EXT ALGEBRA OF NICHOLS ALGEBRAS OF TYPE A_2

XIAOLAN YU AND YINHUO ZHANG

ABSTRACT. We give the full structure of the Ext algebra of a Nichols algebra of type A_2 by using the Hochschild-Serre spectral sequence. As an application, we show that the pointed Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ with Dynkin diagrams of type A, D, or E, except for A_1 and $A_1 \times A_1$ with the order $N_J > 2$ for at least one component J, are wild.

INTRODUCTION

For an algebra R over a field $\Bbbk$, its homological properties, such as the Calabi-Yau property [14], AS-regularity [16], support varieties [21], etc. rely exclusively on the structure of its Ext algebra $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$.

Nichols algebras play an important role in the classification of pointed Hopf algebras [3, 4, 5, 11]. They are braided Hopf algebras in certain braided monoidal categories. In [5], the authors showed that if H is a finite dimensional pointed Hopf algebra such that its coradical is an abelian group with order not divisible by primes less than 11, then H is isomorphic to a deformation of the bosonization of a Nichols algebra of finite Cartan type. Thus the study of Nichols algebras not only helps us to classify pointed Hopf algebras, but also helps us to understand more about the properties of pointed Hopf algebras. In two recent papers [12, 18] support varieties of modules over Hopf algebras are introduced. It turns out that support varieties are useful tools to study homological properties and representations of (braided) Hopf algebras. To define and to compute support varieties over a (braided) Hopf algebra we need first to understand the Ext algebra of the (braided) Hopf algebra. In [1], the author raised the question of when the Ext algebra of a Nichols algebra is still a Nichols algebra. These facts motivate us to study the structure of the Ext algebra of a Nichols algebra in this paper.

As a first attempt to explore the structure of the Ext algebras for further study, we will give the full structure of the Ext algebra of a Nichols algebra of type

2000 *Mathematics Subject Classification.* 16E40, 16W30.

Key words and phrases. Nichols algebra, Ext algebra.

A_2. First we use the Hochschild-Serre spectral sequence to get a basis of the Ext algebra. We then construct the first segment of the minimal projective resolution of $\Bbbk$ and give the relations that hold in the Ext algebra. We calculate the dimensions to verify that these relations are complete (see Theorems 2.12 and 2.13) . The relations are braided commutative, which coincides with what have been proved in [19], where the authors also showed that the cohomology ring of a finite dimensional pointed Hopf algebra of finite Cartan type is finitely generated. Having the generators and relations of the Ext algebra, we can show that the Ext algebra of a Nichols algebra is not a Nichols algebra in general (see Proposition 2.15). However, the quotient algebra of the Ext algebra modulo the ideal generated the nilpotents can be a Nichols algebra (see Proposition 2.16). This partially answers one of the questions raised in [1, Sec. 2.1].

Finite dimensional pointed Hopf algebras with abelian group coradicals have support varieties [12, 19]. For a pointed Hopf algebra A of type A_2, the support variety of $\Bbbk$ over A is isomorphic to the variety of $\Bbbk$ over the associated graded algebra with respect to a certain filtration of A. This can be showed by using the full structure of the Ext algebra of the Nichols algebra of type A_2. Finally, we apply our main results to show that in many cases, the pointed Hopf algebras $u(\mathcal{D}, \lambda, \mu)$ constructed in [5] are wild (Proposition 2.18).

Acknowledgement. The first author deeply thanks Dr. Jiwei He for helpful discussions and useful comments.

1. Preliminaries and Notations

Throughout the paper, we fix an algebraically closed field $\Bbbk$ with char$\Bbbk \neq 2$. All algebras are assumed to be finite dimensional and all modules are assumed to be finitely generated unless otherwise stated.

1.1. Nichols Algebras and pointed Hopf algebra of Cartan type.

In [5], the authors classified finite dimensional pointed Hopf algebras whose coradicals are abelian groups. We need the following terminology:

- a finite abelian group Γ;
- a Cartan matrix $(a_{ij}) \in \mathbb{Z}^{\theta\times\theta}$ of finite type, where $\theta \in \mathbb{N}$;
- a set $\mathcal{X}$ of connected components of the Dynkin diagram corresponding to the Cartan matrix (a_{ij}). If $1 \leqslant i, j \leqslant \theta$, then $i \sim j$ means that they belong to the same connected component;

- elements $g_1, \cdots, g_\theta \in \Gamma$ and characters $\chi_1, \cdots, \chi_\theta \in \widehat{\Gamma}$ such that

$$\chi_j(g_i)\chi_i(g_j) = \chi_i(g_i)^{a_{ij}}, \ \chi_i(g_i) \neq 1, \ \text{for all } 1 \leqslant i, j \leqslant \theta. \tag{1}$$

The collection $\mathcal{D}(\Gamma, (g_i)_{1\leqslant i\leqslant\theta}, (\chi_i)_{1\leqslant i\leqslant\theta}, (a_{ij})_{1\leqslant i,j\leqslant\theta})$ is called a *datum of finite Cartan type* for Γ.

For simplicity, we define $q_{ij} = \chi_j(g_i)$. Then equation (1) reads as

$$q_{ij}q_{ji} = q_{ii}^{a_{ij}}, \ \ q_{ii} \neq 1, \ \text{for all } 1 \leqslant i, j \leqslant \theta. \tag{2}$$

From now on, we assumt that for all $1 \leqslant i \leqslant \theta$,

$$\begin{aligned} &q_{ii} \text{ has odd order, and} \\ &\text{the order of } q_{ii} \text{ is prime to 3, if } i \text{ lies in a component } G_2. \end{aligned} \tag{3}$$

Since $q_{ij}q_{ji} = q_{ii}^{a_{ij}}$, $1 \leqslant i, j \leqslant \theta$, the order of q_{ii} is constant in each component $J \in \mathcal{X}$ of the Dynkin diagram. Let N_J denote this common order.

Given a datum $\mathcal{D}$, we define a braided vector space as follows. Let V be a Yetter-Drinfeld module over the group algebra $\Bbbk\Gamma$ with basis $x_i \in V_{g_i}^{\chi_i}$, $1 \leqslant i \leqslant \theta$. Then V is a braided vector space of diagonal type whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \ \ 1 \leqslant i, j \leqslant \theta. \tag{4}$$

Let $\lambda = (\lambda_{ij})_{1\leqslant i<j\leqslant n, i\nsim j}$ be a set of scalars, such that

$$\lambda_{ij} = 0 \ \text{ if } g_i g_j = 1 \text{ or } \chi_i\chi_j \neq \varepsilon,$$

where ε is the identity in $\widehat{\Gamma}$. The set of scalars $\lambda = (\lambda_{ij})_{1\leqslant i,j\leqslant n, i\nsim j}$ are called *linking parameters*. The algebra $U(\mathcal{D}, \lambda)$ is defined to be the quotient Hopf algebra of the smash product $\Bbbk\langle x_1, \cdots, x_\theta\rangle \# \Bbbk\Gamma$ modulo the ideal generated by the following relations

(Serre relations)	$(\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0,$	$1 \leqslant i, j \leqslant \theta, \ i \neq j, \ i \sim j,$
(linking relations)	$x_i x_j - \chi_j(g_i) x_j x_i = \lambda_{ij}(1 - g_i g_j),$	$1 \leqslant i < j \leqslant \theta, \ i \nsim j,$

where ad_c is the braided adjoint representation defined in [4, Sec. 1.4].

Let Φ be the root system corresponding to the Cartan matrix (a_{ij}) with $\Pi = \{\alpha_1, \cdots, \alpha_\theta\}$ a set of fixed simple roots. Let Φ_J, $J \in \mathcal{X}$, be the root system of the component J. Assume that $\mathcal{W}$ is the Weyl group of the root system Φ. We fix a reduced decomposition of the longest element

$$w_0 = s_{i_1} \cdots s_{i_p}$$

of $\mathcal{W}$ as a product of simple reflections. Then the positive roots Φ^+ are precisely the followings

$$\beta_1 = \alpha_{i_1},\ \ \beta_2 = s_{i_1}(\alpha_{i_2}), \cdots, \beta_p = s_{i_1}\cdots s_{i_{p-1}}(\alpha_{i_p}).$$

If $\beta_i = \sum_{j=1}^{\theta} m_j\alpha_j$, then we define

$$g_{\beta_i} = g_1^{m_1}\cdots g_\theta^{m_\theta} \text{ and } \chi_{\beta_i} = \chi_1^{m_1}\cdots\chi_\theta^{m_\theta}.$$

Similarly, we write $q_{\beta_j\beta_i} = \chi_{\beta_i}(g_{\beta_j})$.

Let x_{β_j}, $1 \leqslant j \leqslant p$, be the root vectors as defined in [5, Sce. 2.1]. Let $(\mu_\alpha)_{\alpha\in\Phi^+}$ be a set of scalars, such that

$$\mu_\alpha = 0 \ \text{ if } g_\alpha^{N_J} = 1 \text{ or } \chi_\alpha^{N_J} \neq \varepsilon,\ \ \alpha \in \Phi_J^+, J \in \mathcal{X}.$$

This set of scalars are called *root vector parameters*. The finite dimensional Hopf algebra $u(\mathcal{D}, \lambda, \mu)$ is the quotient of $U(\mathcal{D}, \lambda)$ modulo the ideal generated by

$$\text{(root vector relations)} \qquad x_\alpha^{N_J} - u_\alpha(\mu), \qquad \alpha \in \Phi_J^+, J \in \mathcal{X},$$

where $u_\alpha(\mu) \in \Bbbk\Gamma$ is defined inductively on Φ^+ as in [5, Sec 4.2].

Let V be the braided vector space defined as in (4). The Nichols algebra $\mathcal{B}(V)$ associated to V is a braided Hopf algebra in the category of Yetter-Drinfeld modules over $\Bbbk\Gamma$. By [5, Thm. 5.1], it is generated by $x_1, \cdots, x_\theta$ subject to relations

$$(\mathrm{ad}_c\, x_i)^{1-a_{ij}}(x_j) = 0,\ \ 1 \leqslant i, j \leqslant \theta,\ \ i \neq j,$$

$$x_\alpha^{N_J} = 0,\ \ \alpha \in \Phi_J^+, J \in \mathcal{X}.$$

The details about Nichols algebras can be found in [4].

