $a, b \in A(\mathbb{D})$. Taking $b \equiv 0$ and $a(z) = z^n$ in (2.1) we find that for all $n \geq 1$,

$$\int_{\mathbb{T}} z^n d\mu(z) = 0,$$

while for $n = 0$ we have $\int_{\mathbb{T}} d\mu = 1$. Thus the measure ν defined by $d\nu = d\mu - d\theta/2\pi$ has the property that

$$\int_{\mathbb{T}} z^n d\nu(z) = 0$$

for all $n \geq 0$. By the F. and M. Riesz theorem, there exists a function $k \in H^1(\mathbb{D})$ with $k(0) = 0$ such that $d\nu = k^*d\theta/2\pi$. Solving for $d\mu$ and replacing k by $k + 1$, we find

$$(2.2) \quad d\mu = k^*d\theta/2\pi$$

with $k \in H^1$ and $k(0) = 1$.

Next, in (2.1) we take $a \equiv 0$ and $b(z) = z^n$ to obtain for $n \geq 1$

$$\int_{\mathbb{T}} z^n \phi(z) d\mu = 0,$$

while for $n = 0$ we have $\int_{\mathbb{T}} \phi d\mu = \lambda$. Arguing as above with μ replaced by $\phi d\mu$ we obtain a function $\ell \in H^1$ with $\ell(0) = \lambda$ and

$$(2.3) \quad \phi d\mu = \ell^*d\theta/2\pi.$$

Multiplying both sides of (2.2) by ϕ and comparing with (2.3) we see that $\phi k^* = \ell^*$ a.e. on \mathbb{T} . Since $k(0) = 1$, k is not identically zero, so k^* is non-zero a.e. on \mathbb{T} . Thus if we set $h = \ell/k$,

$$\phi = h^*$$

a.e. on \mathbb{T} , and $h(0) = \ell(0)/k(0) = \lambda$. Since every function in H^1 can be represented as the quotient of two functions holomorphic and bounded in \mathbb{D} (this is true more generally for functions in the Nevanlinna class - see [3], Theorem 2.1), the same is true for h . Thus we may write $h = f/g$ with f, g bounded, establishing (2).

Now suppose that (2) holds, so that $h = f/g$ with f, g bounded and holomorphic in \mathbb{D} , $h(z) = \lambda$, and $h^* = \phi$ a.e in \mathbb{T} . Note $g(z) \neq 0$. Given $a, b \in A(\mathbb{D})$, we have

$$\begin{aligned} |a(z) + b(z)\lambda| &= |a(z) + b(z)f(z)/g(z)| \\ &= \frac{1}{|g(z)|} \cdot |g(z)a(z) + b(z)f(z)| \\ &= \frac{1}{|g(z)|} \|ga + bf\|_{\infty} \\ &\leq \frac{1}{|g(z)|} \cdot \|g^*\|_{L^{\infty}(\mathbb{T})} \cdot \|a + bf^*/g^*\|_{L^{\infty}(\mathbb{T})} \\ &= C\|a + b\phi\|_{\mathbb{T}} \end{aligned}$$

taking $C = C_z = \|g^*\|_{L^\infty(\mathbb{T})}/|g(z)|$. Here we have used the facts that $ga + bf \in H^\infty$ and that f^*/g^* agrees a.e. on \mathbb{T} with the continuous function ϕ . This shows that (1) holds, and completes the proof of Theorem 2.1. \square

Remark 2.2. Condition (2) can be interpreted as saying that there is a meromorphic function h whose graph over the disk passes through the point $(z, \lambda) \in \mathbb{C}^2$ and whose “boundary” lies on the graph γ of ϕ over \mathbb{T} , in the sense that $(e^{i\theta}, h^*(e^{i\theta})) \in \gamma$ for almost all θ .

Corollary 2.3. *Let $\Omega \subset \mathbb{D}$ be open, and assume $\phi \in C(\Omega \cup \mathbb{T})$. Suppose that for each $z \in \Omega$ there exists a constant C_z such that for all $a, b \in A(\mathbb{D})$,*

$$|a(z) + b(z)\phi(z)| \leq C_z \|a + b\phi\|_{\mathbb{T}}.$$

Then there exist $f, g \in H^\infty$ such that $h = f/g$ satisfies $h^ = \phi$ a.e. on \mathbb{T} and $h = \phi$ on Ω . In particular, ϕ is holomorphic in Ω and extends to be meromorphic in \mathbb{D} .*

