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# Portfolio Insurance and model uncertainty*

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**Abstract.** Some real-world insurance products contain a minimum-wealth or an income-stream guarantee, both of which have to be met irrespective of capital market conditions. Therefore, sellers of such products are well advised to pursue a portfolio strategy that can meet these minimum investment goals if they want to avoid additional cash payments. Portfolio Insurance seems to be the solution to this portfolio problem.

However, this paper shows that Portfolio Insurance cannot protect minimum investment goals because its strategies are fitted to a particular form of market risk. Decision makers do not know for sure (with probability one) what the true form of market risk is (model uncertainty); thus model uncertainty makes Portfolio Insurance fail.

**Key words:** Model uncertainty, Portfolio selection, Minimum-wealth or income-stream guarantee, Portfolio Insurance.

**JEL Classification Numbers:** G11.

## 1 Preliminaries

### *1.1 Introduction to the problem*

In real-world financial markets, there are insurance products that offer their buyers a minimum-wealth or an income-stream guarantee. Two prominent examples are money-back guarantees at maturity (guaranteed minimum wealth), known in

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Germany as "Riester products", and life annuities (guaranteed income streams). By definition, sellers of these products are obliged to meet their guarantees irrespective of capital market conditions. Therefore, sellers are well advised to pursue a portfolio strategy that is able to fulfill these minimum investment goals if they want to avoid additional cash payments.

Portfolio Insurance seems to offer a solution to this portfolio problem: if wealth approaches a level that endangers minimum-wealth or income-stream guarantees, Portfolio Insurance reduces the amount invested in risky assets to (almost) zero and invests (nearly) all funds in the riskless asset. In other circumstances, Portfolio Insurance engages in a more aggressive investment style.

It is the objective of this paper to show that Portfolio Insurance cannot protect minimum investment goals. This is because in the real world, decision makers do not know for sure (with probability one) what the true form of market risk is; in other words, decision makers are subject to model uncertainty. But Portfolio Insurance strategies are fitted to a particular form of market risk, and thus they overlook model uncertainty.

As proof of this statement, for several (isolated) forms of market risk (Step 1) portfolio strategies are calculated that are designed to defend a guaranteed minimum wealth (Option Based Portfolio Insurance) and a guaranteed income stream ((Constant) Proportion Portfolio Insurance). In Step 2, these strategies are examined further to identify which of them are able to meet minimum investment goals even if the decision maker is subject to model uncertainty.

Option Based Portfolio Insurance calls for duplication of the put option implied by the minimum-wealth guarantee. Although a duplication portfolio can be adapted to cope with several sources of market risk, it is fitted to a concrete form of market risk and can handle this form of market risk only. For that reason, when there is model uncertainty, i.e., several forms of market risk are possible, there is only one trivial strategy able to defend guaranteed minimum wealth: invest the present value of guaranteed wealth in the riskless asset.

A more sophisticated strategy is possible with (Constant) Proportion Portfolio Insurance. Assume an ex ante unknown number of stock market crashes which all have a minimum jump amplitude $\varphi_{\mathrm{extr}} > -1$. Then (Constant) Proportion Portfolio Insurance can be based on this worst-case scenario and minimum investment goals can be defended even when there is model uncertainty with a portfolio strategy that does more than simply invest the present value of the income stream in the riskless asset.

With respect to definition of model uncertainty and focusing on worst-case scenarios, this analysis uses the methodology of Anderson, Hansen, and Sargent (2000). However, their line of argumentation must be modified to work with minimum investment goals. Anderson, Hansen, and Sargent (2000) argue with an explicit preference for model similarity, so-called robustness, but do not specifically deal with meeting minimum investment goals.

With respect to its results, this analysis supplements the Portfolio Insurance literature by showing how to adapt Option Based Portfolio Insurance to an environment with several sources of market risk by using roll-over Option Based Portfolio Insurance. Thereby, Rubinstein (1985) is generalized, a hypothesis formulated in

Geman (1992) corrected, and an idea in Leland (1992) proven. In addition, (Constant) Proportion Portfolio Insurance strategies are modified to work in a stochastic volatility environment, thus extending Black and Jones (1987).

This article is organized as follows. The remainder of Section 1 gives some definitions and outlines the framework of the model used. Section 2 deals with minimum-wealth guarantees under several forms of market risk, Section 3 conducts the same analysis for guaranteed income streams. Section 4 concludes the paper; an Appendix follows.

### *1.2 Definition and particularization of model uncertainty*

In the real world, decision makers do not know for sure (with probability one) what the true form of market risk is; in other words, they are subject to model uncertainty. Following Anderson, Hansen, and Sargent (2000, p. 6), model uncertainty is itself a form of uncertainty. Thus, the total uncertainty confronting a decision maker is a combination of model uncertainty and market risk. Technically,[1] model uncertainty results in a term $g(t)$, which, when added to the increment of market risk, describes the increment of total uncertainty, i.e., increment of total risk $= g(t)+$ increment of market risk. Yet, the decision maker does not know the distribution of $g(t)$. This is why the term "model uncertainty" is used instead of "model risk". To summarize, model uncertainty can be defined as an unforeseeable shift in the shape (and not just the parameters) of market risk. This shift, however, has to be close to the assumed model of market risk as Anderson, Hansen, and Sargent (2000, p. 9) and Cagetti, Hansen, Sargent, and Williams (2002, p. 374) point out; otherwise the assumed model of market risk would be rejected empirically.

So far, this notion of model uncertainty is too abstract to be useful in devising concrete portfolio strategies. Therefore, I particularize the shift of the shape of market risk as a shift of the underlying stochastic process. This shift might involve homogenous model uncertainty (intra-model-shift, i.e., from diffusion to diffusion or jump/diffusion to jump/diffusion) or heterogenous model uncertainty (inter-model-shift, i.e., from diffusion to jump/diffusion). The integration of heterogenous model uncertainty denotes a difference to existing particularizations of model uncertainty in continuous time, which all (see Maenhout, 2001, p. 13; Trojani and Vanini, 2001, p. 7; Uppal and Wang, 2002, p. 10) work within homogenous model uncertainty. – The problem with heterogenous model uncertainty is that it seems to violate Cagetti, Hansen, Sargent, and Willliam's (2002) empirical detection criterion. Two arguments can be made that indicate this conclusion might be too hasty. First, the inclusion of jumps does not necessarily change stocks' distribution. For example, if 1 + jump amplitude is logarithmic-normally distributed as in Merton (1976), investor's wealth has the same type of distribution as in the case of a (pure) geometric Brownian motion. Second, Bates (2000, p. 182) points out that there are two models vying to explain negative skewness in stock returns after the

[1] Alternatively, this statement can be formulated as follows: the distribution function of market risk is subject to a multiplicative distortion – a Radon-Nikodym derivative unequal to one (see Anderson, Hansen, and Sargent, 2000, p. 22; Uppal and Wang, 2002, p. 4).

’87 crash: stochastic volatility and jumps. Since crashes and stochastic volatility describe the same phenomenon, they are rather difficult to discriminate empirically.