Corollary 5.2 in [5] showed that the associated graded Hopf algebra $\mathrm{Gr}u(\mathcal{D}, \lambda, \mu)$ of the algebra $u(\mathcal{D}, \lambda, \mu)$ with respect to the coradical filtration is $u(\mathcal{D}, 0, 0)$. Moreover, we have that $\mathcal{B}(V)\#\Bbbk\Gamma \cong u(\mathcal{D}, 0, 0)$. More detailed discussion about the algebras $U(\mathcal{D}, \lambda)$ and $u(\mathcal{D}, \lambda, \mu)$ can be found in [5].

The following set

$$\{x_{\beta_1}^{a_1}\cdots x_{\beta_p}^{a_p} \mid 1 \leqslant a_i < N_J,\quad \beta_i \in \Phi_J^+,\quad 1 \leqslant i \leqslant p\}$$

forms a PBW basis of the Nichols algebra $\mathcal{B}(V)$ [5]. As in [19, Sec. 2], define a degree on each element as

$$\deg x_{\beta_1}^{a_1}\cdots x_{\beta_p}^{a_p} = (\sum a_i ht(\beta_i), a_p, \cdots, a_1) \in \mathbb{N}^{p+1},$$

where $ht(\beta_i)$ is the height of the positive root β_i. That is, if $\beta_i = \sum_{j=1}^{\theta} m_j\alpha_j$, then $ht(\beta_i) = \sum_{j=1}^{\theta} m_j$. Order the elements in $\mathbb{N}^{p+1}$ as follows

$$(a_{p+1}, a_p, \cdots, a_1) < (b_{p+1}, b_p, \cdots, b_1) \text{ if and only if there is some } 1 \leqslant k \leqslant p+1, \text{ such that } a_i = b_i \text{ for } i \geqslant k \text{ and } a_{k+1} < b_{k+1}. \tag{5}$$

By [10, Thm. 9.3], similar to the proof of Lemma 2.4 in [19], we obtain the following lemma.

Lemma 1.1. *In the Nichols algebra $\mathcal{B}(V)$, for $j > i$, we have*

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{\boldsymbol{a}\in\mathbb{N}^p} \rho_{\boldsymbol{a}} x_{\beta_1}^{a_1} \cdots x_{\beta_p}^{a_p}, \tag{6}$$

where $\rho_{\boldsymbol{a}} \in \Bbbk$ and $\rho_{\boldsymbol{a}} \neq 0$ only when $\boldsymbol{a} = (a_1, \cdots, a_p)$ satisfies that $a_k = 0$ for $k \leqslant i$ and $k \geqslant j$.

Therefore, if we order PBW basis elements by degree as in (5), we obtain a filtration on the Nichols algebra $\mathcal{B}(V)$. The associated graded algebra $\mathbb{G}\mathrm{r}\mathcal{B}(V)$ is generated by the root vectors x_{β_i}, $1 \leqslant i \leqslant p$, subject to the relations

$$[x_{\beta_i}, x_{\beta_j}]_c = 0, \text{ for all } i < j;$$

$$x_{\beta_i}^{N_J} = 0, \ \beta_i \in \Phi_J^+, \ 1 \leqslant i \leqslant p.$$

1.2. **Complexity and varieties.** We follow the definitions and the notations in [12]. Let A be a finite dimensional Hopf algebra and $\mathrm{H}^*(A, \Bbbk) := \mathrm{Ext}_A^*(\Bbbk, \Bbbk)$. The vector space $\mathrm{H}^*(A, \Bbbk)$ is an associative graded algebra under the Yoneda product. The subalgebra $\mathrm{H}^{ev}(A, \Bbbk)$ of $\mathrm{H}^*(A, \Bbbk)$ is defined as

$$\mathrm{H}^{ev}(A, \Bbbk) = \oplus_{n=0}^{\infty} \mathrm{H}^{2n}(A, \Bbbk).$$

The algebra $\mathrm{H}^{ev}(A, \Bbbk)$ is commutative, since $\mathrm{H}^*(A, \Bbbk)$ is graded commutative. In the following, we say that a Hopf algebra A satisfies the *assumption* (**fg**) if the following conditions hold:

- **(fg1)** The algebra $\mathrm{H}^{ev}(A, \Bbbk)$ is finitely generated.
- **(fg2)** The $\mathrm{H}^{ev}(A, \Bbbk)$-module $\mathrm{Ext}_A^*(M, N)$ is finitely generated for any two finite dimensional A-modules M and N.

Under the assumption (**fg**), the *variety* $\mathcal{V}_A(M, N)$ for A-modules M and N is defined as

$$\mathcal{V}_A(M, N) := \mathrm{MaxSpec}(\mathrm{H}^{ev}(A, \Bbbk)/I(M, N)),$$

where $I(M, N)$ is the annihilator of the action of $\mathrm{H}^{ev}(A, \Bbbk)$ on $\mathrm{Ext}_A^*(M, N)$. It is a homogeneous ideal of $\mathrm{H}^{ev}(A, \Bbbk)$. The *support variety* of M is defined as $\mathcal{V}_A(M) = \mathcal{V}_A(M, M)$. By [19, Thm 6.3], a finite dimensional pointed Hopf algebra of the form $u(\mathcal{D}, \lambda, \mu)$ satisfies the assumption (**fg**).

For a graded vector space $V^\bullet = \oplus_{n \in \mathbb{Z} \geqslant 0} V^n$, the growth rate $\gamma(V^\bullet)$ is defined as

$$\gamma(V^\bullet) = min\{c \in \mathbb{Z}, c \geqslant 0 \mid \exists b \in \mathbb{R}, \text{such that } \dim V^n \leqslant bn^{c-1}, \text{for all } n \geqslant 0\}.$$

Let M be an A-module and $P_* : \quad \cdots \to P_1 \to P_0 \to M \to 0$ a minimal projective resolution of M. Then the growth rate $\gamma(P_*)$ is defined to be the *complexity* $\mathrm{cx}_A(M)$ of M.

2. Ext algebras

It is clear that each Nichols algebra can be written as a twisted tensor product of a set of Nichols algebras, such that each of them satisfies that the Dynkin diagram associated to the Cartan matrix is connected. In [6], the authors showed that the Ext algebra of a twisted tensor algebra is essentially the twisted tensor algebra of the Ext algebras. Therefore, we only need to discuss the case where the Dynkin diagram is connected. Now we calculate the Ext algebra of a Nichols algebra of type A_2.

Let N be an integer, and let $\bar{q}$ be a primitive root of 1 of order N. Let q_{ij}, $1 \leqslant i, j \leqslant 2$ be roots of 1, such that

$$q_{11} = q_{22} = \bar{q}, \quad q_{12}q_{21} = \bar{q}^{-1}.$$

Let V be a 2-dimensional vector space with basis x_1 and x_2, whose braiding is given by

$$c(x_i \otimes x_j) = q_{ij} x_j \otimes x_i, \ \ 1 \leqslant i, j \leqslant 2.$$

Then V is a braided vector space of type A_2.

2.1. Case $N = 2$.

As discussed in [2], the Nichols algebra $R = \mathcal{B}(V)$ is isomorphic to the algebra generated by x_1 and x_2, with relations

$$x_1x_2x_1x_2 + x_2x_1x_2x_1 = 0, \ \ x_1^2 = x_2^2 = 0.$$

The dimension of R is 8.

Its Ext algebra can be calculated directly via the minimal projective resolution of $\Bbbk$.

Throughout, for an algebra R, we write elements in the free module R^n, $n \geqslant 1$, as row vectors. A morphism $f : R^m \to R^n$ is described by an $m \times n$ matrix.

Proposition 2.1. *Let* $R = \mathcal{B}(V)$ *be the algebra mentioned before, then the algebra* $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$ *is generated by* $\mathfrak{a}_1$*,* $\mathfrak{a}_2$ *and* $\mathfrak{b}$ *with* $\deg \mathfrak{a}_1 = \deg \mathfrak{a}_2 = 1$ *and* $\deg \mathfrak{b} = 2$*, subject to the relations*

$$\mathfrak{a}_2\mathfrak{a}_1 = \mathfrak{a}_1\mathfrak{a}_2 = 0, \quad \mathfrak{a}_1\mathfrak{b} = \mathfrak{b}\mathfrak{a}_1, \quad \mathfrak{a}_2\mathfrak{b} = \mathfrak{b}\mathfrak{a}_2.$$

Proof. We claim that the following complex is the minimal projective resolution of $\Bbbk$.

$$\cdots P_n \xrightarrow{d_n} P_{n-1} \to \cdots P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \to \Bbbk, \tag{7}$$

where $P_n = R^{n+1}$ and d_n is defined as

$$d_n = \begin{pmatrix} x_1 & & & & & & \\ x_2x_1x_2 & x_1 & & & & & \\ & \cdots & & & & & \\ & & x_2x_1x_2 & x_1 & & & \\ & & & x_2 & x_1x_2x_1 & & \\ & & & & \cdots & & \\ & & & & & x_2 & x_1x_2x_1 \\ & & & & & & x_2 \end{pmatrix},$$

when n is odd and

$$d_n = \begin{pmatrix} x_1 & & & & & & & \\ x_2x_1x_2 & x_1 & & & & & & \\ & \cdots & & & & & & \\ & & x_2x_1x_2 & x_1 & & & & \\ & & & x_2x_1x_2 & x_1x_2x_1 & & & \\ & & & & x_2 & x_1x_2x_1 & & \\ & & & & & \cdots & & \\ & & & & & & x_2 & x_1x_2x_1 \\ & & & & & & & x_2 \end{pmatrix},$$

when n is even. Especially, $d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. It is routine to check that (7) is indeed a complex. Now we use induction to prove the exactness. It is clear that the minimal projective resolution starts as

$$R^3 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \to \Bbbk \to 0,$$

where $d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $d_2 = \begin{pmatrix} x_1 & \\ x_2x_1x_2 & x_1x_2x_1 \\ & x_2 \end{pmatrix}$. Assume that the complex (7) is exact up to P_n. If n is odd, then

$$\begin{aligned} & \dim(\operatorname{Ker} d_n) \\ = \; & (1 + \dim P_1 + \dim P_3 + \cdots \dim P_n) - (\dim P_0 + \dim P_2 + \cdots \dim P_{n-1}) \\ = \; & 4n + 5. \end{aligned}$$

Since the dimension of R is small, we can calculate the dimension of the submodule $\operatorname{Im} d_{n+1}$ of P_n directly, it is also $4n + 5$. Then the complex is exact at P_{n+1}. If n is even, by a similar discussion, we can also conclude that the complex is exact at P_{n+1}. In this case $\dim(\operatorname{Ker} d_n) = 4n + 7$. We have that $\operatorname{Im} d_n \subseteq \operatorname{rad} P_{n-1}$ for each $i \geqslant 0$. Therefore, the complex (7) is the minimal projective resolution of $\Bbbk$. Since $\Bbbk$ is a simple module, we have

$$\operatorname{Hom}_R(P_n, \Bbbk) \cong \operatorname{Ext}^n_R(\Bbbk, \Bbbk) \tag{8}$$

as vector spaces for each $n \geqslant 0$. Let $\mathfrak{a}_1,\mathfrak{a}_2 \in \operatorname{Hom}_R(P_1, \Bbbk)$ be the functions dual to $(1, 0)$ and $(0, 1)$ respectively and $\mathfrak{b} \in \operatorname{Hom}_R(P_2, \Bbbk)$ be the function dual to $(0, 1, 0)$.