Proof. By Theorem 2.1, for each fixed $z \in \Omega$, there exist $f, g \in H^\infty$ such that if $h = f/g$, we have $h^* = \phi$ a.e. on \mathbb{T} and $h(z) = \phi(z)$. The functions f, g and h depend *a priori* on the point z , but note that the condition $h^* = \phi$ a.e. uniquely determines h : if h_1, h_2 are both quotients of H^∞ functions and $h_1^* = h_2^*$ a.e. on \mathbb{T} , then $h_1 = h_2$. We therefore obtain a single function h , and (we may choose) a single pair of bounded functions f, g , so that $h = f/g$ on \mathbb{D} and $h(z) = \phi(z)$ for all $z \in \Omega$. Thus both ϕ and h are in fact holomorphic in Ω , and ϕ extends meromorphically (as h) to \mathbb{D} . \square

Remark 2.4. Let F be a discrete subset of \mathbb{D} , and $\Omega = \mathbb{D} \setminus F$. Corollary 2.3 then gives a characterization of functions meromorphic in \mathbb{D} whose poles are contained in the set F . In particular, taking F to be the empty set, we obtain a generalization of Wermer’s result Theorem 1.1 in which the function ϕ is not required to be real-analytic on \mathbb{T} .

3. PROJECTIVE HULLS OF GRAPHS

In this section we consider the possibility of a converse to Theorem 1.2. Our notation is as in the Introduction: $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, we fix a function ϕ continuous on $\mathbb{D}^* \cup \mathbb{T}$, and Σ, γ are the graphs of ϕ over \mathbb{D}^* and \mathbb{T} , respectively.

We begin with some necessary preliminaries from pluripotential theory. Given a domain $\Omega \subset \mathbb{C}^n$, we let $\text{PSH}(\Omega)$ be the set of functions plurisubharmonic in Ω , i.e., the set of upper semi-continuous functions $u : \Omega \rightarrow [-\infty, \infty)$ which are subharmonic when restricted to (each

component of) $L \cap \Omega$, for any complex line L in \mathbb{C}^n . If g is a function holomorphic in Ω , then $\log |g| \in \text{PSH}(\Omega)$. We will use the fact that the maximum of a finite subset of $\text{PSH}(\Omega)$ is also plurisubharmonic in Ω .

For a compact set $K \subset \mathbb{C}^n$, set

$$V_K(z) = \sup\left\{\frac{1}{\deg(p)} \log |p(z)| : p \text{ is a polynomial with } \|p\|_K \leq 1\right\}.$$

It is clear from the characterization (1.4) of the projective hull that

$$\hat{K} \cap \mathbb{C}^n = \{z : V_K(z) < \infty\},$$

and that if $z \in \hat{K} \cap \mathbb{C}^n$, then $V_K(z) = \log C_z$, where C_z is the least constant such that (1.4) holds. Each of the functions $u = \log |p| / \deg p$ appearing in the definition of V_K satisfies $u \leq 0$ on K and is easily seen to be a member of the Lelong class $L(\mathbb{C}^n)$ of plurisubharmonic functions of logarithmic growth defined by

$$(3.1) \quad L(\mathbb{C}^n) = \{u \in \text{PSH}(\mathbb{C}^n) : u(z) \leq \log(1 + |z|) + c, \forall z \in \mathbb{C}^n\}$$

where in (3.1) the constant c is allowed to depend on the function u . For K compact we may also obtain (see [13] or [17]) the extremal function V_K by taking the supremum over the larger class:

$$V_K(z) = \sup\{u(z) : u \in L(\mathbb{C}^n), u \leq 0 \text{ on } K\}.$$

Note that $V_K \geq 0$, and thus, by replacing each function $\log |p| / \deg p$ by $\max\{0, \log |p| / \deg p\}$ we may realize $V_K(z)$ as an upper envelope of *continuous* functions in the Lelong class.

A set $E \subset \mathbb{C}^n$ is said to be *pluripolar* if there exists a function u which is plurisubharmonic on \mathbb{C}^n and not identically equal to $-\infty$, such that $E \subset \{z \in \mathbb{C}^n : u(z) = -\infty\}$. By a result of Josefson [7], this is equivalent to the statement that E is *locally pluripolar*, i.e., for each $z_0 \in E$, there is an open neighborhood U of z_0 and $u \in \text{PSH}(U)$ with $E \cap U \subset \{z \in U : u(z) = -\infty\}$. It follows that any complex analytic subvariety of an open subset of \mathbb{C}^n is pluripolar.