To further illustrate model uncertainty, it is useful to distinguish model uncertainty from estimation risk. Under estimation risk, the parameters of a stochastic process change stochastically over time like, e.g., in Merton’s (1973) stochastic volatility model. However, the presence of two such types of risk does not necessarily change the shape of market risk. This can again be seen from Merton’s (1973) stochastic volatility model: different estimates for parameters of the stochastic process clearly mean that estimation risk is present, but the stochastic process itself remains a process under stochastic volatility, i.e., does not change its shape to, e.g., a jump/diffusion process.

## *1.3 Framework of the analysis*

### 1.3.1 General assumptions

To analyze the consequences of model uncertainty on portfolio selection under minimum investment goals, it is a good idea to exclude all other circumstances that might endanger the minimum investment goals. Therefore, I fall back on the standard assumptions of continuous-time finance and minimum investment goal literature:[2]

1. Capital markets are free of arbitrage and perfect, i.e., short selling constraints or transaction costs do not interfere with meeting minimum investment goals.
2. Trading happens in continuous time, i.e., a potential lack of transaction speed does not hurt minimum investment goals.
3. There is a riskless asset in the market with dynamics
   $$dP(t) = rP(t)dt \tag{1}$$
   where $P(t)$ denotes the price of the riskless asset at time $t$, $r$ its interest rate per unit time, and $dt$ a time period of infinitesimal length.
4. There is one risky asset (stock market index), i.e., there is no basis risk that endangers minimum investment goals.

In addition to these four standard assumptions is one more added to reflect the particularization of model uncertainty developed in this paper:

5. The price process of the stock index is subject to model uncertainty since it can follow a geometric Brownian motion (one source of market risk), a stochastic volatility model similar to that of Merton (1973) (two homogenous sources of risk due to estimation risk), or a combined jump/diffusion model (infinitesimal and non-infinitesimal price changes and thus two heterogenous sources of market risk); decision makers do not know the probability with which each type of price process will occur.

To begin with the reference model (or with $g(t) = 0$ in Anderson, Hansen, and Sargent’s (2000) terms), a geometric Brownian motion can be formalized as

$$dS(t) = \alpha\, S(t)\; d\,t + \sigma\; S(t)\, d\; z(t) \tag{2}$$

[2] See, e.g., Merton (1973), as well as Black and Perold (1992).

where $S(t)$ denotes the price of the stock index at time $t$, $dS(t)$ its infinitesimal price change, $\alpha$ its per unit time mean, $\sigma$ its per unit time standard deviation, and $dz(t)$ the increment of a Wiener process.

Under a combined jump/diffusion process the index evolves according to (or with $g(t) =$ the stochastic differential equation for a Poisson-driven process; see Tapiero, 1998, p. 255)

$$dS(t) = \alpha\, S(t)\; d\, t + \sigma\; S(t)\, d\; z(t) \tag{3}$$

with probability $1 - \lambda dt$ (diffusion case)

$$\Delta S(t) = S(t^{-})(1 + \varphi(t)) - S(t^{-})$$

with probability $\lambda dt$ (jump case)

with $\Delta$ signifying a large jump-induced, i.e., non-infinitesimal price change, $\lambda dt$ denoting the probability[3] that a jump occurs between time $t$ and $t+dt$, i.e., the exact number and dates of jumps are ex ante unknown, $\varphi(t)$ depicting the stochastic jump amplitude, and $t^{-}$ meaning a point in time immediately before time $t$; moreover, jump and diffusion risk are uncorrelated.

Finally, a stochastic volatility model similar to that of Merton (1973, p. 873) reads

$$d\,\sigma(t) = \alpha_\sigma\, \sigma(t)\, d\,t + \sigma_\sigma\, \sigma(t)\, d\, z_\sigma(t) \tag{4}$$

with $\alpha_\sigma$ and $\sigma_\sigma$ denoting per unit time mean and standard deviation of $\frac{d\,\sigma(t)}{\sigma(t)}$, the relative change in volatility, and $d\, z_\sigma(t)$ the increment of a Wiener process that is correlated with d z(t).

Consequently, one obtains the following dynamics of the index under stochastic volatility (or with $g(t) = S(t)\;(\sigma(t) - \sigma)\; d\, z(t)$):

$$dS(t) = \alpha\, S(t)\; d\, t + \sigma(t)\; S(t)\, d\; z(t) \tag{5}$$

### 1.3.2 Minimum investment goals

In this paper two legally different minimum investment goals are considered: minimum-wealth guarantees and guaranteed income streams. Are these guarantees different from an economic point of view as well and will thus have to be analyzed separately?

Although wealth can be transformed into an annuity, a minimum-wealth guarantee is not the same as an income-stream guarantee because this annuity turned wealth might not last over the uncertain lifespan of an individual, whereas a guaranteed income stream covers the entire lifespan of an individual. – Likewise, just

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[3] As $\lambda dt$ just states the probability that a jump occurs assuming that a combined jump/diffusion model is the correct specification of the stock price model, this specification of a jump probability does not contradict the statement that decision makers are unable to specify probabilities for a certain stock price process under model uncertainty. Thus, it should not be confused with an assertion on the probability that the combined jump/diffusion model itself is valid.

because it is possible to defend an income stream and an income stream can be accumulated to a wealth level, this fact does not mean the same is true of a minimum-wealth guarantee. The income stream simply needs some re-investment in order to be transferred into wealth; the search for a non-trivial portfolio strategy that guarantees minimum wealth starts anew. For these reasons, both types of guarantees must be examined.

#### 1.3.3 Methodology

The literature (see in particular Anderson, Hansen, and Sargent (2000)) contains a general solution principle for portfolio strategies even under model uncertainty: follow that portfolio strategy that performs best in a reasonable worst-case capital market scenario. Therefore, I will develop portfolio strategies under minimum investment goals along the lines of Anderson, Hansen, and Sargent (2000).

However, the details of their argumentation must be modified to work under minimum investment goals. Anderson, Hansen, and Sargent (2000) take model uncertainty into account by adding an explicit preference for model similarity, so-called robustness, to the objective function of the decision problem. Minimum investment goals constitute an extreme preference for robustness in that they show zero tolerance (penalty term of $-\infty$) below the minimum investment goal and full tolerance (no penalty term) above, i.e., for these wealth regions a preference for robustness is unnecessary. Hence, minimum investment goals are better integrated into the decision problem by adding a strict minimum-wealth or an income-stream constraint to the decision problem instead of modifying the objective function itself. – An investment strategy based on a normal preference for robustness is undoubtedly more conservative than one without any preference for robustness, but might still be too aggressive if there are minimum investment goals that must be met.

Using this modification of the decision problem, I identify that portfolio strategy under minimum investment goals that performs best in a reasonable worst-case capital market scenario (portfolio strategy in the spirit of Anderson, Hansen, and Sargent (2000)) as follows: I calculate in a first step the optimum portfolio strategy under minimum investment goals for each of the (isolated) three price processes – as if there was no model uncertainty. In a second step, I confront the portfolio strategies derived from the (isolated) prices processes with model uncertainty, i.e., I determine from the set of isolated strategies those that are able to defend minimum investment goals in the worst-case scenario that model uncertainty incorporates.