Let f_i, g_i and h_i be the morphisms described by the following matrices:

$$f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 1 & 0 \\ 0 & x_2x_1 \\ 0 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$g_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0 & 0 \\ x_1x_2 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$h_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then we have the following commutative diagrams:

$$\begin{array}{ccccccccc} P_3 & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & \Bbbk\ , \\ \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & \searrow & & & \\ P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & \Bbbk & & \end{array}$$

$$\begin{array}{ccccccccc} P_3 & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & \Bbbk\ , \\ \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & \searrow & & & \\ P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & \Bbbk & & \end{array}$$

$$\begin{array}{ccccccccc} P_3 & \xrightarrow{d_3} & P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & \Bbbk\ . \\ \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_2} & \searrow & & & & & \\ P_1 & \xrightarrow{d_2} & P_0 & \xrightarrow{d_1} & \Bbbk & & & & \end{array}$$

These commutative diagrams show that the relation listed in the proposition hold.

Let U be the algebra generated by $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{b}$ subject to the relations listed in the proposition. When n is odd, U_n has a basis

$$\{\mathfrak{a}_1^n, \mathfrak{a}_1^{n-2}\mathfrak{b}, \cdots, \mathfrak{a}_1\mathfrak{b}^{\frac{n-1}{2}}, \mathfrak{a}_2\mathfrak{b}^{\frac{n-1}{2}}, \cdots, \mathfrak{a}_2^{n-2}\mathfrak{b}, \mathfrak{a}_2^n\}$$

and when n is even, U_n has a basis

$$\{\mathfrak{a}_1^n, \mathfrak{a}_1^{n-2}\mathfrak{b}, \cdots, \mathfrak{a}_1\mathfrak{b}^{\frac{n}{2}-1}, \mathfrak{b}^{\frac{n}{2}}, \mathfrak{a}_2\mathfrak{b}^{\frac{n}{2}-1}, \cdots, \mathfrak{a}_2^{n-2}\mathfrak{b}, \mathfrak{a}_2^n\}.$$

They are functions dual to $(1, 0\cdots, 0)$, $\cdots$, $(0, \cdots, 0, 1)$ respectively in the projective resolution (7). We have

$$\begin{aligned} \dim U_n &= n+1 \\ &= \dim \operatorname{Hom}_R(P_n/(\operatorname{rad} P_n), \Bbbk) \\ &= \dim \operatorname{Hom}_R(P_n, \Bbbk) \\ &= \dim \operatorname{Ext}_R^n(\Bbbk, \Bbbk), \end{aligned}$$

where the last equation follows from equation (8). So we have $\operatorname{Ext}_R^*(\Bbbk, \Bbbk) = U$, which completes the proof of the proposition. □

2.2. **Case $N \geqslant 3$.** In this case, the Nichols algebra $R = \mathcal{B}(V)$ is the algebra generated by x_1 and x_2 subject to the relations

$$x_1^2x_2 - (q_{12} + q_{12}q_{11})x_1x_2x_1 + q_{12}^2q_{22}x_2x_1^2 = 0,$$
$$x_2^2x_1 - (q_{21} + q_{21}q_{22})x_2x_1x_2 + q_{21}^2q_{22}x_2x_1^2 = 0,$$
$$x_1^N = x_2^N = (x_1x_2 - q_{12}x_2x_1)^N = 0.$$

The dimension of R is N^3.

In the rest of the paper, we set $y = x_1x_2 - q_{12}x_2x_1$. From the above relations, we obtain that

$$q_{21}x_1y - yx_1 = 0, \quad x_2y - q_{21}yx_2 = 0.$$

Let α_1 and α_2 be the two simple roots. The element $\alpha_1\alpha_2\alpha_1$ is a reduced decomposition of the longest element in the Weyl group $\mathcal{W}$ and $\{\alpha_1, \alpha_1+\alpha_2, \alpha_2\}$ are the positive roots. The corresponding root vectors are just x_1, y and x_2. So the set

$$\{x_1^{a_1}y^{a_2}x_2^{a_3},\ 0 \leqslant a_i < N, i = 1, 2, 3\}$$

forms a PBW basis of R. The graded algebra $\mathbb{G}\mathrm{r}R$ corresponding to R is isomorphic to the algebra generated by x_1, y and x_2 subject to the relations

$$x_1y = q_{21}^{-1}yx_1, \quad x_1x_2 = q_{12}x_2x_1, \quad yx_2 = q_{21}^{-1}x_2y,$$
$$x_1^N = y^N = x_2^N = 0.$$

We first show that the algebra $\operatorname{Ext}_R^*(\Bbbk, \Bbbk)$ is generated in degree 1 and 2.

A connected graded algebra is called a $\mathcal{K}_2$ *algebra* if the algebra $\mathrm{EXT}_R^*(\Bbbk,\Bbbk)$ is generated by $\mathrm{EXT}_R^1(\Bbbk,\Bbbk)$ and $\mathrm{EXT}_R^2(\Bbbk,\Bbbk)$. Here $\mathrm{EXT}_R^*(-,-)$ denotes the functor on graded category [8, Definition. 1.1].

Remark 2.2. *For a finite dimensional connected algebra R, we have*

$$\mathrm{Ext}_R^*(\Bbbk,\Bbbk) \cong \mathrm{EXT}_R^*(\Bbbk,\Bbbk).$$

In the following we just identify them.

Let S be the subalgebra of R generated by x_1 and y. To be more precise, it is isomorphic to the algebra generated by x_1 and y subject to the relations

$$yx_1 = q_{21}x_1y, \qquad x_1^N = y^N = 0.$$

Lemma 2.3. *The algebra $R = \mathcal{B}(V)$ is $\mathcal{K}_2$.*

Proof. The algebra R is isomorphic to the graded Ore extension $R \cong S[x_2;\sigma,\delta]$, where σ is the graded algebra automorphism of S defined by $\sigma(x_1) = q_{12}^{-1}x_1$ and $\sigma(y) = q_{21}y$ and δ is the degree $+1$ graded σ-derivation of S defined by $\delta(x_1) = -q_{12}^{-1}y$ and $\delta(y) = 0$. By [8, Thm 10.2], the $\mathcal{K}_2$ property is preserved under graded Ore extension. From [19, Thm. 4.1], we can see that S is $\mathcal{K}_2$. Therefore, the algebra R is $\mathcal{K}_2$. $\square$

The subalgebra S is a normal subalgebra of R (we refer to [15, Appendix] for the definition of normal subalgebras). Now set $\overline{R} = R/(RS^+)$, where S^+ is the augmentation ideal of S. That is, $\overline{R} = k[x_2]/(x_2^N)$. We use the Hochschild-Serre spectral sequence (cf. [15])

$$E_2^{pq} = \mathrm{Ext}_{\overline{R}}^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk,\Bbbk)) \Longrightarrow \mathrm{Ext}_R^{p+q}(\Bbbk,\Bbbk) \tag{9}$$

to calculate the Ext algebra of R. We show that $E_2 = E_\infty$.

The spectral sequence is constructed as follows. Let

$$\cdots \to Q_1 \to Q_0 \to \Bbbk \to 0$$

and

$$\cdots \to P_1 \to P_0 \to \Bbbk \to 0$$

be free resolutions of ${}_{\overline{R}}\Bbbk$ and ${}_R\Bbbk$ respectively. There is a natural $\overline{R}$-module action on $\mathrm{Hom}_S(P_q,\Bbbk)$ for $q \geqslant 0$. We form a double complex

$$E_0^{pq} = \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Hom}_S(P_q,\Bbbk)).$$

By taking the vertical homology and then the horizontal homology, we have

$$E_1^{pq} = \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Ext}_S^q(\Bbbk,\Bbbk))$$

and

$$E_2^{pq} = \mathrm{Ext}^p_{\overline{R}}(\mathbb{k}, \mathrm{Ext}^q_S(\mathbb{k}, \mathbb{k})).$$

Now we construct a free resolution of $\mathbb{k}$ over R, which is a filtered complex. The corresponding graded complex is the minimal projective resolution of $\mathbb{k}$ over $\mathbb{G}\mathrm{r}R$.

