On the metric dimension of line graphs

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Abstract

Let G be a (di)graph. A set W of vertices in G is a *resolving set* of G if every vertex u of G is uniquely determined by its vector of distances to all the vertices in W. The *metric dimension* $\mu(G)$ of G is the minimum cardinality of all the resolving sets of G. Cáceres et al. [3] computed the metric dimension of the line graphs of complete bipartite graphs. Recently, Bailey and Cameron [1] computed the metric dimension of the line graphs of complete graphs. In this paper we study the metric dimension of the line graph $L(G)$ of G. In particular, we show that $\mu(L(G)) = |E(G)| - |V(G)|$ for a strongly connected digraph G except for directed cycles, where $V(G)$ is the vertex set and $E(G)$ is the edge set of G. As a corollary, the metric dimension of de Brujin digraphs and Kautz digraphs is given. Moreover, we prove that $\lceil \log_2 \Delta(G) \rceil \leq \mu(L(G)) \leq |V(G)| - 2$ for a simple connected graph G with at least five vertices, where $\Delta(G)$ is the maximum degree of G. Finally, we obtain the metric dimension of the line graph of a tree in terms of its parameters.

Key words: Metric dimension; resolving set; line graph; de Brujin digraph; Kautz digraph.

1 Introduction

Let G be a (di)graph. We often write $V(G)$ for the vertex set of G and $E(G)$ for the edge set of G. A (di)graph G is (strongly) connected if for any two distinct vertices u and v of G, there exists a path from u to v. In this paper we only consider finite strongly connected digraphs, or undirected simple connected graphs. For two vertices u and v of G, we denote the distance from u to v by $d_G(u, v)$. A *resolving set* of G is a set of vertices $W = \{w_1, \dots, w_m\}$ such that for each $u \in V(G)$, the vector $D(u|W) = (d_G(u, w_1), \dots, d_G(u, w_m))$ uniquely determines u. The *metric dimension* of G, denoted by $\mu(G)$, is the minimum cardinality of all the resolving sets of G.

Metric dimension of graphs was introduced in the 1970s, independently by Harary and Melter [10] and by Slater [13]. Metric dimension of digraphs was first studied by Chartrand et al. in [5] and further in [6]. Fehr et al. [8] investigated the metric dimension of Cayley digraphs. In graph theory, metric dimension is a parameter that has appeared in various applications, as diverse as network discovery and verification [2], strategies for the Mastermind game [7], combinatorial optimization [12]

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and so on. It was noted in [9, p. 204] and [11] that determining the metric dimension of a graph is an NP-complete problem.

Let $L(G)$ denote the line graph of a (di)graph G. For the complete bipartite graph $K_{m,n}$, Cáceres et al. [3] proved that

$$\mu(L(K_{m,n})) = \begin{cases} \lfloor \frac{2(m+n-1)}{3} \rfloor, & m \leq n \leq 2m-1, n \geq 2, \\ n-1, & n \geq 2m. \end{cases}$$

For the complete graph K_n when $n \geq 6$, Bailey and Cameron [1] proved that $\mu(L(K_n)) = \lceil \frac{2n}{3} \rceil$.

Motivated by these results, in this paper we study the metric dimension of the line graph of a (di)graph. In Section 2, we show that $\mu(L(G)) = |E(G)| - |V(G)|$ for a strongly connected digraph G except for directed cycles. As a corollary, the metric dimension of de Brujin digraphs and Kautz digraphs, which are two families of famous networks, is given. In Section 3, we prove that $\lceil \log_2 \Delta(G) \rceil \leq \mu(L(G)) \leq |V(G)| - 2$ for a connected graph G with at least five vertices, where $\Delta(G)$ is the maximum degree of G. Finally, we obtain the metric dimension of the line graph of a tree in terms of its parameters.

2 Line graph of a digraph

Let G be a digraph. For a directed edge $a = (x, y)$ of G, we say that x is the *head* of a and y is the *tail* of a; we also say that a is the *out-going edge* of x and the *in-coming edge* of y. For $x \in V(G)$, we denote the set of all out-going edges of x by $E_G^+(x)$ and the set of all in-coming edges of x by $E_G^-(x)$. The *line graph* of G is the digraph $L(G)$ with the edges of G as its vertices, and where (a, b) is a directed edge in $L(G)$ if and only if the tail of a is the head of b in G. For two distinct vertices $a = (x_1, x_2), b = (y_1, y_2)$ of $L(G)$, we have

$$d_{L(G)}(a, b) = d_G(x_2, y_1) + 1. \tag{1}$$

Note that $\mu(L(G)) = 1$ if G is a directed cycle.