## 2 Minimum-wealth guarantees (Option Based Portfolio Insurance)

### *2.1 Decision problem and general structure of portfolio strategies that solve it*

A minimum-wealth guarantee promises its buyer (for the sake of simplicity, I assume that there is just one buyer) at the beginning of his retirement at $T$ to pay back his investment if the world is in bad states, or to capitalize on a positive wealth development if the world is in good states; formally:

$$\max\{W(T), K\} \tag{6}$$

where $W(T)$ denotes total wealth of the seller's portfolio, and $K$ is the guaranteed minimum wealth (the so-called floor).

The seller of this guarantee is characterized with the help of a fairly stylized model:[4] a risk averse decision maker who wants to create a payoff structure over time. As payoffs are comprised of money that is no longer available for investment purposes, the payoff structure can be particularized as a consumption stream. To achieve his goals, the seller determines his consumption $(C(t), t \geq 0)$ and portfolio strategy $(w(t), t \geq 0$, where $w(t)$ denotes the weight invested in the risky index).

Because the seller is obliged to meet the guarantee irrespective of capital market conditions, the minimum-wealth guarantee (eq. (6)) enters the seller's decision problem in form of a minimum terminal wealth constraint – assuming that the seller does not want to put additional funds into his company just to reach the required wealth level and that bankruptcy is excluded.

Formalizing these verbal descriptions, the decision problem of the seller of the minimum-wealth guarantee reads

$$\underset{C(t),\, w(t)}{Max} \; E_0 \left\{ \int_0^\infty e^{-\rho\, t} \, \frac{C(t)^\gamma}{\gamma} \, dt \right\} \tag{7}$$

s.t.: $W(T) \geq K$
wealth dynamics according to eqs. (2), (3), or (5)

where $\rho$ denotes the time preference rate of the seller, and $1 - \gamma$ his (constant) relative risk aversion.

According to Grossman and Zhou (1996, p. 1379), the general structure of the portfolio strategy $w(t)$ that solves the decision problem (7) consists of two parts: the (usual) expected utility maximizing portfolio and a correction portfolio that takes the constraint into account. The reasons for this portfolio structure are as follows. The constraint is identical with an investment in the index and the purchase of a (implied) put option $F^{\mathrm{impl}}$ with strike price $K$ and maturity $T$ because it can be written as

$$\max\{W(T), K\} = W(T) + \underbrace{\max\{K - W(T), 0\}}_{F^{\mathrm{impl}}} \tag{8}$$

Since real-world options do not necessarily offer the desired strike price or the adequate maturity, the implied put's payoff at time $T$ has to be duplicated by a dynamic portfolio strategy. This means that the solution to the decision problem (7) involves as an integral part a well-known portfolio concept, namely, Option Based Portfolio Insurance.

In other words, the task of finding a portfolio strategy, w(t), that solves the decision problem under minimum terminal wealth constraints boils down to the question of whether Option Based Portfolio Insurance can be implemented successfully. Therefore, the expected utility maximizing portfolio, which stems from the objective function of the decision problem, will not be examined further.

[4] Further institutional details do not offer additional insights regarding the interplay between minimum-wealth guarantees and model uncertainty, as will shortly become clear.

### *2.2 Option Based Portfolio Insurance under isolated price processes*

So far, just the basic structure of portfolio strategies that solve the decision problem (7) has been identified. In this section, Option Based Portfolio Insurance is particularized for each of the (isolated) three price processes as if there was no model uncertainty (Step 1 of the analysis).

#### 2.2.1 "Classical" Option Based Portfolio Insurance under different volatility scenarios

Leland (1980) and Rubinstein and Leland (1981) have demonstrated under geometric Brownian motion that the price of the implied put option at every time $t$, and thus also at maturity $T$, can be duplicated by trading in the index and the riskless asset because implied option and index depend linearly on the same single source of market risk. The duplication portfolio consists of investing the number $F_S^{\text{impl}}$ in stocks (delta of the implied put option calculated with the help of the Black/Scholes formula) and the amount $F^{\text{impl}} - F_S^{\text{impl}}\, S$ in the riskless asset.

However, under combined jump/diffusion processes (see Merton, 1976) and stochastic volatility (see Johnson and Shanno, 1987; Hull and White, 1987), duplication fails. There are two sources of market risk, but only one risky asset. Therefore, decision makers can eliminate one source of market risk only, leaving them fully exposed with respect to the other source of market risk.

#### 2.2.2 Roll-over Option Based Portfolio Insurance under geometric Brownian motion

Unlike "classical" Option Based Portfolio Insurance, roll-over Option Based Portfolio Insurance employs traded options on the same underlying, but with shorter maturities, to duplicate the implied put. After the first set of options matures, roll-over Option Based Portfolio Insurance switches to the next set of options and so forth, until the implied put becomes due at time $T$.

To put this verbal description into practice, consider the dynamics of every (implied or traded) derivative $F^j$ under geometric Brownian motion:

$$d\,F^j(t) = F_t^j\,d\,t + F_S^j\,\alpha\,S(t)\,d\,t + F_S^j\,\sigma\,S(t)\,d\,z(t) + \frac{1}{2}F_{SS}^j\,\sigma^2\,S^2(t)\,d\,t \tag{9}$$

Complete replication of the implied put $F^{\text{impl}}$ requires that the duplication portfolio must replicate the stochastic component $F_S^j\,\sigma\,S(t)\,d\,z(t)$. Since a geometric Brownian motion contains only one source of market risk, one (traded) option $F^i$ is needed (see Rubinstein, 1985, p. 46):

$$N^i(t)\cdot F_S^i\,\sigma\;S(t)\,d\,z(t) = F_S^{\text{impl}}\,\sigma\,S(t)\,d\,z(t) \tag{10}$$

Therefore, the duplication portfolio consists of the number

$$N^i(t) = \frac{F_S^{\text{impl}}}{F_S^i} \tag{11}$$

### 2.2.3 Roll-over Option Based Portfolio Insurance under combined jump and diffusion risk

The price dynamics of every (implied or traded) derivative $F^j$ under combined jump/diffusion risk follows

$$d\,F^j(t) = F_t^j\,d\,t + F_S^j\,\alpha\,S(t)\,d\,t + F_S^j\,\sigma\,S(t)\,d\,z(t) + \frac{1}{2}F_{SS}^j\,\sigma^2\,S^2(t)\,d\,t \qquad (12)$$

with probability $1-\lambda dt$ (diffusion case)

$$F^j\left(S(t^-)\,(1+\varphi(t))\right) - F^j(S(t^-))$$

with probability $\lambda dt$ (jump case)

Since there are two sources of market risk (jump and diffusion risk), two (traded) options are required to duplicate the implied put option:

- diffusion risk

$$N^i(t)\cdot F_S^i\,\sigma\;S(t)\,d\,z(t) + N^k(t)\cdot F_S^k\,\sigma\;S(t)\,d\,z(t) = F_S^{\text{impl}}\,\sigma\,S(t)\,d\,z(t) \qquad (13a)$$

- jump risk

$$N^i(t)\varDelta F^i + N^k(t)\varDelta F^k = \varDelta F^{\text{impl}} \qquad (13b)$$