Let $\sigma, \tau : \mathbb{N} \to \mathbb{N}$ be the functions defined by

$$\sigma(a) = \begin{cases} 1, & \text{if } a \text{ is odd;} \\ N-1, & \text{if } a \text{ is even} \end{cases}$$

and

$$\tau(a) = \begin{cases} \frac{a-1}{2}N+1, & \text{if } a \text{ is odd;} \\ \frac{a}{2}N, & \text{if } a \text{ is even.} \end{cases}$$

Let

$$P_\bullet : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \cdots \to P_1 \to P_0 \tag{10}$$

be a complex of free R-modules constructed as follows. For each triple (a_1, a_2, a_3), let $\Phi(a_1, a_2, a_3)$ be a free generator for P_n with $n = a_1 + a_2 + a_3$. Set

$$P_n = \oplus_{a_1+a_2+a_3=n} R\Phi(a_1, a_2, a_3)(\tau(a_1) + 2\tau(a_2) + \tau(a_3), \tau(a_3), \tau(a_2), \tau(a_1)).$$

Here, (-,-,-,-) denotes the degree shift. The differentials are defined by

$$\partial(\Phi(a_1, a_2, a_3)) = \begin{cases} (\delta_1 + \delta_2 + \delta_3)(\Phi(a_1, a_2, a_3)), & \text{if } a_2 \text{ is odd;} \\ (\delta_1 + \delta_2 + \tilde{\delta}_2 + \delta_3)(\Phi(a_1, a_2, a_3)), & \text{if } a_2 \text{ is even.} \end{cases}$$

The maps δ_i, $1 \leqslant i \leqslant 3$ and $\tilde{\delta}_2$ are defined as follows.

Put

$$\begin{aligned} \delta_1(\Phi(a_1, a_2, a_3)) &= x_1^{\sigma(a_1)}\Phi(a_1 - 1, a_2, a_3), \quad \text{if } a_1 > 0; \\ \delta_2(\Phi(a_1, a_2, a_3)) &= (-1)^{a_1} q_{21}^{-\sigma(a_2)\tau(a_1)} y^{\sigma(a_2)}\Phi(a_1, a_2 - 1, a_3), \quad \text{if } a_2 > 0; \\ \delta_3(\Phi(a_1, a_2, a_3)) &= (-1)^{a_1+a_2} q_{12}^{\sigma(a_3)\tau(a_1)} q_{21}^{-\sigma(a_3)\tau(a_2)} x_2^{\sigma(a_3)}\Phi(a_1, a_2, a_3 - 1), \text{ if } a_3 > 0; \\ \tilde{\delta}_2(\Phi(a_1, a_2, a_3)) &= D\Phi(a_1 - 1, a_2 + 1, a_3 - 1), \quad \text{if } a_1, a_3 > 0, a_2 \text{ is even,} \end{aligned}$$

where D is an element in R such that

$$Dy = -q_{21}^{\tau(a_1-1)} q_{12}^{\sigma(a_3)\tau(a_1-1)} q_{21}^{-\sigma(a_3)\tau(a_2)} [x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]_c.$$

The existence of such element D will be explained in Lemma 2.4. For $i = 1, 2, 3$, if $a_i = 0$, set $\delta_i(\Phi(a_1, a_2, a_3)) = 0$. If $a_1 = 0$ or $a_3 = 0$, set $\tilde{\delta}_2(\Phi(a_1, a_2, a_3)) = 0$.

Lemma 2.4. *The element y is a right divisor of $[x_1^{\sigma(a_1)}, x_2^{\sigma(a_3)}]$.*

(1) *If* $a_1, a_3 > 0$ *are odd, then*

$$\tilde{\delta}_2(\Phi(a_1,a_2,a_3)) = -q_{21}^{-\frac{a_2}{2}N}\Phi(a_1-1,a_2+1,a_3-1).$$

(2) *If* $a_1 > 0$ *is odd and* $a_3 > 0$ *is even, then*

$$\begin{aligned}&\tilde{\delta}_2(\Phi(a_1,a_2,a_3))\\ =\ & q_{12}^{(N-1)\frac{a_1-1}{2}N}q_{21}^{-(N-1)\frac{a_2}{2}N}\bar{q}q_{21}^{-(N-2)}q_{21}^{\frac{a_1-1}{2}N}x_2^{N-2}\Phi(a_1-1,a_2+1,a_3-1).\end{aligned}$$

(3) *If* $a_1 > 0$ *is even and* $a_3 > 0$ *is odd, then*

$$\tilde{\delta}_2(\Phi(a_1,a_2,a_3)) = q_{21}^{-\frac{a_2}{2}N}x_1^{N-2}\Phi(a_1-1,a_2+1,a_3-1).$$

(4) *If* $a_1, a_3 > 0$ *are even, then*

$$\begin{aligned}&\tilde{\delta}_2(\Phi(a_1,a_2,a_3))\\ =\ & -q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)}q_{21}^{-(N-1)\frac{a_2}{2}N}q_{21}^{\frac{a_1-2}{2}N+1}\\ &\quad(k_1x_1^{N-2}x_2^{N-2}+\cdots+k_{N-2}y^{N-3}x_1x_2+k_{N-1}y^{N-2})\Phi(a_1-1,a_2-1,a_3-1)\\ =\ & -q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)}q_{21}^{-(N-1)\frac{a_2}{2}N}q_{21}^{\frac{a_1-2}{2}N+1}\\ &\quad(l_1x_2^{N-2}x_1^{N-2}+\cdots+l_{N-2}y^{N-3}x_2x_1+l_{N-1}y^{N-2})\Phi(a_1-1,a_2-1,a_3-1),\end{aligned}$$

where

$$\begin{aligned}[x_1^{N-1},x_2^{N-1}]_c &= k_1yx_1^{N-2}x_2^{N-2}+\cdots+k_{N-2}y^{N-2}x_1x_2+k_{N-1}y^{N-1}\\ &= l_1yx_2^{N-2}x_1^{N-2}+\cdots+l_{N-2}y^{N-2}x_2x_1+l_{N-1}y^{N-1},\end{aligned}$$

with $k_i, l_i \in \Bbbk, 1 \leqslant i \leqslant N-1$.

Proof. (1) is easy to see. (2) and (3) follow from the following two equations,

$$[x_1^{N-1},x_2]_c = (1+\bar{q}^{-1}+\cdots+\bar{q}^{-N+1})x_1^{N-2}y = -\bar{q}x_1^{N-2}y$$

and

$$[x_1,x_2^{N-1}]_c = (1+\bar{q}^{-1}+\cdots+\bar{q}^{-N+1})yx_2^{N-2} = -\bar{q}yx_2^{N-2} = -\bar{q}q_{21}^{2-N}x_2^{N-2}y.$$

For (4), by Lemma 2.5 below, both $\{x_1^{a_1}y^{a_2}x_2^{a_3}\}$ and $\{x_2^{a_3}y^{a_2}x_1^{a_1}\}$, $0 \leqslant a_i < N$, $i = 1,2,3$, are bases of R. Using an easy induction, we can see that $[x_1^{N-1},x_2^{N-1}]_c$ can be expressed as

$$\begin{aligned}[x_1^{N-1},x_2^{N-1}]_c &= x_1^{N-1}x_2^{N-1}-q_{12}^{(N-1)^2}x_2^{N-1}x_1^{N-1}\\ &= k_1yx_1^{N-2}x_2^{N-2}+\cdots+k_{N-2}y^{N-2}x_1x_2+k_{N-1}y^{N-1}\\ &= l_1yx_2^{N-2}x_1^{N-2}+\cdots+l_{N-2}y^{N-2}x_2x_1+l_{N-1}y^{N-1},\end{aligned}$$

with $k_i, l_i \in \Bbbk$, $1 \leqslant i \leqslant N-1$. Observe that y commutes with $x_1^tx_2^t$ and $x_2^tx_1^t$ for $t \geqslant 0$. Then the result follows. □

Lemma 2.5. *Both the sets*

$$\{x_2^{a_3}y^{a_2}x_1^{a_1}\} \text{ and } \{x_1^{a_1}y^{a_2}x_2^{a_3}\},$$

$0 \leqslant a_i < N, i = 1,2,3$ *form bases of the algebra* R.

Proof. It is clear for the set $\{x_1^{a_1}y^{a_2}x_2^{a_3}\}$. For the set $\{x_2^{a_3}y^{a_2}x_1^{a_1}\}$, it is easy to see that

$$x_2^{a_3}y^{a_2}x_1^{a_1} = q_{21}^{a_1a_2+a_2a_3}q_{12}^{-a_1a_3}x_1^{a_1}y^{a_2}x_2^{a_3} + \sum_{i=1}^{min\{a_1,a_2,a_3\}} k_i x_1^{a_1-i}y^{a_2-i}x_2^{a_3-i}$$

and

$$x_1^{a_1}y^{a_2}x_2^{a_3} = q_{21}^{-a_1a_2-a_2a_3}q_{12}^{a_1a_3}x_2^{a_3}y^{a_2}x_1^{a_1} + \sum_{i=1}^{min\{a_1,a_2,a_3\}} l_i x_2^{a_3-i}y^{a_2-i}x_1^{a_1-i}$$

with each $k_i, l_i \in \Bbbk$. So $\{x_2^{a_3}y^{a_2}x_1^{a_1}\}$ also form a basis of R. □

Proposition 2.6. *The complex (10) is a projective resolution of* $\Bbbk$ *over* R*, the corresponding graded complex is the minimal projective resolution of* $\Bbbk$ *over* $\mathbb{G}\mathrm{r}R$.

Proof. It is routine to check that (10) is indeed a complex. We see it in Appendix 3.1. The differentials preserve the filtration and the corresponding graded complex is just the minimal projective resolution of $\Bbbk$ over $\mathbb{G}\mathrm{r}R$ as constructed in [19, Sec. 4]. Since the filtration is finite, the complex $P_\bullet$ is exact by [7, Chapter 2, Lemma 3.13]. Therefore, $P_\bullet$ is a free resolution of $\Bbbk$ over R. □

In the following, we will forget the shifting on the modules in the complex (10). It is clear that it is still a projective resolution of $\Bbbk$ over R. The only difference is that the differentials are not of degree 0. We denote this complex by $P_\bullet$ as well.