Theorem 2.1 *If G is a strongly connected digraph except for directed cycles, then*

$$\mu(L(G)) = |E(G)| - |V(G)|.$$

Proof. Let R be a resolving set of $L(G)$ with the minimum cardinality. For each vertex x of G, since G is strongly connected, $E_G^-(x) \neq \emptyset$. If $|E_G^-(x)| \geq 2$, pick two distinct edges $a, b \in E_G^-(x)$. For any $c \in V(L(G)) \setminus \{a, b\}$, since $d_{L(G)}(a, c) = d_{L(G)}(b, c)$, $a \in R$ or $b \in R$. It follows that $|E_G^-(x) \cap R| \geq |E_G^-(x)| - 1$. If $|E_G^-(x)| = 1$, the above inequality is directed. By $R = \dot{\cup}_{x \in V(G)} (E_G^-(x) \cap R)$, we obtain

$$\mu(L(G)) = |R| \geq \sum_{x \in V(G)} (|E_G^-(x)| - 1) = |E(G)| - |V(G)|. \tag{2}$$

Let W be a set obtained from $E(G)$ by deleting one in-coming edge of each vertex of G. Since G is not a directed cycle, $W \neq \emptyset$. We shall prove that W is a resolving set of $L(G)$. It suffices to show that, for any two distinct edges $a = (x_1, x_2)$ and $b = (y_1, y_2)$ in $E(G)\backslash W$, there exists an edge $c \in W$ such that

$$d_{L(G)}(a, c) \neq d_{L(G)}(b, c). \tag{3}$$

Let A denote the set of all the heads of each edge of W. Pick $z_0 \in A$ satisfying $d_G(x_2, z_0) \leq d_G(x_2, z)$ for any $z \in A$.

Case 1. $d_G(x_2, z_0) \neq d_G(y_2, z_0)$. Pick $c \in E_G^+(z_0) \cap W$. By (1), (3) holds.

Case 2. $d_G(x_2, z_0) = d_G(y_2, z_0)$. Owing to $a, b \notin W$, $x_2 \neq y_2$, which implies $z_0 \neq x_2$. Let $P_{x_2,z_0} = (v_0 = x_2, v_1, \ldots, v_k = z_0)$ be a shortest path from x_2 to z_0 and $P_{y_2,z_0} = (u_0 = y_2, u_1, \ldots, u_k = z_0)$ be a shortest path from y_2 to z_0. Suppose i denotes the minimum index such that $v_i = u_i$. Since $d_G(x_2, v_{i-1}) < d_G(x_2, v_i) \leq d_G(x_2, z_0)$, we have $v_{i-1} \notin A$, which implies $(v_{i-1}, v_i) \notin W$. Hence $(u_{i-1}, u_i) \in W$ and $u_{i-1} \in A$. Pick $c = (u_{i-1}, u_i)$. By (1), we have

$$\begin{aligned} d_{L(G)}(a, c) &= d_G(x_2, u_{i-1}) + 1 \\ &\geq d_G(x_2, z_0) + 1 \\ &= d_G(y_2, z_0) + 1 \\ &\geq d_G(y_2, u_i) + 1 \\ &= d_{L(G)}(b, c) + 1 \\ &> d_{L(G)}(b, c), \end{aligned}$$

so (3) holds.

Therefore, W is a resolving set of $L(G)$ with size $|E(G)| - |V(G)|$, which implies that $\mu(L(G)) \leq |E(G)| - |V(G)|$. By (2), the desired result follows. □

Let K_d be the complete digraph with d vertices. A *flowered complete digraph* of order d, denoted by K_d^+, is a digraph obtained from K_d by appending a self-loop at each vertex. Let

$$B(d, 1) = K_d^+,\ B(d, n) = L(B(d, n-1));$$
$$K(d, 1) = K_{d+1},\ K(d, n) = L(K(d, n-1)).$$

Then $B(d, n)$ is the *de Brujin digraph* and $K(d, n)$ is the *Kautz digraph*. By [14, Chapter 3], $B(d, n)$ and $K(d, n)$ are strongly connected and

$$|V(B(d, n))| = d^n,\ |E(B(d, n))| = d^{n+1};$$
$$|V(K(d, n))| = d^n + d^{n-1},\ |E(K(d, n))| = d^{n+1} + d^n.$$

As a corollary of Theorem 2.1, we get the metric dimension of de Brujin digraphs and Kautz digraphs, respectively.

Corollary 2.2 *Let integers $d \geq 2$ and $n \geq 1$. Then*

(i) $\mu(B(d, n)) = d^{n-1}(d-1)$;

(ii) $\mu(K(d, n)) = \begin{cases} d, & \text{if } n = 1, \\ d^{n-2}(d^2 - 1), & \text{if } n \geq 2. \end{cases}$

3 Line graph of a graph

Let G be a graph with at least two vertices. The *line graph* of G is the graph $L(G)$ with the edges of G as its vertices, and where two edges of G are adjacent in $L(G)$ if and only if they are adjacent in G.