It is at this point that the advantage of roll-over Option Based Portfolio Insurance, compared to "classical" Option Based Portfolio Insurance, becomes obvious. Following Ross (1976), increasing the number of derivatives makes the market "more complete", i.e., allows for more sources of market risk to be duplicated, although the number of spot market instruments does not change. Consequently, the desired duplication portfolio under combined jump/diffusion risk, first, exists and, second, reads (from eqs. (13a) and (13b))

$$N^i(t) = \frac{F_S^{\text{impl}}\,\varDelta\,F^k - F_S^k\,\varDelta\,F^{\text{impl}}}{F_S^i\,\varDelta\,F^k - F_S^k\,\varDelta\,F^i} \qquad (14a)$$

$$N^k(t) = \frac{F_S^i\,\varDelta\,F^{\text{impl}} - F_S^{\text{impl}}\,\varDelta\,F^i}{F_S^i\,\varDelta\,F^k - F_S^k\,\varDelta\,F^i} \qquad (14b)$$

Equations (14a) and (14b) correct a common statement (see, e.g., Geman, 1992, p. 187; Zhou and Kavee, 1988, p. 54), namely, that Option Based Portfolio Insurance does not work in a jump/diffusion environment. These equations demonstrate that this statement is true for "classical" Portfolio Insurance only, not for roll-over Option Based Portfolio Insurance strategies.

Equations (14a) and (14b) also elaborate on Leland's (1992, p. 155) idea of extending Option Based Portfolio Insurance to an environment of combined jump/diffusion risk, and generalize a result of Rubinstein (1985, p. 49), who uses just one option $F^i$ to duplicate the implied put. His result, however, only holds if and only if both $F_S = a\cdot F_S^i$ (with $a$ an arbitrary constant) and $\varDelta F = a\cdot \varDelta F_S^i$ are true.

2.2.4 Roll-over Option Based Portfolio Insurance under stochastic volatility

The price of every derivative $F^j$ in a market under stochastic volatility has the following dynamics:

$$
\begin{aligned}
d\,F^j(t) = {} & F_t^j dt + F_S^j \alpha S(t)dt + F_S^j \sigma(t)S(t)dz(t) + \frac{1}{2}F_{SS}^j \sigma^2(t)S^2(t)dt \\
& + F_\sigma^j \alpha_\sigma \sigma(t)dt + F_\sigma^j \sigma_\sigma \sigma(t)dz_\sigma + \frac{1}{2}F_{\sigma\sigma}^j \sigma_\sigma^2 \sigma^2(t)dt \\
& + F_{\sigma S}^j cov_{\sigma S}\sigma(t)S(t)dt
\end{aligned}
\qquad (16)
$$

where $cov_{\sigma S}$ denotes the covariance per unit time between $\sigma(t)$ and $S(t)$.

To eliminate both sources of market risk (diffusion and volatility risk), the duplication portfolio of the implied put consists of two options,

- diffusion risk

$$
N^i(t) \cdot F_S^i\,\sigma(t)\;S(t)\,d\,z(t) + N^k(t) \cdot F_S^k\,\sigma(t)\;S(t)\,d\,z(t) \qquad \text{(16a)}
$$
$$
= F_S^{\text{impl}}\,\sigma(t)\,S(t)\,d\,z(t)
$$

- stochastic volatility risk

$$
N^i(t) \cdot F_\sigma^i\,\sigma_\sigma\;\sigma(t)\,d\,z_\sigma(t) + N^k(t) \cdot F_\sigma^k\,\sigma_\sigma\;\sigma(t)\,d\,z_\sigma(t) \qquad \text{(16b)}
$$
$$
= F_\sigma^{\text{impl}}\,\sigma_\sigma\,\sigma(t)\,d\,z_\sigma(t)
$$

which numbers can be obtained by solving eqs. (16a) and (16b):

$$
N^i(t) = \frac{F_S^{\text{impl}}\,F_\sigma^k - F_S^k\,F_\sigma^{\text{impl}}}{F_S^i\,F_\sigma^k - F_S^k\,F_\sigma^i} \qquad \text{(17a)}
$$

$$
N^k(t) = \frac{F_S^i\,F_\sigma^{\text{impl}} - F_S^{\text{impl}}\,F_\sigma^i}{F_S^i\,F_\sigma^k - F_S^k\,F_\sigma^i} \qquad \text{(17b)}
$$

Again, eqs. (17b) and (17b) are a practical illustration of Leland's (1992, p. 155) idea of adapting Option Based Portfolio Insurance to stochastic volatility; the equations correct Geman (1992, p. 187) in that only "classical" Option Based Portfolio Insurance does not work under stochastic volatility; and generalize Rubinstein (1985, p. 49), who uses just one option $F^i$ to achieve duplication, meaning that he must assume that $F_S = a \cdot F_S^i$ implies $F_\sigma = a \cdot F_\sigma^i$.

### *2.3 Consequences of model uncertainty to Option Based Portfolio Insurance*

Having calculated Option Based Portfolio Insurance strategies for each (isolated) price process, the second step of the analysis becomes possible: the confrontation of (isolated) Option Based Portfolio Insurance strategies with model uncertainty, i.e., the determination of those portfolio strategies from the set of isolated strategies that are able to defend minimum investment goals in the worst-case scenario that model uncertainty incorporates.

Analysis of "classical" Option Based Portfolio Insurance under several price process specifications, and thus model uncertainty, has shown that "classical" Option Based Portfolio Insurance can duplicate the desired option only under geometric Brownian motion. Or, more generally, "classical" Option Based Portfolio

Insurance has to operate in an environment characterized by just one source of risk. Whenever there is a second source of uncertainty (combined jump/diffusion processes, estimation risk, or model uncertainty), “classical” Option Based Portfolio Insurance cannot assure guaranteed minimum wealth. In other words, “classical” Option Based Portfolio Insurance is completely unable to cope with model uncertainty.

In comparison, roll-over Option Based Portfolio Insurance is able to duplicate options under all three classes of (isolated) price processes considered. However, each duplication portfolio requires different numbers $N(t)$ for different price processes. Under geometric Brownian motion, there is just one option involved (eq. (11)); however, jumps need two options to finish duplication and the duplication portfolio under jumps cannot coincide with that under geometric Brownian motion (eqs. (14a) and (14b) on the one hand and eq. (11) on the other hand diverge). Basically, the same argument holds for stochastic volatility, wherefore the duplication portfolios under geometric Brownian motion (eq. (11)) and stochastic volatility (eqs. (17a) and (17b)) are different. Moreover, the number of options held under stochastic volatility (eqs. (17a) and (17b)) and jumps (eqs. (14a) and (14b)) are unequal; the reaction of the option price to a change of volatility ($F_\sigma =$ linear change of the option price) does not coincide with its price movement due to a stock price jump ($\Delta F =$ non-linear change of the option price).

One could argue that increasing the number of options used for duplication purposes might circumvent these problems. That is, to cope with model uncertainty, the following strategy could be applied: use three options so as to be able to manage three types of market risks – normal risk, jump risk, and stochastic volatility risk. The problem with this argument is that it assumes that all three sources of market risk are present at the same time. Thus, it does not constitute the worst-case scenario for situations when all three risks may or may not be present at the same time. In fact, under minimum-wealth guarantees there is no longer “the” worst-case scenario in the spirit of Anderson, Hansen, and Sargent (2000). Instead, there is just one price process that allows for duplication.