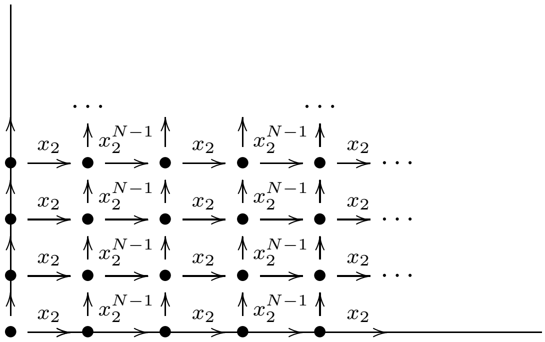
It is well-known that the following complex is the minimal projective resolution of $\Bbbk$ over $\overline{R} = k[x_2]/(x_2^N)$.

$$Q_\bullet : \cdots \to \overline{R} \xrightarrow{x_2^{N-1}} \overline{R} \xrightarrow{x_2} \overline{R} \xrightarrow{x_2^{N-1}} \overline{R} \xrightarrow{x_2} \overline{R} \to \Bbbk.$$

Therefore, we have

$$\begin{aligned} E_0^{pq} &= \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Hom}_S(P_q, \Bbbk)) \\ &= \mathrm{Hom}_S(\oplus_{a_1+a_2+a_3=q} R\Phi(a_1,a_2,a_3), \Bbbk) \\ &= \oplus_{a_1+a_2+a_3=q}\overline{R}\Phi(a_1,a_2,a_3), \end{aligned}$$

since $\mathrm{Hom}_S(R,\Bbbk) \cong \overline{R}$. The double complex reads as follows



The vertical differentials are induced from the differentials of the complex (10).

By taking the vertical homology, we have $E_1^{pq} = \mathrm{Hom}_{\overline{R}}(Q_p, \mathrm{Ext}_S^q(\Bbbk, \Bbbk))$. Following from [19], the algebra $\mathrm{Ext}_S^*(\Bbbk, \Bbbk)$ is generated by $\mathfrak{u}_1$, $\mathfrak{u}_y$, $\mathfrak{w}_1$ and $\mathfrak{w}_y$, where $\deg \mathfrak{u}_1 = \deg \mathfrak{u}_y = 2$ and $\deg \mathfrak{w}_1 = \deg \mathfrak{w}_y = 1$, subject to the relations

$$\mathfrak{w}_y \mathfrak{w}_1 = -q_{21} \mathfrak{w}_1 \mathfrak{w}_y, \ \ \mathfrak{w}_1^2 = \mathfrak{w}_y^2 = 0,$$

$$\mathfrak{w}_y \mathfrak{u}_1 = q_{21}^N \mathfrak{u}_1 \mathfrak{w}_y, \ \ \mathfrak{w}_1 \mathfrak{u}_1 = \mathfrak{u}_1 \mathfrak{w}_1, \ \ \mathfrak{w}_y \mathfrak{u}_y = \mathfrak{u}_y \mathfrak{w}_y, \ \ \mathfrak{w}_1 \mathfrak{u}_y = q_{21}^{-N} \mathfrak{u}_y \mathfrak{w}_1,$$

$$\mathfrak{u}_y \mathfrak{u}_1 = q_{21}^{N^2} \mathfrak{u}_1 \mathfrak{u}_y.$$

We use the notations $\mathfrak{u}_i$ and $\mathfrak{w}_i$ in place of the notations ξ_i and η_i used in [19]. Note that $\mathfrak{w}_1^2 = \mathfrak{w}_y^2 = 0$ holds since we assume that the characteristic of the field $\Bbbk$ is 0. It should also be noticed that the Ext algebra in [19] is the opposite algebra here.

As described in the appendix of [15], there is an action of $\overline{R}$ on $\mathrm{Ext}_S^*(\Bbbk, \Bbbk)$ given by

$$x_2(\mathfrak{u}_y) = x_2(\mathfrak{u}_1) = 0, \ \ x_2(\mathfrak{w}_y) = \mathfrak{w}_1, \text{ and } x_2(\mathfrak{w}_1) = 0.$$

This action is a derivation on $\mathrm{Ext}_S^*(\Bbbk, \Bbbk)$. That is, $x_2(\mathfrak{u}\mathfrak{w}) = x_2(\mathfrak{u})\mathfrak{w} + \mathfrak{u} x_2(\mathfrak{w})$ for $\mathfrak{u}, \mathfrak{w} \in \mathrm{Ext}_S^*(\Bbbk, \Bbbk)$.

The following lemma gives a basis of $\mathrm{Ext}_{\overline{R}}^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk, \Bbbk))$.

Lemma 2.7. *As a vector space,* $\mathrm{Ext}_{\overline{R}}^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk, \Bbbk))$ *has a basis as follows*

$$\begin{cases} \mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_1, & 2(i+j)+1 = q, \quad q \text{ is odd and } p \text{ is even}; \\ \mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_y, & 2(i+j)+1 = q, \quad q \text{ is odd and } p \text{ is odd}; \\ \mathfrak{u}_1^i \mathfrak{u}_y^j (\mathfrak{w}_1 \mathfrak{w}_y)^k, & k = 0, 1 \text{ and } 2(i+j)+2k = q, \quad q \text{ is even.} \end{cases}$$

Proof. Let $E = \mathrm{Ext}_S^*(\Bbbk, \Bbbk)$. The lemma follows directly from the following facts:

(i) If q is odd, then $\{\mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_1 | i, j \geqslant 0, 2(i+j)+1 = q\}$ forms a basis of $x_2 E^q$ and $\{e \in E^q | x_2 e = 0\}$.

(ii) If q is even, then $x_2 E^q = 0$.

(iii) $x_2^{N-1} E = 0$. □

Proposition 2.8. *The spectral sequence*

$$E_2^{p,q} = \mathrm{Ext}_{\overline{R}}^p(\Bbbk, \mathrm{Ext}_S^q(\Bbbk, \Bbbk)) \Longrightarrow \mathrm{Ext}_R^{p+q}(\Bbbk, \Bbbk)$$

satisfies $E_2 = E_\infty$.

Proof. The elements $\mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_y$ and $\mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_1$ are represented by

$$x_2^{N-2} \Phi(2i+1, 2j, 0) + q_{12}^{-(j+1)} x_2^{N-1} \Phi(2i, 2j+1, 0)$$

and

$$x_2^{N-1} \Phi(2i+1, 2j, 0),$$

while $\mathfrak{u}_1^i \mathfrak{u}_y^j$ and $\mathfrak{u}_1^i \mathfrak{u}_y^j \mathfrak{w}_1 \mathfrak{w}_y$ are represented by

$$x_2^{N-1} \Phi(2i, 2j, 0) \text{ and } x_2^{N-1} \Phi(2i+1, 2j+1, 0)$$

in E_0. In other words, all the elements in E_0 representing the elements in E_2 are mapped to 0 under the horizontal differentials. We conclude that $E_2 = E_\infty$. □

We now can determine the dimension of $\mathrm{Ext}_R^*(\Bbbk, \Bbbk)$. This dimension depends on the parity of n.

Corollary 2.9. *We have*

$$\dim \mathrm{Ext}_R^n(\Bbbk, \Bbbk) = \begin{cases} \frac{3n^2+8n+5}{8}, & \text{if } n \text{ is odd;} \\ \frac{3n^2+10n+8}{8}, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Set $E^n = \oplus_{p+q=n} E_2^{pq} = \oplus_{p+q=n} \operatorname{Ext}_R^p(\Bbbk, \operatorname{Ext}_S^q(\Bbbk, \Bbbk))$. By Lemma 2.7, we can illustrate the dimensions of E_2^{pq} with the following table:

		$\cdots$	$\cdots$			
4	4	4	4	4	4	$\cdots$
7	7	7	7	7	7	$\cdots$
3	3	3	3	3	3	$\cdots$
5	5	5	5	5	5	$\cdots$
2	2	2	2	2	2	$\cdots$
3	3	3	3	3	3	$\cdots$
1	1	1	1	1	1	$\cdots$
1	1	1	1	1	1	$\cdots$.

Therefore, when n is odd,

$$\begin{aligned} \dim E^n &= (1 + 2 + \cdots + \tfrac{n+1}{2} + 1 + 3 + \cdots + n) \\ &= \tfrac{3n^2+8n+5}{8}. \end{aligned}$$

When n is even,

$$\begin{aligned} \dim E^n &= (1 + 2 + \cdots + \tfrac{n}{2} + 1 + 3 + \cdots + n + 1) \\ &= \tfrac{3n^2+10n+8}{8}. \end{aligned}$$

By Proposition 2.8, we have $E_2 = E_\infty$, so $\dim \operatorname{Ext}_R^n(\Bbbk, \Bbbk) = \dim E^n$. This completes the proof. □

Now we give the first segment of the minimal projective resolution of a Nichols algebra of type A_2.

The algebra R is a local algebra. Thus projective R-modules are free. Let

$$R^{n_4} \to R^{n_3} \to R^{n_2} \to R^{n_1} \to R^{n_0} \to \Bbbk \to 0$$

be the first segment of the minimal projective resolution. Since $\Bbbk$ is a simple module, we have

$$\begin{aligned} \dim \operatorname{Ext}_R^i(\Bbbk, \Bbbk) &= \dim \operatorname{Hom}_R(R^{n_i}, \Bbbk) \\ &= \dim \operatorname{Hom}_R((R/(\operatorname{rad} R))^{n_i}, \Bbbk) \\ &= n_i. \end{aligned}$$

From the computation of the dimensions of $\mathrm{Ext}_R^*(\Bbbk,\Bbbk)$ in Corollary 2.9, we can see that the minimal projective resolution begins as

$$R^{12} \to R^7 \to R^5 \to R^2 \to R \to \Bbbk \to 0.$$

We give the differentials in the following proposition.

As in the construction of $\tilde{\delta}_2$ in §2.2, let $\overline{D}$ be the element in R such that $\overline{D}y = [x_1^{N-1}, x_2^{N-1}]_c$.