Using a result of Guedj and Zeriahi [5], Harvey and Lawson prove ([6], Corollary 4.4) the following fundamental relationship between pluripolarity and the projective hull:

Proposition 3.1. *For a compact set $K \subset \mathbb{P}^n$,*

$$K \text{ is pluripolar} \Leftrightarrow \hat{K} \text{ is pluripolar} \Leftrightarrow \hat{K} \neq \mathbb{P}^n.$$

If $\gamma \subset \mathbb{C}^n$ is a real-analytic curve, then γ is locally contained in a complex analytic variety, and so is pluripolar. However there are (as we noted in the introduction) smooth curves $\gamma \subset \mathbb{C}^2$ with γ not pluripolar, implying $\hat{\gamma} = \mathbb{P}^2$. (The paper [1] explores conditions under which a

smooth curve is pluripolar.) It is therefore natural, in considering the converse of Theorem 1.2, to impose the condition that γ is pluripolar. With this assumption we have:

Proposition 3.2. *Suppose $\phi \in C(\mathbb{D}^* \cup \mathbb{T})$ and that γ is pluripolar. If $\Sigma \subset \hat{\gamma}$, then ϕ is holomorphic in \mathbb{D}^* .*

Proof. A result of Shcherbina [14] states that if Ω is a domain in \mathbb{C}^n , and $f \in C(\Omega)$, then the graph of f over Ω is pluripolar if and only if f is holomorphic in Ω . Our assumption on γ implies that $\hat{\gamma}$ is a proper subset of \mathbb{C}^2 , and so γ , and therefore $\hat{\gamma}$ is pluripolar. Since Σ is contained in $\hat{\gamma}$, it too is pluripolar. By Shcherbina's result, ϕ is holomorphic in \mathbb{D}^* . \square

Given Proposition 3.2 and our previous remarks, it is natural to make the following conjecture (cf. Wermer's question mentioned in the Introduction):

Conjecture 3.3. *If γ is pluripolar and $\Sigma \subset \hat{\gamma}$, then ϕ is meromorphic in \mathbb{D} .*

As a first step we prove that if the extremal function V_γ is harmonic on Σ , then ϕ is in fact meromorphic.

Lemma 3.4. *Under the hypotheses of Proposition 3.2, if $z \mapsto V_\gamma(z, \phi(z))$ is harmonic on \mathbb{D}^* , then ϕ is meromorphic in \mathbb{D} .*

Proof. Let $h(z) = -V_\gamma(z, \phi(z))$. Since $V_\gamma \geq 0$, h is harmonic and bounded above on \mathbb{D}^* , and thus extends to be subharmonic on \mathbb{D} (the union of \mathbb{D}^* with the polar set $\{0\}$) by defining $h(0) = \limsup_{z \rightarrow 0} h(z)$. Then $\Delta u = \alpha \delta_0$ for some constant α , where δ_0 is the unit point mass at the origin, so

$$\beta(z) := h(z) - \alpha \frac{\log |z|}{2\pi}$$

is in fact harmonic on \mathbb{D} , and therefore is bounded below near the origin. It follows that there exists a constant $m \geq 0$ such that

$$(3.2) \quad V_\gamma(z, \phi(z)) \leq -m \log |z|$$

Recall that $V_\gamma(z, \phi(z)) = \log C_z$, where C_z is the least constant such that $|p(z)| \leq C_z^{\deg p} \|p\|_\gamma$ for all polynomials p . Taking $p(z, w) = w$ we obtain for all z near the origin

$$|\phi(z)| \leq C_z \|\phi\|_\mathbb{T} = \exp(V_\gamma(z, \phi(z))) \|\phi\|_\mathbb{T} \leq C |z|^{-m}$$

implying that ϕ has a pole of order at most m at $z = 0$. \square

Note that while the assumption $\Sigma \subset \hat{\gamma}$ implies that V_γ is finite on Σ , *a priori* we have no reason to believe that V_γ has any regularity, e.g., is even locally bounded, on Σ . However, by making a stronger assumption on $\hat{\gamma}$ we can prove that V_γ is harmonic and thereby establish the following special case of Conjecture 3.3:

Theorem 3.5. *If $\phi \in C(\mathbb{D}^* \cup \mathbb{T})$ and $\Sigma = \hat{\gamma} \cap (\mathbb{D}^* \times \mathbb{C})$, then ϕ is meromorphic in \mathbb{D} .*

The proof of Theorem 3.5 follows closely the reasoning in the proof of Theorem 7.3 of [6], which is itself an adaptation of arguments of Sadullaev in [12]. It is based upon the following lemma.