In summary, roll-over Option Based Portfolio Insurance can solve the duplication problem for all (isolated) price processes and hence is an improvement over “classical” Option Based Portfolio Insurance. However, model uncertainty makes roll-over Option Based Portfolio Insurance unable to adequately deal with guaranteed minimum wealth. – This observation delivers an alternative proof of a statement made by Avramov (2001, p. 21) that model uncertainty seemed to be more important than estimation risk.

These results have one remarkable consequence for Option Based Portfolio Insurance (portfolio strategies designed to defend minimum-wealth guarantees) under model uncertainty. Since duplication is fitted to one particular price process, there is only one trivial strategy able to defend guaranteed minimum wealth: invest the present value of the guaranteed wealth in the riskless asset.[5] This outcome holds irrespective of whether there is homogenous or heterogonous model uncer-

[5] Technically speaking, this strategy equals a super-replicating strategy in its extreme, i.e., most expensive, form. It assumes, however, that there is a (globally) riskless asset. If there is just an asset with a local riskfree rate, as is the case in the real world, even the trivial strategy

tainty and by construction does not depend on sellers' coefficients of risk aversion. Nevertheless it does not imply that total wealth must be invested in the riskless asset because total investment consists of the sum of the utility maximizing portfolio and the correction portfolio. Instead, the correction portfolio, i.e., the portfolio that deals with the minimum terminal wealth constraint, is identical with a riskless investment.

## 3 Guaranteed income stream ((Constant) Proportion Portfolio Insurance)

### *3.1 Decision problem and general structure of portfolio strategies that solve it*

By selling a life annuity, the seller guarantees that the buyer can withdraw every period the amount $K\,dt$ as long as he lives. As opposed to minimum-wealth guarantees, $K$ (the so-called floor) now denotes withdrawal per unit time, i.e., a rate, and not an amount of money.

The seller of this guarantee is the same stylized decision maker who sold the minimum-wealth guarantee (see Sect. 2.1) and he faces the following formalized decision problem:

$$\underset{C(t),\,w(t)}{Max}\; E_0\left\{\int_0^{\infty} e^{-\rho\,t}\,\frac{C(t)^{\gamma}}{\gamma}\,dt\right\} \tag{18}$$

s.t.: $C(t) \geq C_{\min} = K$
wealth dynamics according to eq. (2), (3), or (5)

Two remarks might prove useful to distinguish the decision problem under income-stream guarantees (18) from that under minimum-wealth guarantees (7). First, assuming that a seller of an income-stream guarantee has an infinite planning horizon simplifies the setup because it avoids the problem of specifying a date at which the buyer of the annuity will die, thus ending the obligation to pay the annuity. Under minimum-wealth guarantees, specification of the buyer's planning horizon $T$ is an integral part of the problem; hence it cannot be circumvented. Second, to simplify calculations, the payment for the annuity is integrated into the seller's consumption constraint because both consumption and payment for the annuity are withdrawals. There is one disadvantage to this procedure: it assumes that the seller encounters a positive utility by meeting the guarantee. However, since this utility is the (absolute) minimum utility, the seller's incentive to achieve higher utility levels by pursuing an adequate portfolio and consumption strategy is not hampered by considering this (combined) consumption constraint.

According to Black and Jones (1987), the portfolio strategy, $w(t)$, that solves the decision problem (18) is distinguished by the fact that the seller of the guarantee invests under some circumstances a multiple of total wealth in the risky asset, under other circumstances a multiple of $W(t) - \frac{K}{r}$; in other words, he pursues a so-called (Constant) Proportion Portfolio Insurance strategy.

of investing the present value of guaranteed wealth in the (local) riskless asset might miss this minimum investment goal.

This strategy can be illustrated as follows. If at the beginning of each period (of infinitesimal length) the amount $\frac{K}{r}$ can be invested in the riskless asset, the income-stream guarantee will be defended. This means the investment in the risky asset must equal zero whenever $W(t)$ reaches $\frac{K}{r}$. Therefore, a portfolio strategy that continuously re-balances a portfolio that invests a multiple of $W(t) - \frac{K}{r}$ in the risky asset ((Constant) Proportion Portfolio Insurance) will indeed be able to defend the income stream guarantee.

### *3.2 (Constant) Proportion Portfolio Insurance under isolated price processes*

So far, just the basic structure of portfolio strategies that solve the decision problem (18) has been identified; the circumstances under which (Constant) Proportion Portfolio Insurance is applied and the calculation of the multiplier $m$ are yet to be discussed. Both will be covered in this subsection using (isolated) price process, i.e., as if there was no model uncertainty (Step 1 of the analysis).

#### 3.2.1 (Constant) Proportion Portfolio Insurance under geometric Brownian motion

Black and Perold (1992, pp. 420, 425) particularize (Constant) Proportion Portfolio Insurance under geometric Brownian motion as follows:

$$w(t) \cdot W(t) = \begin{cases} \dfrac{1}{1-\gamma} \cdot \dfrac{\alpha - r}{\sigma^2} \cdot W(t) & \text{for } W(t) \geq W^+ \\ \underbrace{\dfrac{1}{1-\gamma'} \cdot \dfrac{\alpha - r}{\sigma^2}}_{m} \cdot \left[ W(t) - \dfrac{K}{r} \right] & \text{for } W(t) < W^+ \end{cases} \tag{19}$$

with $W^+ = \frac{K}{r} \cdot \frac{\gamma - 1}{\gamma - \gamma'}$.

According to eq. (19), the optimum portfolio weight of the risky asset calls for a division of the portfolio strategy into two parts: above wealth level $W^+$, decision makers follow the well-known portfolio strategy under geometric Brownian motion (see Merton, 1969). Below the critical wealth level $W^+$, (Constant) Proportion Portfolio Insurance is used. Obviously, the circumstances under which a decision maker follows (Constant) Proportion Portfolio Insurance under geometric Brownian motion read: wealth does not exceed level $W^+$. – This outcome is rather intuitive since "high" wealth levels are able to bear "high" losses without putting pressure on the guarantee.

The multiplier $m$ depends on mean and variance of the index as well as on the risk aversion parameter $\gamma'$.[6] $\gamma'$ is a function of the (model-exogenously specified) risk aversion parameter $\gamma$ of the decision maker and thus investor-specific. However, it deviates from $\gamma$ because it is determined model-endogenously to finetune the portfolio strategy if wealth is low and the withdrawal constraint might soon become binding.

[6] $\gamma'$ can be calculated from eq. (A.5a). This section, however, focuses on the fundamentals of the optimum solution and not on its details. Therefore, the explicit calculation of $\gamma'$ is omitted.