Proposition 2.10. *Let R be a Nichols algebra of type A_2. The following sequence provides the first segment of the minimal projective resolution of $\Bbbk$ over R,*

$$R^{12} \xrightarrow{d_4} R^7 \xrightarrow{d_3} R^5 \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \to \Bbbk \to 0, \tag{11}$$

where the differentials are given by the following matrices:

$$d_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$d_2 = \begin{pmatrix} x_1^{N-1} & 0 \\ -(q_{12}+\bar{q}q_{12})x_1x_2 + \bar{q}q_{12}^2x_2x_1 & x_1^2 \\ -q_{12}y^{N-1}x_2 & y^{N-1}x_1 \\ x_2^2 & \bar{q}q_{21}^2x_1x_2 - (q_{21}+\bar{q}q_{21})x_2x_1 \\ 0 & x_2^{N-1} \end{pmatrix},$$

$$d_3 = \begin{pmatrix} x_1 & 0 & 0 & 0 & 0 \\ q_{12}^Nx_2 & x_1^{N-2} & 0 & 0 & 0 \\ 0 & 0 & x_2 & q_{12}q_{21}^{N-1}y^{N-1} & 0 \\ 0 & x_2 & 0 & x_1 & 0 \\ 0 & -q_{21}^{1-N}y^{N-1} & x_1 & 0 & 0 \\ 0 & 0 & 0 & q_{12}^Nx_2^{N-2} & x_1 \\ 0 & 0 & 0 & 0 & x_2 \end{pmatrix},$$

$$d_4 = \left(\begin{array}{c:c} A_1 & A_2 \\ \hdashline A_3 & A_4 \end{array}\right),$$

where

$$A_1 = \begin{pmatrix} x_1^{N-1} & 0 & 0 & 0 \\ -(q_{12}^{1+N}+\bar{q}q_{12}^{1+N})x_1x_2 + \bar{q}q_{12}^{2+N}x_2x_1 & x_1^2 & 0 & 0 \\ -q_{12}^{1+N}y^{N-1}x_2 & y^{N-1}x_1 & 0 & 0 \\ q_{12}^Nx_2^2 & \bar{q}q_{21}^2x_1x_2 - (q_{21}+\bar{q}q_{21})x_2x_1 & 0 & q_{21}^Nx_1^{N-1} \\ 0 & x_2^{N-1} & 0 & -q_{12}^{-N^2+2N}\overline{D} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ q_{12}^{-N^2+N} x_1^{N-1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & q_{12}^{-N^2+N} x_1^{N-1} & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 0 & x_1^2 & -\bar{q}^{-1} q_{12}^N y^{N-1} x_1 \\ 0 & 0 & y^{N-1} x_1 & 0 \\ 0 & 0 & \bar{q} q_{21}^2 x_1 x_2 - (q_{21} + \bar{q} q_{21}) x_2 x_1 & q_{21}^{N-1} y^{N-1} x_2 \\ 0 & 0 & 0 & q_{12}^{2N} x_2^{N-1} \\ 0 & 0 & q_{12}^{N^2} x_2^{N-1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A_4 = \begin{pmatrix} -(q_{12} + \bar{q} q_{12}) x_1 x_2 + \bar{q} q_{12}^2 x_2 x_1 & 0 & 0 \\ -q_{12} y^{N-1} x_2 & 0 & 0 \\ x_2^2 & 0 & 0 \\ 0 & -(q_{12} + \bar{q} q_{12}) x_1 x_2 + \bar{q} q_{12}^2 x_2 x_1 & x_1^2 \\ 0 & -q_{12} y^{N-1} x_2 & y^{N-1} x_1 \\ 0 & x_2^2 & \bar{q} q_{21}^2 x_1 x_2 - (q_{21} + \bar{q} q_{21}) x_2 x_1 \\ 0 & 0 & x_2^{N-1} \end{pmatrix}.$$

Proof. It is routine to check that (11) is indeed a complex. But we need to mention that the following two equations hold

$$\overline{D} x_1 - x_1^{N-1} x_2^{N-2} = 0,$$

$$x_2^{N-1} x_1^{N-2} - q_{12}^{-N^2+2N} \overline{D} x_2 = 0.$$

These equations follow from Lemma 2.5 and the equations

$$[x_1^{N-1}, x_2^{N-1}]_c x_1 = y x_1^{N-1} x_2^{N-2},$$

$$[x_1^{N-1}, x_2^{N-1}]_c x_2 = q_{12}^{N^2-2N} y x_2^{N-1} x_1^{N-2}.$$

The complex (11) is homotopically equivalent to the first segment of the resolution $P_\bullet$ (without shifting) constructed in Section 2. Therefore, it is exact. □

Remark 2.11. *In* [17, Theorem 6.1.3]*, the authors give a set of linearly independent 2-cocycles on R, indexed by the positive roots. In the resolution (11), the functions dual to* $(1,0,0,0,0)$*,* $(0,0,1,0,0)$ *and* $(0,0,0,0,1)$ *are just those 2-cocycles, corresponding to the positive roots* α_1*,* $\alpha_1+\alpha_2$ *and* α_2 *respectively.*

Now we give our main theorems about the structure of the Ext algebra of a Nichols algebra of type A_2.

Theorem 2.12. *Let R be a Nichols algebra of type A_2 with $N = 3$, then* $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$ *is generated by* $\mathfrak{a}_i$, $\mathfrak{b}_i$, $\mathfrak{c}_i$, $i = 1, 2$ *and* $\mathfrak{b}_y$ *with*

$$\deg \mathfrak{a}_i = 1, \quad \deg \mathfrak{b}_i = \deg \mathfrak{b}_y = \deg \mathfrak{c}_i = 2,$$

subject to the relations

$$\mathfrak{a}_1^2 = \mathfrak{a}_2^2 = 0, \quad \mathfrak{a}_1\mathfrak{a}_2 = \mathfrak{a}_2\mathfrak{a}_1 = 0,$$

$$\mathfrak{a}_1\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{a}_1, \quad \mathfrak{a}_1\mathfrak{b}_y = q_{12}^3\mathfrak{b}_y\mathfrak{a}_1, \quad \mathfrak{a}_1\mathfrak{b}_2 = q_{12}^3\mathfrak{b}_2\mathfrak{a}_1,$$

$$\mathfrak{a}_1\mathfrak{c}_1 = \bar{q}^2 q_{12}\mathfrak{c}_1\mathfrak{a}_1, \quad \mathfrak{a}_1\mathfrak{c}_2 = \bar{q}q_{12}^2\mathfrak{c}_2\mathfrak{a}_1,$$

$$q_{12}^3\mathfrak{a}_2\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{a}_2, \quad q_{12}^3\mathfrak{a}_2\mathfrak{b}_y = \mathfrak{b}_y\mathfrak{a}_2, \quad \mathfrak{a}_2\mathfrak{b}_2 = \mathfrak{b}_2\mathfrak{a}_2,$$

$$\mathfrak{a}_2\mathfrak{c}_1 = \bar{q}q_{21}^2\mathfrak{c}_1\mathfrak{a}_2, \quad \bar{q}^2 q_{12}\mathfrak{a}_2\mathfrak{c}_2 = \mathfrak{c}_2\mathfrak{a}_2,$$

$$\bar{q}^2 q_{12}\mathfrak{a}_2\mathfrak{b}_1 = \mathfrak{a}_1\mathfrak{c}_1, \quad \mathfrak{a}_1\mathfrak{b}_2 = \bar{q}^2 q_{12}\mathfrak{a}_2\mathfrak{c}_2, \quad \mathfrak{c}_1\mathfrak{a}_2 = \mathfrak{c}_2\mathfrak{a}_1,$$

$$\mathfrak{b}_1\mathfrak{c}_2 = q_{12}^6\mathfrak{c}_1^2, \quad q_{12}^6\mathfrak{b}_2\mathfrak{c}_1 = \mathfrak{c}_2^2, \quad \mathfrak{b}_1\mathfrak{b}_2 = q_{12}^3\mathfrak{c}_1\mathfrak{c}_2, \quad \mathfrak{c}_1\mathfrak{c}_2 = q_{12}^3\mathfrak{c}_2\mathfrak{c}_1,$$

$$\mathfrak{b}_1\mathfrak{b}_y = q_{12}^9\mathfrak{b}_y\mathfrak{b}_1, \quad \mathfrak{b}_1\mathfrak{b}_2 = q_{12}^9\mathfrak{b}_2\mathfrak{b}_1, \quad \mathfrak{b}_y\mathfrak{b}_2 = q_{12}^9\mathfrak{b}_2\mathfrak{b}_y,$$

$$q_{12}^3\mathfrak{c}_1\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{c}_1, \quad \mathfrak{c}_1\mathfrak{b}_y = q_{12}^3\mathfrak{b}_y\mathfrak{c}_1, \quad \mathfrak{c}_1\mathfrak{b}_2 = q_{12}^6\mathfrak{b}_2\mathfrak{c}_1,$$

$$q_{12}^6\mathfrak{c}_2\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{c}_2, \quad q_{12}^3\mathfrak{c}_2\mathfrak{b}_y = \mathfrak{b}_y\mathfrak{c}_2, \quad \mathfrak{c}_2\mathfrak{b}_2 = q_{12}^3\mathfrak{b}_2\mathfrak{c}_2.$$

Theorem 2.13. *Let R be a Nichols algebra of type A_2 with $N > 3$, then* $\mathrm{Ext}^*_R(\Bbbk, \Bbbk)$ *is generated by* $\mathfrak{a}_i$, $\mathfrak{b}_i$ *and* $\mathfrak{c}_i$, $i = 1, 2$ *and* $\mathfrak{b}_y$ *with*

$$\deg \mathfrak{a}_i = 1, \quad \deg \mathfrak{b}_i = \deg \mathfrak{b}_y = \deg \mathfrak{c}_i = 2,$$

subject to the relations

$$\mathfrak{a}_1^2 = \mathfrak{a}_2^2 = 0, \quad \mathfrak{a}_1\mathfrak{a}_2 = \mathfrak{a}_2\mathfrak{a}_1 = 0,$$

$$\mathfrak{a}_1\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{a}_1, \quad \mathfrak{a}_1\mathfrak{b}_y = q_{12}^N\mathfrak{b}_y\mathfrak{a}_1, \quad \mathfrak{a}_1\mathfrak{b}_2 = q_{12}^N\mathfrak{b}_2\mathfrak{a}_1,$$

$$q_{12}^N\mathfrak{a}_2\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{a}_2, \quad q_{12}^N\mathfrak{a}_2\mathfrak{b}_y = \mathfrak{b}_y\mathfrak{a}_2, \quad \mathfrak{a}_2\mathfrak{b}_2 = \mathfrak{b}_2\mathfrak{a}_2,$$