If G has at most four vertices, it is routine to compute the metric dimension of $L(G)$. Next we shall consider the case $|V(G)| \geq 5$.

Theorem 3.1 *If G is a connected graph with at least five vertices, then*

$$\lceil \log_2 \Delta(G) \rceil \leq \mu(L(G)) \leq |V(G)| - 2,$$

where $\Delta(G)$ is the maximum degree of G.

Proof. Let v be a vertex of degree $\Delta(G)$, and let $\{f_1, \ldots, f_{\Delta(G)}\}$ be the set of all the edges incident to v. Suppose $W = \{e_1, \ldots, e_{\mu(L(G))}\}$ is a resolving set of $L(G)$ with the minimum cardinality. For each $j \in \{1, \ldots, \mu(L(G)\}$, let $d_j = \min\{d_G(v, w) | w \text{ is incident to } e_j\}$. Then $d_{L(G)}(f_i, e_j)$ is d_j or $d_j + 1$. Therefore, the size of $\mathcal{D} = \{D(f_i|W) | i = 1, \ldots, \Delta(G)\}$ is at most $2^{\mu(L(G))}$. Since $D(f_i|W) \neq D(f_k|W)$ for $i \neq k$, $\Delta(G) \leq 2^{\mu(L(G))}$, which implies the lower bound.

Suppose $|V(G)| = 5$. If G is isomorphic to the path P_5 or the cycle C_5, since $\mu(L(P_5)) = 1$ and $\mu(L(C_5)) = 2$, the upper bound is directed. If G is not isomorphic to P_5 or C_5, then G has a subgraph S isomorphic to $K_{1,3}$. Since $E(S)$ is a resolving set of $L(G)$, $\mu(L(G)) \leq 3$, which implies the upper bound.

Now suppose $|V(G)| \geq 6$. Let T be a spanning tree of G, and let v be a vertex of degree 1 in T. Suppose T_1 is the subgraph of T induced on $V(T) \backslash \{v\}$. We shall prove that $E(T_1)$ is a resolving set of $L(G)$. It suffices to show that, for any two distinct edges $a, b \in E(G) \backslash E(T_1)$, there exists an edge $e \in E(T_1)$ such that

$$d_{L(G)}(a, e) \neq d_{L(G)}(b, e). \tag{4}$$

Case 1. a or b is not incident to v. Without loss of generality, suppose a is not incident to v. Let $a = uu'$. Then there exists a unique path $P_{u,u'} = (u_0 = u, u_1, \ldots, u_k = u')$ between u and u' in T where $k \geq 2$. If b is not adjacent to u_0u_1, then (4) holds for $e = u_0u_1 \in E(T_1)$; If b is not adjacent to $u_{k-1}u_k$, then (4) holds for $e = u_{k-1}u_k \in E(T_1)$. Now we assume that b is adjacent to both u_0u_1 and $u_{k-1}u_k$.

Case 1.1. $k = 2$. Then b is incident to u_1. Suppose $b = u_1x$, where $x \in V(G) \backslash \{u_0, u_1, u_2\}$. Let $S = \{u_0, u_1, u_2, x\}$ and $\overline{S} = V(T_1) \backslash S$. Since $|V(T_1)| = |V(G)| - 1 \geq 5$, there exists an edge $e \in [S, \overline{S}]_{T_1}$, where $[S, \overline{S}]_{T_1}$ is the set of edges between S and $\overline{S}$ in T_1. If e is incident to u_0 or u_2, then $d_{L(G)}(a, e) = 1$ and $d_{L(G)}(b, e) = 2$; If e is incident to u_1 or x, then $d_{L(G)}(a, e) = 2$ and $d_{L(G)}(b, e) = 1$. So (4) holds.

Case 1.2. $k \geq 3$. Note that b is incident to u_1 or u_{k-1}. Without loss of generality, assume that b is incident to u_1. Let $e = u_1u_2 \in E(T_1)$. Then $d_{L(G)}(a, e) = 2 \neq 1 = d_{L(G)}(b, e)$, (4) holds.

Case 2. Both a and b are incident to v. Let $a = vx$, $b = vy$, $S = \{x, y\}$ and $\overline{S} = V(T_1) \backslash S$. Pick $e \in [S, \overline{S}]_{T_1}$. Note that e is not incident to v. Similar to Case 1.1, e satisfies (4).

Therefore, $E(T_1)$ is a resolving set of $L(G)$ with size $|V(G)|-2$, and the upper bound is valid. □

The lower bound in Theorem 3.1 can be attained if G is a path. The fact that $\mu(L(K_{1,n})) = n-1$ implies that the upper bound in Theorem 3.1 is tight. It seems to be difficult to improve the bound for general graphs. However, for a tree T, we can obtain the metric dimension of $L(T)$ in terms of some parameters of T.