### 3.2.2 (Constant) Proportion Portfolio Insurance under combined jump/diffusion risk

Under combined jump/diffusion processes, the optimum investment in the risky asset reads (see Sect. A of the Appendix)

$$w(t) \cdot W(t) = \underbrace{-\frac{1}{\varphi_{\text{extr}}}}_{m} \cdot \left( W(t^-) - \frac{K}{r} \right) \tag{20}$$

As opposed to the situation under geometric Brownian motion, the optimum portfolio strategy under combined jump/diffusion processes (eq. (20)) does not distinguish between a critical and an uncritical region; thus it does not switch between more or less conservative portfolio strategies. This is easily explained by the fact that there are two completely different sources of market risk and it is impossible to define a critical wealth level that works under both types of risk. Jumps lead to potentially higher losses and thus call for higher wealth levels than diffusion processes. Ex ante, however, the loss in each period induced by the jump is unknown because the jump amplitude is defined as a percentage of future wealth. – Consequently, under combined jump/diffusion risk, the decision maker will never deviate from (Constant) Proportion Portfolio Insurance.

The multiplier $m$ is specified as the minimum jump amplitude ($\varphi_{\text{extr}}$).[7] Choosing $W(t^-) - \frac{K}{r}$ as the basis of the risky investment is not conservative enough – the large price movements caused by jumps can nevertheless violate the floor. Therefore, the multiplier must be restricted from above in each period. This fact makes the multiplier independent of investors' coefficients of risk aversion.

By adapting (Constant) Proportion Portfolio Insurance to jumps, eq. (20) corrects the statement by Black and Jones (1987, p. 49) that the performance of (Constant) Proportion Portfolio Insurance is inappropriate under jumps. (Constant) Proportion Portfolio Insurance can be modified to work under combined jump/diffusion risk.

### 3.2.3 (Constant) Proportion Portfolio Insurance under stochastic volatility

The optimum investment in the risky asset under stochastic volatility can be determined as follows[8]

$$w(t) \cdot W(t) = \underbrace{\left[ \frac{1}{1-\delta} \cdot \frac{\alpha - r}{\sigma^2(t)} + \frac{1}{1-\delta} \frac{B_\sigma(\sigma(t))}{B(\sigma(t))} \cdot \frac{cov_{\sigma W}(t)}{\sigma^2(t)} \right]}_{m} \left[ W(t) - \frac{K}{r} \right] \tag{21}$$

Similar to the situation under combined jump/diffusion risk, the optimum portfolio strategy under stochastic volatility (eq. (21)) does not distinguish between a critical

[7] Concentrating on jumps with negative amplitude implies that the risky asset is not sold short in the optimum. Otherwise, the "minimum jump amplitude" would be the maximum price increase of the risky asset. Yet, selling the risky asset short would signify that the only risky asset in the market would be worse than the riskless asset. For that reason, this case is excluded from further analysis.

[8] A proof can be found in Sect. B of the Appendix. – Similar to $\gamma'$, $\delta$ can be obtained from eq. (A.5a).

and an uncritical region, i.e., does not switch between more or less conservative strategies. Since there are two completely different sources of market risk, it is impossible to define a critical wealth level that works under both types of risk. A high volatility leads to potentially higher losses and thus calls for higher wealth levels than a low volatility. Ex ante, however, the volatility level is unknown and thus the critical wealth level. Consequently, under stochastic volatility (estimation risk), a decision maker will never deviate from (Constant) Proportion Portfolio Insurance.

The multiplier $m$ depends on mean and variance of the index, terms evaluating the stochastic volatility risk, and on the risk aversion parameter $\delta$. $\delta$ is a function of the (model-exogenously specified) risk aversion parameter $\gamma$ of the decision maker and thus investor-specific. However, it deviates from $\gamma$ because it is determined model-endogenously to finetune the portfolio strategy if wealth is low and the withdrawal constraint might soon become binding.

In summary, eq. (21) adapts (Constant) Proportion Portfolio Insurance strategies – to my knowledge for the first time in literature – to work in a stochastic volatility environment. The equation also makes obvious that the term Constant Proportion Portfolio Insurance is no longer justified. The multiplier $m$ depends on parameters of the stochastic volatility risk; as such, it varies with time. The term time-varying Proportion Portfolio Insurance seems more appropriate.

### *3.3 Consequences of model uncertainty to (Constant) Proportion Portfolio Insurance*

Having set forth the circumstances under which (Constant) Proportion Portfolio Insurance is applied and the calculation of the multiplier $m$ for (isolated) price processes, the second step of the analysis becomes possible: the confrontation of (Constant) Proportion Portfolio Insurance strategies with model uncertainty, i.e., the determination of those portfolio strategies from the set of isolated strategies that are able to defend minimum investment goals in the worst-case scenario that model uncertainty incorporates.

A pure geometric Brownian motion calls for (Constant) Proportion Portfolio Insurance to be applied only when wealth is low (see eq. (19)), whereas combined jump/diffusion risk (eq. (20)) and stochastic volatility (estimation) risk (eq. (21)) apply (Constant) Proportion Portfolio Insurance to all wealth levels. To cope with model uncertainty under income-stream guarantees hence means to fall back on the most conservative representation of a portfolio strategy, i.e., to follow (Constant) Proportion Portfolio Insurance at all wealth levels.

In a similar way, the multiplier that works even under model uncertainty can be characterized: it is the smallest multiplier that occurs under all three (isolated) price processes: $-\frac{1}{\phi_{\mathrm{extr}}}$. Jumps, therefore, depict a clear-cut worst-case scenario (see Sect. B of the Appendix for a proof) under income-stream guarantees as opposed to minimum-wealth guarantees where no (real) worst-case scenario was observable.

These conclusions regarding (Constant) Proportion Portfolio Insurance (optimum portfolio strategies under income-stream guarantees) are in stark contrast to those of Option Based Portfolio Insurance (portfolio strategies designed to defend minimum-wealth guarantees). First, (Constant) Proportion Portfolio Insurance (usually) does not have to rely on investing the present value of the floor in the riskless asset, which makes income-stream guarantees easier to defend than minimum-wealth guarantees. Only if a stock exchange does not specify[9] an upper limit for stock price movements will the minimum jump amplitude read $\varphi_{\text{extr}} = -1$ and make that part of (Constant) Proportion Portfolio Insurance that defends the guarantee indistinguishable from a buy and hold strategy in the riskless asset.[10] Second, there is a huge difference between portfolio strategies under homogenous and heterogenous model uncertainty. Under homogenous model uncertainty, the multiplier $m$ and, hence, the portfolio strategy, is investor-specific. Under heterogenous model uncertainty, the multiplier $m$ stems from the minimum jump amplitude, thus making (Constant) Proportion Portfolio Insurance independent of sellers' coefficients of risk aversion in that it yields strategies identical for all types of decision makers.

## 4 Conclusion

This paper began by stating that in real-world market financial markets there are insurance products that offer their buyers a minimum-wealth or an income-stream guarantee irrespective of capital market conditions, i.e., stock price processes assumed. Therefore, sellers of these products are well advised to pursue a portfolio strategy that is able to meet these minimum investment goals if they want to avoid additional cash payments. Portfolio Insurance seems to offer a solution to this portfolio problem.

However, it was the objective of this paper to show that Portfolio Insurance cannot protect minimum investment goals because it overlooks a real-world phenomenon: model uncertainty.