$$\mathfrak{a}_1\mathfrak{c}_2 = \bar{q}q_{12}^2\mathfrak{c}_2\mathfrak{a}_1, \quad \mathfrak{a}_2\mathfrak{c}_1 = \bar{q}q_{21}^2\mathfrak{c}_1\mathfrak{a}_2,$$

$$\mathfrak{a}_1\mathfrak{c}_1 = \mathfrak{c}_1\mathfrak{a}_1 = \mathfrak{c}_2\mathfrak{a}_2 = \mathfrak{a}_2\mathfrak{c}_2 = 0, \quad \mathfrak{c}_1\mathfrak{a}_2 = \mathfrak{c}_2\mathfrak{a}_1,$$

$$\mathfrak{c}_1^2 = \mathfrak{c}_2^2 = \mathfrak{c}_1\mathfrak{c}_2 = \mathfrak{c}_2\mathfrak{c}_1 = 0,$$

$$\mathfrak{b}_1\mathfrak{b}_y = q_{12}^{N^2}\mathfrak{b}_y\mathfrak{b}_1, \quad \mathfrak{b}_1\mathfrak{b}_2 = q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_1, \quad \mathfrak{b}_y\mathfrak{b}_2 = q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_y,$$

$$q_{12}^N\mathfrak{c}_1\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{c}_1, \quad \mathfrak{c}_1\mathfrak{b}_y = q_{12}^N\mathfrak{b}_y\mathfrak{c}_1, \quad \mathfrak{c}_1\mathfrak{b}_2 = q_{12}^{2N}\mathfrak{b}_2\mathfrak{c}_1,$$

$$q_{12}^{2N}\mathfrak{c}_2\mathfrak{b}_1 = \mathfrak{b}_1\mathfrak{c}_2, \quad q_{12}^N\mathfrak{c}_2\mathfrak{b}_y = \mathfrak{b}_y\mathfrak{c}_2, \quad \mathfrak{c}_2\mathfrak{b}_2 = q_{12}^N\mathfrak{b}_2\mathfrak{c}_2.$$

Proof of Theorems 2.12 and 2.13 We prove Theorem 2.12. Theorem 2.13 can be proved similarly. Consider the minimal resolution (11) showed in Proposition 2.10, we have $\mathrm{Ext}^1_R(\Bbbk,\Bbbk)=\mathrm{Hom}_R(R^2,\Bbbk)$ and $\mathrm{Ext}^2_R(\Bbbk,\Bbbk)=\mathrm{Hom}_R(R^5,\Bbbk)$, since $\Bbbk$ is a simple module. Let $\mathfrak{a}_1,\mathfrak{a}_2\in\mathrm{Ext}^1_R(\Bbbk,\Bbbk)$ be the functions dual to $(1,0)$ and $(0,1)$ respectively. Let $\mathfrak{b}_1,\mathfrak{c}_1,\mathfrak{b}_y,\mathfrak{c}_2,\mathfrak{b}_2\in\mathrm{Ext}^2_R(\Bbbk,\Bbbk)$ be the functions dual to $(1,0,0,0,0),\cdots,(0,0,0,0,1)$ respectively. The relations listed in the theorem can be verified by constructing suitable commutative diagrams, we do this in Appendix 3.2. Let U be an algebra generated by $\mathfrak{b}_1$, $\mathfrak{b}_y$, $\mathfrak{b}_2$ and $\mathfrak{a}_i$, $\mathfrak{c}_i$, $i=1,2$, subject to the relations listed in the theorem. Then any element in U can be written as a linear combination of elements of the form $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}\mathfrak{a}_i^{a_i}$, $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_i^{c_i}$ and $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_1\mathfrak{a}_2$, with b_1, b_2, $b_3\geqslant 0$, a_i, $c_i\in\{0,1\}$, $i=1,2$.

By Lemma 2.3, the algebra U is a quotient of $\mathrm{Ext}^*_R(\Bbbk,\Bbbk)$. When n is odd,

$$\begin{aligned}\dim U_n &= (\tfrac{n-1}{2}+2)(\tfrac{n-1}{2}+1)+\tfrac{1}{2}(\tfrac{n-1}{2})(\tfrac{n-1}{2}+1)\\ &= \tfrac{3n^2+8n+5}{8}.\end{aligned}$$

When n is even,

$$\begin{aligned}\dim U_n &= (\tfrac{n}{2})(\tfrac{n}{2}+1)+\tfrac{1}{2}(\tfrac{n}{2}+1)(\tfrac{n}{2}+2)\\ &= \tfrac{3n^2+10n+8}{8}.\end{aligned}$$

It follows from Corollary 2.9 that $\dim U_n=\dim\mathrm{Ext}^n_R(\Bbbk,\Bbbk)$, for all $n\geqslant 0$, so $U=\mathrm{Ext}^*_R(\Bbbk,\Bbbk)$, which completes the proof of the theorem. □

Remark 2.14. *In* [19, Thm 5.4], *the authors showed that the Ext algebra of a Nichols algebra of finite Cartan type is braided commutative. This coincides with the results we obtain in Theorems 2.12 and 2.13.*

Now we can answer the question whether the Ext algebra of a Nichols algebra is still a Nichols algebra. In general, the answer is negative.

Proposition 2.15. *The Ext algebra of a Nichols algebra of type A_2 is not a Nichols algebra.*

Proof. We consider the case $N=2$ first. Denote the Ext algebra by E. From Proposition 2.1, E is generated by $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{b}$ subject to the relations

$$\mathfrak{a}_2\mathfrak{a}_1=\mathfrak{a}_1\mathfrak{a}_2=0,\quad \mathfrak{a}_1\mathfrak{b}=\mathfrak{b}\mathfrak{a}_1,\quad \mathfrak{a}_2\mathfrak{b}=\mathfrak{b}\mathfrak{a}_2.$$

If E is a Nichols algebra with respect to some braided vector space V, then $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{b}$ should form a basis of V. This is because as an algebra, a Nichols algebra $\mathcal{B}(V)$ is generated by elements in V. With relation $\mathfrak{a}_2\mathfrak{a}_1=\mathfrak{a}_1\mathfrak{a}_2$, $\mathfrak{a}_1\mathfrak{b}=\mathfrak{b}\mathfrak{a}_1$ and $\mathfrak{a}_2\mathfrak{b}=\mathfrak{b}\mathfrak{a}_2$, the vector space V is of diagonal type. This contradicts to the relation $\mathfrak{a}_2\mathfrak{a}_1=\mathfrak{a}_1\mathfrak{a}_2=0$. Therefore, E is not a Nichols algebra. By a similar argument, we can conclude that when $N \geqslant 3$, the Ext algebra is not a Nichols algebra either. $\square$

However, we have the following positive result.

Proposition 2.16. *Let R be a Nichols algebra of type A_2 with $N > 3$. Then $\mathrm{Ext}^*_R(\Bbbk,\Bbbk)/\mathcal{N}$ is a Nichols algebra of diagonal type, where $\mathcal{N}$ is the ideal generated by nilpotent elements.*

Proof. From the proof of Theorem 2.13, the elements $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}\mathfrak{a}_i^{a_i}$, $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_i^{c_i}$ and $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_1\mathfrak{a}_2$, with b_1, b_2, $b_3 \geqslant 0$, a_i, $c_i \in \{0,1\}$, $i = 1,2$ form a basis of $\mathrm{Ext}^*_R(\Bbbk,\Bbbk)$. With the relation listed in that theorem, the elements $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}\mathfrak{a}_i$, $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_i$ and $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_y}\mathfrak{b}_2^{b_2}\mathfrak{c}_1\mathfrak{a}_2$ are nilpotent. However, linear combination of elements $\mathfrak{b}_1^{b_1}\mathfrak{b}_y^{b_2}\mathfrak{b}_2^{b_3}$ are not nilpotent. Then the algebra $\mathrm{Ext}^*_R(\Bbbk,\Bbbk)/\mathcal{N}$ is generated by $\mathfrak{b}_1$, $\mathfrak{b}_y$ and $\mathfrak{b}_2$ subject to the relations

$$\mathfrak{b}_1\mathfrak{b}_y = q_{12}^{N^2}\mathfrak{b}_y\mathfrak{b}_1,\quad \mathfrak{b}_1\mathfrak{b}_2 = q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_1,\quad \mathfrak{b}_y\mathfrak{b}_2 = q_{12}^{N^2}\mathfrak{b}_2\mathfrak{b}_y.$$

It is obvious that it is a Nichols algebra of diagonal type with Cartan matrix of type $A_1 \times A_1 \times A_1$. $\square$

The following corollary is a direct consequence from Theorem 2.12 and 2.13.

Corollary 2.17. *Let $A = u(\mathcal{D}, 0, \mu)$ be a pointed Hopf algebra of type A_2 with $N \geqslant 3$ and $R = \mathcal{B}(V)$ the corresponding Nichols algebra. Then*

$$\mathrm{cx}_R(\Bbbk) = \mathrm{cx}_A(\Bbbk) = 3.$$

In addition, $\mathcal{V}_A(\Bbbk) \cong \mathcal{V}_{(\mathbb{G}\mathrm{r}R)\#\Bbbk\Gamma}(\Bbbk)$.