Lemma 3.6. *Under the hypotheses of Theorem 3.5, assume $u \in L(\mathbb{C}^2) \cap C(\mathbb{C}^2)$ with $u \leq 0$ on γ . Fix an open disk D with $\bar{D} \subset \mathbb{D}^*$, and let v be the harmonic extension to D of the restriction of $u(z, \phi(z))$ to ∂D . Then for every $\epsilon > 0$ there exists $u_\epsilon \in L(\mathbb{C}^2) \cap C(\mathbb{C}^2)$ with $u_\epsilon \leq 0$ on γ and*

$$(3.3) \quad u_\epsilon(z, \phi(z)) \geq \max\{u(z, \phi(z)), v(z) - \epsilon\}$$

for all $z \in \bar{D}$.

Proof. For $z \in \bar{D}$ and $w \in \mathbb{C}$ set $\tilde{v}(z, w) = v(z)$. By the continuity of u and v we may choose $\rho > 0$ sufficiently small so that

$$(3.4) \quad \tilde{v} - \epsilon < u \text{ on } X := \{(z, w) : z \in \partial D, |w - \phi(z)| \leq \rho\}.$$

Since $\Sigma = \hat{\gamma} \cap (\mathbb{D}^* \times \mathbb{C})$, and $V_\gamma = \infty$ on $\mathbb{C}^2 \setminus \hat{\gamma}$, for each point $q \in (\mathbb{D}^* \times \mathbb{C}) \setminus \Sigma$, we may choose a function $U_q \in L(\mathbb{C}^2) \cap C(\mathbb{C}^2)$ with $U_q \leq 0$ on γ and $U_q(q) > \|v\|_{\bar{D}}$. Thus by compactness of $Y := \{(z, w) : z \in \bar{D}, |w - \phi(z)| = \rho\}$, there exist finitely many functions $U_1, \dots, U_N \in L(\mathbb{C}^2) \cap C(\mathbb{C}^2)$ so that

$$(3.5) \quad U := \max\{U_1, \dots, U_N\} > \|v\|_{\bar{D}} = \|\tilde{v}\|_Y \text{ on } Y$$

and $U \leq 0$ on γ . Then $U \in L(\mathbb{C}^2) \cap C(\mathbb{C}^2)$. Set $\Omega = \{(z, w) : z \in D, |w - \phi(z)| < \rho\}$, and define

$$u_\epsilon = \begin{cases} \max\{U, u, \tilde{v} - \epsilon\} & \text{on } \Omega \\ \max\{U, u\} & \text{on } \mathbb{C}^2 \setminus \Omega. \end{cases}$$

Since $\partial\Omega = X \cup Y$, equations (3.4) and (3.5) imply that u_ϵ is continuous. Moreover, U and u are plurisubharmonic on \mathbb{C}^2 while \tilde{v} is plurisubharmonic on Ω , so by a standard “gluing” result (see [8], Corollary 2.9.15) u_ϵ is plurisubharmonic on \mathbb{C}^2 . Since both U and u are of logarithmic growth, and less than or equal to zero on γ , the same is true of u_ϵ (note $\gamma \subset \mathbb{C}^2 \setminus \Omega$). Finally, $\bar{D} \times \mathbb{C} \subset \bar{\Omega}$, so (3.3) clearly holds. \square

Proof of Theorem 3.5. By Lemma 3.4, to show that ϕ is meromorphic, it suffices to show that $V_\gamma(z, \phi(z))$ is harmonic in \mathbb{D}^* . To this end, fix an arbitrary disk D with $\overline{D} \subset \mathbb{D}^*$ and a point $z_0 \in D$. We may choose a sequence $\{u_m\} \subset L(\mathbb{C}^2) \cap C(\mathbb{C}^2)$ with $u_m \leq 0$ on γ such that $\lim_{m \rightarrow \infty} u_m(z_0) = V_\gamma(z_0, \phi(z_0))$. Replacing u_m with $\max\{u_1, \dots, u_m\}$, we may assume that the sequence u_m is increasing. Let v_m be the harmonic extension to D of the restriction of $u_m(z, \phi(z))$ to ∂D . Note that since $\{u_m\}$ is increasing, the sequence $\{v_m\}$ is also increasing. Taking $\epsilon = 1/m$, Lemma 3.6 gives a function $\tilde{u}_m = u_{m, 1/m}$ with