Option Based Portfolio Insurance (portfolio strategies designed to defend minimum-wealth guarantees) calls for duplication of the put option implied by the minimum-wealth guarantee. Although a duplication portfolio can be adapted to cope with several sources of market risk, it is fitted to a concrete form of market risk and can handle this form of market risk only. For that reason, when there is model uncertainty, i.e., several forms of market risk are possible, there is only one strategy able to defend guaranteed minimum wealth: invest the present value of guaranteed wealth in the riskless asset.

A more sophisticated strategy is possible with (Constant) Proportion Portfolio Insurance (portfolio strategies to defend income-stream guarantees). Assume an ex ante unknown number of stock market crashes which all have a minimum jump amplitude $\varphi_{\text{extr}} > -1$. Then (Constant) Proportion Portfolio Insurance can be

[9] See Roll (1989, p. 54) for an overview on stock exchanges that have limits for maximum possible price changes per day.

[10] As opposed to minimum-wealth guarantees, the riskless investment under income-stream guarantees will be able to defend the floor even if there is only a local riskless asset. (Constant) Proportion Portfolio Insurance works with a riskless investment that matures after one period. Therefore, it is not subject to the uncertainty imposed by the local riskfree rate.

based on this worst-case scenario and minimum investment goals can be defended even when there is model uncertainty with a portfolio strategy that does more than simply invest the present value of the income stream in the riskless asset.

These results on the performance of Portfolio Insurance strategies can be applied to judge the benefits of products with a minimum-wealth or an income-stream guarantee.

Minimum-wealth guarantees can be defended under the real-world phenomenon of model uncertainty only by following a trivial portfolio strategy – trivial in the sense that the minimum investment goal is achieved by riskless investment. For that reason, buyers of minimum-wealth guarantees are buying a product that performs like a riskless investment. Sellers of minimum-wealth guarantees cannot fall back on their superior portfolio selection skills because employing a sophisticated portfolio strategy, i.e., a strategy that involves risky investment, automatically means speculating on model uncertainty. This speculation, though, differs from that of "normal" portfolio selection. Whereas "normal" portfolio selection tries to forecast stocks' means in particular, which is difficult as Merton (1980) has shown, speculation on model uncertainty must determine the probabilities that a certain class of price processes is the true one; this job might be an even tougher task. In summary, minimum-wealth guarantees offer, at least in the stylized world of this paper, no advantage to either buyer or seller.

Since meeting guaranteed income streams allows for non-trivial portfolio strategies under the real-world phenomenon of model uncertainty, both buyers and sellers of such products can capitalize on the sellers' superior portfolio selection skills. Thus, in the stylized world of this paper, guaranteed-income stream products are valuable to both buyer and seller.

## Appendix

### *A. A derivation of Constant Proportion Portfolio Insurance strategies under combined jump/diffusion processes*

After having specified the wealth dynamics in the decision problem (18) with a combined jump/diffusion process, the following Hamilton/Jacobi/Bellman equation is obtained (see, e.g., Ahn and Thompson, 1988, p. 158):

$$\begin{aligned} 0 \;=\; & \underset{C(t),\,w(t)}{Max}\left\{ e^{-\rho t}\frac{C(t)^{\gamma}}{\gamma} + J_t \right. \\ & + J_W\,(\alpha - r)\,w(t)W(t) + J_W\,(rW(t) - C(t)) \\ & \left. + \frac{1}{2}J_{WW}w^2(t)\sigma^2W^2(t) + \lambda\, E\left\{ J\left[(1 + w(t)\,\varphi(t))\,W(t)\right] - J \right\} \right\} \\ s.t.:\; & C(t) \geq C_{\min} = K \\ & \lim_{t\to\infty} J\left[W(t), t\right] = 0 \end{aligned} \tag{A.1}$$

Following the line of argument developed by Black and Jones (1992) and Merton (1993, p. 186), a possible solution of (the constrained) problem (A.1) can be found by splitting the task of determining $w(t)$ into two related, but unconstrained problems: the determination of the optimum portfolio strategy for a critical and an uncritical region.

To verify that this is indeed a solution to problem (A.1), three steps are needed:

1. Derivation of the optimum portfolio weight for the critical region.
2. Derivation of the optimum portfolio weight for the uncritical region.
3. Determination of a critical wealth level that separates the critical (c) from the uncritical (u) region.

This third step serves in particular to verify that the two-step procedure delivers an admissible solution, i.e., keeps the indirect utility function $J[.]$ continuous and twice differentiable (as required by problem (A.1)).

*Step 1*
The decision maker's budget equation reads in every period

$$W(t^-) = E(t^-) + E_0(t^-) \tag{A.2}$$

where $E$ denotes the amount invested in the risky index and $E_0$ that invested in the riskless asset.

Jumps entail a sudden and large change of wealth. Therefore, immediately after a jump, wealth changes to (after having used the budget constraint (A.2) to substitute out $E_0$)

$$W(t) = E(t^-) \cdot (1 + \varphi(t)) + \left[W(t^-) - E(t^-)\right] \tag{A.3}$$

On the one hand, by definition, (Constant) Proportion Portfolio Insurance does not require a zero investment in the risky asset; instead it invests a multiplier $m$ of $W(t^-) - \frac{K}{r}$ in the risky asset. On the other hand, the guaranteed income stream $K$, the so-called floor, must be defended, which means that at the beginning of each period wealth must be not less than $\frac{K}{r}$. To fulfill both requirements, the multiplier $m$ must be determined in a way that wealth remains above $\frac{K}{r}$ in every period even in the worst-case jump scenario. Denote the minimum jump amplitude (in any period) with $\varphi_{\text{extr}}$, it must hold:

$$W(t) = \frac{K}{r} = \underbrace{m \cdot \left(W(t^-) - \frac{K}{r}\right)}_{E(t^-)} \cdot (1 + \varphi_{\text{extr}}) + \left[W(t^-) - m\left(W(t^-) - \frac{K}{r}\right)\right] \tag{A.4}$$

which yields

$$m = -\frac{1}{\varphi_{\text{extr}}}$$

*Step 2*
I omit a detailed explanation of Step 2 as will become clear immediately.

*Step 3*
The third step involves checking whether $J[.]$ is continuous and twice differentiable. To that end, the following boundary conditions must be met by the candidate

solution:

$$J^c\left[W^+ - \frac{K}{r}\right] = J^u\left[W^+\right] \tag{A.5a}$$

$$J^c_W\left[W^+ - \frac{K}{r}\right] = J^u_W\left[W^+\right] \tag{A.5b}$$

$$J^c_{WW}\left[W^+ - \frac{K}{r}\right] = J^u_{WW}\left[W^+\right] \tag{A.5c}$$

Without having to rely on an explicit calculation of $J[.]$, such a (non-trivial) $W^+$ cannot exist under combined jump/diffusion processes. To see this, define for any period the critical wealth level based on the diffusion component, that is, an infinitesimal price movement. A jump, however, signifies a non-infinitesimal price movement in that this $W^+$ does not keep $J[.]$ continuous in a jump environment. Now consider the reverse case, i.e., base $W^+$ in any period on the minimum jump amplitude. In that case $J[.]$ remains continuous in a jump environment, but will not meet conditions (A.5a) to (A.5c) if the risky asset follows the diffusion component.