Let T be a tree. A vertex of degree 1 in T is called an *end-vertex*. A vertex of degree at least 3 in T is called a *major vertex*. An end-vertex u of T is said to be a *terminal vertex of a major vertex* v of T if $d_T(u,v) < d_T(u,w)$ for every other major vertex w of T. A major vertex v of T is an *exterior major vertex* of T if there exists a terminal vertex of v in T. We denote the set of all the exterior major vertices in T by $\mathrm{EX}(T)$; For $v \in \mathrm{EX}(T)$, we denote the set of all the terminal vertices of v by $\mathrm{TER}(v)$. Let $\sigma(T) = \sum_{v\in \mathrm{EX}(T)} |\mathrm{TER}(v)|$ and $\mathrm{ex}(T) = |\mathrm{EX}(T)|$. Chartrand et al. [4] computed the metric dimension of a tree in terms of $\sigma(T)$ and $\mathrm{ex}(T)$.

Proposition 3.2 ([4]) *If T is a tree that is not a path, then $\mu(T) = \sigma(T) - \mathrm{ex}(T)$.*

Finally, we shall compute the metric dimension of the line graph of a tree. If P is a path, then $\mu(L(P)) = 1$.

Proposition 3.3 *If T is a tree that is not a path, then $\mu(L(T)) = \sigma(T) - \mathrm{ex}(T)$.*

Proof. Let R be a resolving set of $L(T)$ with the minimum cardinality. For a given vertex $v \in \mathrm{EX}(T)$, we claim that

$$\sum_{u\in \mathrm{TER}(v)} |R \cap E(P_{u,v})| \geq |\mathrm{TER}(v)| - 1, \tag{5}$$

where $P_{u,v}$ is the unique path between u and v in T. To the contrary, suppose that there exist two different terminate vertices u_1, u_2 of v such that $R \cap E(P_{u_1,v}) = R \cap E(P_{u_2,v}) = \emptyset$. Let e_1 and e_2 be the edges incident to v in $P_{u_1,v}$ and $P_{u_2,v}$, respectively. For each $e \in R$, we have $d_{L(T)}(e_1, e) = d_{L(T)}(e_2, e)$, contradicting the fact that R is a resolving set of $L(T)$. Hence our claim is valid. Since $|R| \geq \sum_{v\in \mathrm{EX}(T)} \sum_{u\in \mathrm{TER}(v)} |R \cap E(P_{u,v})|$, by (5) we have

$$\mu(L(T)) = |R| \geq \sum_{v\in \mathrm{EX}(T)} (|\mathrm{TER}(v)| - 1) = \sigma(T) - \mathrm{ex}(T). \tag{6}$$

Let W be a set obtained from the end-vertex set of T by deleting one terminal vertex of each exterior major vertex of T. In [4, Theorem 5], Chartrand et al. proved that W is a resolving set of T with size $\sigma(T) - \mathrm{ex}(T)$. Let W_L be the set of all the edges each of which is incident to one vertex of W. Then $|W_L| = |W|$. We will show that W_L is a resolving set of $L(T)$.

For any two distinct edges a and b of T, there exists a unique path

$$(w_0, w_1, \ldots, w_{k-1}, w_k)$$

such that $a = w_0w_1$ and $b = w_{k-1}w_k$. Since $w_0 \neq w_k$, there exists a vertex $w \in W$ such that $d_T(w_0, w) \neq d_T(w_k, w)$. Without loss of generality, assume that $d_T(w_0, w) < d_T(w_k, w)$. Let e be the edge incident to w. Then $e \in W_L$.

Case 1. $w_1 \in V(P_{w_0,w})$. Then

$$d_{L(T)}(a, e) = d_T(w_0, w) - 1 < d_T(w_k, w) - 1 \leq d_{L(T)}(b, e).$$

Case 2. $w_1 \notin V(P_{w_0,w})$. Then $(w_k, w_{k-1}, \ldots, w_1, P_{w_0,w})$ is the unique path between w_k and w. It follows that

$$d_{L(T)}(a, e) = d_T(w_0, w) < d_T(w_{k-1}, w) = d_{L(T)}(b, e).$$

Therefore, W_L is a resolving set of $L(T)$, which implies that $\mu(L(T)) \leq \sigma(T) - \mathrm{ex}(T)$. By (6), the desired result follows. □

Combing Proposition 3.2 and Proposition 3.3, $\mu(T) = \mu(L(T))$ for a tree T. It seems to be interesting to characterize a graph G satisfying $\mu(G) = \mu(L(G))$.

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