$$(3.6) \quad u_m(z, \phi(z)) - 1/m \leq v_m(z) - 1/m \leq \tilde{u}_m(z, \phi(z)) \leq V_\gamma(z, \phi(z))$$

for all $z \in \overline{D}$, the rightmost inequality following from the fact that $\tilde{u}_m \in L(\mathbb{C}^2)$ and $\tilde{u}_m \leq 0$ on γ . By (3.6) and our choice of the sequence u_m ,

$$\lim_{m \rightarrow \infty} v_m(z_0) = V_\gamma(z_0, \phi(z_0)) < \infty.$$

Harnack's Theorem ([9], Theorem 1.3.9) then implies that the sequence v_m converges locally uniformly to a function s harmonic in D ; by (3.6) we have $s(z) \leq V_\gamma(z, \phi(z))$ for all $z \in D$. Now choose any point $z_1 \in D, z_1 \neq z_0$, and repeat the above argument, beginning with an increasing sequence $\{U_m\} \subset L(\mathbb{C}^2) \cap C(\mathbb{C}^2)$ such that

$$(3.7) \quad \lim_{m \rightarrow \infty} U_m(z_1) = V_\gamma(z_1, \phi(z_1)).$$

We may replace U_m by $\max\{U_m, u_m\}$ and retain (3.7), since U_m and u_m are both bounded above by V_γ ; thus we may assume that $U_m \geq u_m$. Arguing as above, we obtain a function t harmonic in D with $t(z_1) = V_\gamma(z_1, \phi(z_1))$ and $t(z) \leq V_\gamma(z, \phi(z))$ for all $z \in D$. Moreover, since $U_m \geq u_m$, we have $t \geq s$ on D . But then

$$V_\gamma(z_0, \phi(z_0)) = s(z_0) \leq t(z_0) \leq V_\gamma(z_0, \phi(z_0)),$$

implying $s(z_0) = t(z_0)$. By the maximum principle, $s \equiv t$ on D , and so the harmonic function $s(z)$ is equal to $V_\gamma(z, \phi(z))$ when $z = z_0, z_1$. But z_1 was an arbitrary point in D , and therefore $s(z) = V_\gamma(z, \phi(z))$ for all $z \in D$, implying $V_\gamma(z, \phi(z))$ is harmonic in D . Since D was arbitrary, $V_\gamma(z, \phi(z))$ is harmonic on \mathbb{D}^* . This completes the proof. \square

Remark 3.7. The hypothesis $\Sigma = \hat{\gamma} \cap (\mathbb{D}^* \times \mathbb{C})$ was used only in the proof of Lemma 3.6, to ensure that (locally) points at some fixed small distance from Σ are disjoint from $\hat{\gamma}$, and therefore $V_\gamma = \infty$ at such points. We do not know if this assumption is necessary in order to conclude that ϕ is meromorphic, but it is in the spirit of certain ‘‘regularity’’ hypotheses of $\hat{\gamma}$ appearing in the work of Harvey and Lawson.

For example, Harvey and Lawson prove the following analogue of Wermer's theorem on the polynomial hull of a curve in \mathbb{C}^n : if a smooth pluripolar curve $\gamma \subset \mathbb{P}^n$ has the property that the closure of $\hat{\gamma}$ has finite two-dimensional Hausdorff measure in a neighborhood of some complex hypersurface, then $\overline{\hat{\gamma}} \setminus \gamma$ is a 1-dimensional complex analytic subvariety of $\mathbb{P}^n \setminus \gamma$ ([6], Theorem 12.9).

Remark 3.8. The condition given in Lemma 3.4 - that V_γ is harmonic on Σ - is sufficient to show that ϕ is meromorphic in \mathbb{D}^* , but may not be necessary. Note that in the proof of Lemma 3.4 we did not use the full strength of the assumption $\Sigma \subset \hat{\gamma}$, which is equivalent to (1.5). Instead, we only used (1.5) for the particular polynomial $p(z, w) = w$, and so it is possible that the conclusion of Theorem 3.5 obtains in cases where V_γ is not harmonic.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, COLLEGE OF THE HOLY CROSS, WORCESTER, MA 01610, U.S.A.

E-mail address: `anderson@mathcs.holycross.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NORTH CAROLINA, CHAPEL HILL, NC 27599, U.S.A.

E-mail address: `cima@email.unc.edu`

DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, IN, 47405, U.S.A.

E-mail address: `nlevenbe@indiana.edu`

DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ LAVAL, QUÉBEC (QC), CANADA G1V 0A6

E-mail address: `ransford@mat.ulaval.ca`