In other words, the only portfolio strategy that guarantees the floor under combined jump/diffusion processes employs the portfolio weight for the critical region and does not switch to a less "cautious" portfolio strategy. In particular, it does not split the optimum portfolio weight into a critical and an uncritical region. Therefore, the portfolio strategy depicted in eq. (20) is obtained, which implicitly sets $W^+$ equal to $W(t)$.

### *B. Derivation of (Constant) Proportion Portfolio Insurance strategies under stochastic volatility*

After having particularized the wealth dynamics in the decision problem (18) with a diffusion process under stochastic volatility, the following Hamilton/Jacobi/Bellman equation is derived (see, e.g., Merton, 1973, p. 875):

$$\begin{aligned} 0 \;=\; & \underset{C(t),w(t)}{Max}\left\{ e^{-\rho t}\frac{C(t)^\gamma}{\gamma} + J_t + J_W\,(\alpha - r)\,w(t)W(t) \right.\\ & +J_W\,(rW(t) - C(t)) \\ & +\frac{1}{2}J_{WW}w^2(t)\sigma^2(t)W^2(t) + J_\sigma\alpha_\sigma\,\sigma(t) \\ & \left. +\frac{1}{2}J_{\sigma\sigma}\sigma_\sigma^2\sigma^2(t) + J_{W\sigma}w(t)\sigma(t)W(t)cov_{\sigma W}(t)\right\} \end{aligned} \tag{B.1}$$

$$\begin{aligned} s.t.:\; & C(t) \geq C_{min} = K \\ & \lim_{t\to\infty} J\left[W(t), t\right] = 0 \end{aligned}$$

To solve problem (B.1), the same procedure as in Section A, above, is utilized.

*Step 1*

Since the guaranteed income stream $K$ must be defended, at the beginning of each period wealth must not be less than $\frac{K}{r}$. Hence, in the critical region, the decision maker pursues (Constant) Proportion Portfolio Insurance, i.e., invests a multiplier $m$ of $W(t) - \frac{K}{r}$ in the risky asset instead of a multiplier $m$ of wealth $W(t)$.

*Step 2*
According to Merton (1973), the optimum portfolio weight (for both parts of the unconstrained problem (B.1)) reads:

$$w^{c/u}(t) = -\frac{J_W^{c/u}}{J_{WW}^{c/u} W(t)} \cdot \frac{\alpha - r}{\sigma^2(t)} - \frac{J_{\sigma W}^{c/u}}{J_{WW}^{c/u} W(t)} \cdot \frac{cov_{\sigma W}(t)}{\sigma(t)} \tag{B.2}$$

with $J^u$ based on $W = W(t)$ (uncritical region), and $J^c$ based on $W = W(t) - \frac{K}{r}$ (critical region).

*Step 3*
To figure out whether $J[.]$ resulting from the candidate solution is continuous and twice differentiable, $J[.]$ has to be determined explicitly. Substituting the portfolio weight (eq. (B.2)) back into problem (B.1), yields the following partial differential equation for $J[.]$:

$$\begin{aligned} 0 \;=\;& e^{-\rho t}\frac{C(t)^\gamma}{\gamma} + J_t + J_W\left(rW(t) - C(t)\right) \qquad \text{(B.3)} \\ & -\frac{1}{2}\left(\frac{\alpha - r}{\sigma(t)}\right)^2 \frac{J_W^2}{J_{WW}} + J_\sigma \alpha_\sigma \sigma(t) + \frac{1}{2} J_{\sigma\sigma}\sigma_\sigma^2 \sigma^2(t) \\ & -\frac{J_W J_{\sigma W}}{J_{WW}}\left(\frac{\alpha - r}{\sigma(t)}\right) \cdot cov_{\sigma W}(t) - \frac{1}{2}\frac{J_{\sigma W}^2}{J_{WW}} \cdot cov_{\sigma W}^2(t) \\ s.t.:\;& C(t) \geq C_{min} = K \\ & \lim_{t\to\infty} J\left[W(t), t\right] = 0 \end{aligned}$$

According to Cox, Ingersoll, and Ross (1985, p. 389), the general solution of the unconstrained partial differential eq. (B.4) for power utility functions reads

$$J\left[W(t), \sigma(t), t\right] = X(\sigma(t), t) \cdot \frac{W^\gamma(t)}{\gamma} + Y(\sigma(t), t) \tag{B.4}$$

with $X$ and $Y$ arbitrary functions of $\sigma(t)$ and $t$.

Transferring these findings to the constrained partial differential equation, define

- for the uncritical region

  $$J^u\left[W(t), \sigma(t), t\right] = A(\sigma(t), t)\frac{W^\gamma(t)}{\gamma} + G(\sigma(t), t) \tag{B.5}$$

  with $A$ and $G$ arbitrary functions of $\sigma(t)$.
- for the critical region

  $$J^c\left[W(t), \sigma(t), t\right] = B(\sigma(t), t)\frac{\left[W(t) - \frac{K}{r}\right]^\delta}{\delta} + D(\sigma(t), t) \tag{B.6}$$

  with $B$ and $D$ arbitrary functions of $\sigma(t)$ as well as $t$, and $\delta$ a risk aversion parameter, which is determined model-endogenously.

Moreover, from substituting eqs. (B.5) and (B.6) into boundary conditions (A.5b) and (A.5c), a candidate for the critical wealth level is obtained:

$$W^{+} = \frac{K}{r}\frac{1-\gamma}{\delta-\gamma} \tag{B.7}$$

where the only remaining unknown, $\delta$, is accessible with the help of boundary condition (A.5a):

$$\begin{aligned} & B(\sigma(t),t)\frac{\left(\frac{K}{r}\frac{1-\gamma}{\delta-\gamma}-\frac{K}{r}\right)^{\delta}}{\delta}+D(\sigma(t),t) \\ & = A(\sigma(t),t)\frac{\left(\frac{K}{r}\frac{1-\gamma}{\delta-\gamma}\right)^{\gamma}}{\gamma}+G(\sigma(t),t) \end{aligned} \tag{B.8}$$

Equations (B.7) and (B.8) contain, however, an incompatibility. On the one hand, $W^{+}$ in eq. (B.7) has been calculated with the help of the Cox/Ingersoll/Ross solution, a solution that assumes constant risk aversion parameters $\gamma$ and $\delta$. On the other hand, $\delta$ must be a function of $\sigma(t)$ according to eq. (B.8). Both statements cannot be true at the same time in that there is no wealth level $W^{+}$ that is part of the Cox/Ingersoll/Ross solution for $J[.]$ and fulfils eq. (B.8). In other words, splitting $J[.]$ into $J^{c}[.]$ and $J^{u}[.]$ does not keep $J[.]$ continuous and twice differentiable and cannot be an admissible solution to (the constraint) eq. (B.4) for power utility functions.

Putting all these arguments together, the optimum portfolio strategy under stochastic volatility employs the portfolio weight for the critical area as the sole optimum portfolio weight – hence implicitly sets $W^{+}$ equal to $W(t)$ – and does not split the optimum portfolio weight into a critical and an uncritical region. Therefore, the portfolio strategy depicted in eq. (21) is obtained.

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