Proof. For the Nichols algebra R, the complexity

$$\mathrm{cx}_R(\Bbbk) = \gamma(\mathrm{Ext}^*_R(\Bbbk,\Bbbk)) = 3$$

follows directly from Proposition 2.9 or Theorems 2.12 and 2.13. By [19, Lemma 6.1], we have

$$\mathrm{H}^*(u(\mathcal{D}, 0, \mu), \Bbbk) \cong \mathrm{H}^*(u(\mathcal{D}, 0, 0), \Bbbk).$$

In addition, we also have

$$\mathrm{Ext}^*_{u(\mathcal{D},0,0)}(\Bbbk,\Bbbk) \cong \mathrm{Ext}^*_R(\Bbbk,\Bbbk)^{\Gamma}.$$

Observe that for each positive root α, some power of $\mathfrak{b}_\alpha$ is invariant under the group action. Indeed, from the discussion in Section 6 in [17], each $\mathfrak{b}_\alpha$ (denoted by f_α there) can be expressed as a function $R^+ \times R^+ \to \Bbbk$. Then we see that $\mathfrak{b}_\alpha^{M_\alpha}$ is Γ-invariant, where M_α is the integer such that $\chi_\alpha^{M_\alpha} = \varepsilon$. Hence,

$\gamma(\mathrm{H}^*(u(\mathcal{D},0,0),\Bbbk) = 3$, which implies that $\mathrm{cx}_A(\Bbbk) = 3$. With the relations in Theorems 2.12 and 2.13, we see that

$$\mathcal{V}_A(\Bbbk) \cong \mathrm{MaxSpec}(\Bbbk[\mathfrak{b}_1^{m_1}, \mathfrak{b}_y^{m_y}, \mathfrak{b}_2^{m_2}]),$$

where m_1, m_y and m_2 are the least integers such that $\mathfrak{b}_1^{m_1}$, $\mathfrak{b}_y^{m_y}$, $\mathfrak{b}_2^{m_2} \in \mathrm{H}^*(u(\mathcal{D},0,0),\Bbbk)$. That is, $\mathcal{V}_A(\Bbbk)$ is isomorphic to the maximal spectrum of the polynomial algebra $\Bbbk[y_1, y_2, y_3]$. By [19, Thm. 4.1] $\mathcal{V}_{\mathbb{G}\mathrm{r}R\#\Bbbk\Gamma}(\Bbbk)$ is also isomorphic to the maximal spectrum of $\Bbbk[y_1, y_2, y_3]$. So $\mathcal{V}_A(\Bbbk) \cong \mathcal{V}_{\mathbb{G}\mathrm{r}R\#\Bbbk\Gamma}(\Bbbk)$. □

To end this section, we give an easy application of the main theorems. We show that a large class of finite dimensional pointed Hopf algebras of finite Cartan type are wild.

Proposition 2.18. *Let $A = u(\mathcal{D},\lambda,\mu)$ be a pointed Hopf algebra such that the components of the Dynkin diagram are of type A, D, or E, except for A_1 and $A_1 \times A_1$, and the order $N_J > 2$ for at least one component. Then A is wild.*

Proof. In view of [12, Thm. 3.1], we only need to prove that $\mathrm{cx}_A(\Bbbk) \geqslant 3$. Using [19, Lemma 6.1] again, we have $\mathrm{cx}_A(\Bbbk) = \mathrm{cx}_{u(\mathcal{D},\lambda,0)}(\Bbbk)$. However, $u(\mathcal{D},\lambda,0)$ contains a Hopf subalgebra B which is of type A_2 with the order $N \geqslant 3$. Thus $\mathrm{cx}_{u(\mathcal{D},\lambda,0)}(\Bbbk) \geqslant \mathrm{cx}_B(\Bbbk) \geqslant 3$ by [12, Prop 2.1]. □

We conjecture that the isomorphism $\mathcal{V}_A \cong \mathcal{V}_{\mathbb{G}\mathrm{r}R\#\Bbbk\Gamma}$ in Corollary 2.17 holds for general finite dimensional pointed Hopf algebra $A = u(\mathcal{D},\lambda,\mu)$ of finite Cartan type.

3. Appendix

3.1. In this subsection, we verify that the complex (10) in §2.2 is indeed a complex.

The following equations follow directly from Lemma 2.4.

$$Dy = \begin{cases} yD, & \text{if } a_1, a_3 \text{ are both even or both odd;} \\ q_{21}^{-N+2} yD, & \text{if } a_1 \text{ even and } a_3 \text{ is odd;} \\ q_{21}^{N-2} yD, & \text{if } a_1 \text{ odd and } a_3 \text{ is even.} \end{cases} \tag{12}$$

It is clear that $\delta_i^2 = 0$ for $i = 1, 2, 3$. So if a_2 is odd,

$$\partial^2(\Phi(a_1,a_2,a_3)) = ((\delta_3\delta_1 + \delta_1\delta_3 + \tilde{\delta}_2\delta_2) + (\delta_2\delta_3 + \delta_3\delta_2) + (\delta_1\delta_2 + \delta_2\delta_1))\Phi(a_1,a_2,a_3).$$

Put

$$\begin{aligned} A &= (\delta_3\delta_1+\delta_1\delta_3+\tilde{\delta}_2\delta_2)\Phi(a_1,a_2,a_3),\\ B &= (\delta_2\delta_3+\delta_3\delta_2)\Phi(a_1,a_2,a_3),\\ C &= (\delta_1\delta_2+\delta_2\delta_1)\Phi(a_1,a_2,a_3). \end{aligned}$$

We show that $A=B=C=0$.

$$\begin{aligned} A &= (\delta_3\delta_1+\delta_1\delta_3+\tilde{\delta}_2\delta_2)\Phi(a_1,a_2,a_3)\\ &= ((-1)^{a_1-1+a_2}q_{12}^{\sigma(a_3)\tau(a_1-1)}q_{21}^{-\sigma(a_3)\tau(a_2)}[x_1^{\sigma(a_1)},x_2^{\sigma(a_3)}]_c\\ &\quad +(-1)^{a_1}q_{21}^{-\tau(a_1)}yD)\Phi(a_1-1,a_2,a_3-1), \end{aligned}$$

where D satisfies that

$$Dy=-q_{21}^{\tau(a_1-1)}q_{12}^{\sigma(a_3)\tau(a_1-1)}q_{21}^{-\sigma(a_3)\tau(a_2-1)}[x_1^{\sigma(a_1)},x_2^{\sigma(a_3)}]_c.$$

That is,

$$q_{12}^{\sigma(a_3)\tau(a_1-1)}q_{21}^{-\sigma(a_3)\tau(a_2-1)}[x_1^{\sigma(a_1)},x_2^{\sigma(a_3)}]_c+q_{21}^{-\tau(a_1-1)}Dy=0.$$

Hence,

$$q_{12}^{\sigma(a_3)\tau(a_1-1)}q_{21}^{-\sigma(a_3)\tau(a_2)}[x_1^{\sigma(a_1)},x_2^{\sigma(a_3)}]_c+q_{21}^{-\sigma(a_3)}q_{21}^{-\tau(a_1-1)}Dy=0.$$

By equation (12), we have $q_{21}^{-\sigma(a_3)}q_{21}^{-\tau(a_1-1)}Dy=q_{21}^{-\tau(a_1)}yD$. So

$$\begin{aligned} A &= ((-1)^{a_1-1+a_2}q_{12}^{\sigma(a_3)\tau(a_1-1)}q_{21}^{-\sigma(a_3)\tau(a_2)}[x_1^{\sigma(a_1)},x_2^{\sigma(a_3)}]_c\\ &\quad +(-1)^{a_1}q_{21}^{-\tau(a_1)}yD)\Phi(a_1-1,a_2,a_3-1)\\ &= ((-1)^{a_1-1+a_2}q_{12}^{\sigma(a_3)\tau(a_1-1)}q_{21}^{-\sigma(a_3)\tau(a_2)}[x_1^{\sigma(a_1)},x_2^{\sigma(a_3)}]_c\\ &\quad +(-1)^{a_1}q_{21}^{-\sigma(a_3)}q_{21}^{-\tau(a_1-1)}Dy)\Phi(a_1-1,a_2,a_3-1)\\ &= 0. \end{aligned}$$

The equations $B=0$ and $C=0$ can be verified directly. For example,

$$\begin{aligned} B &= (\delta_2\delta_3+\delta_3\delta_2)(\Phi(a_1,a_2,a_3))\\ &= ((-1)^{a_1}q_{21}^{-\sigma(a_2)\tau(a_1)}y^{\sigma(a_2)}(-1)^{a_1+a_2-1}q_{12}^{\sigma(a_3)\tau(a_1)}q_{21}^{-\sigma(a_3)\tau(a_2-1)}x_2^{\sigma(a_3)}\\ &\quad +(-1)^{a_1+a_2}q_{12}^{\sigma(a_3)\tau(a_1)}q_{21}^{-\sigma(a_3)\tau(a_2)}x_2^{\sigma(a_3)}(-1)^{a_1}q_{21}^{-\sigma(a_2)\tau(a_1)}y^{\sigma(a_2)})\\ &\quad \Phi(a_1,a_2-1,a_3-1)\\ &= 0, \end{aligned}$$

since $\tau(a_2-1)+\sigma(a_2)=\tau(a_2)$.

If a_2 is even, then

$$\begin{aligned} \partial^2(\Phi(a_1,a_2,a_3)) &= ((\delta_1\delta_3+\delta_3\delta_1+\delta_2\tilde{\delta}_2)+(\delta_1\delta_2+\delta_2\delta_1)+(\delta_3\delta_2+\delta_2\delta_3)\\ &\quad +(\tilde{\delta}_2\delta_1+\delta_1\tilde{\delta}_2)+(\tilde{\delta}_2\delta_3+\delta_3\tilde{\delta}_2))\Phi(a_1,a_2,a_3). \end{aligned}$$

The equation $(\delta_1\delta_3+\delta_3\delta_1+\delta_2\tilde{\delta}_2)\Phi(a_1,a_2,a_3)=0$ follows directly from the definition of $\tilde{\delta}_2$. As in the case in which a_2 is odd,

$$(\delta_2\delta_3+\delta_3\delta_2)\Phi(a_1,a_2,a_3)=0 \text{ and } (\delta_1\delta_2+\delta_2\delta_1)\Phi(a_1,a_2,a_3)=0$$

can be also verified via a straightforward computation. Now, we show that $(\tilde{\delta}_2\delta_1+\delta_1\tilde{\delta}_2)\Phi(a_1,a_2,a_3)=0$ case by case, using Lemma 2.4.

Case (i) a_1 and a_3 are both odd,

$$\begin{aligned}&(\tilde{\delta}_2\delta_1+\delta_1\tilde{\delta}_2)\Phi(a_1,a_2,a_3)\\=\;&(x_1(q_{21}^{-\frac{a_2}{2}}x_1^{N-2})-q_{21}^{-\frac{a_2}{2}}x_1^{N-1})\Phi(a_1-2,a_2+1,a_3)\\=\;&0.\end{aligned}$$

Case (ii) a_1 is odd and a_3 is even,

$$\begin{aligned}&(\tilde{\delta}_2\delta_1+\delta_1\tilde{\delta}_2)\Phi(a_1,a_2,a_3)\\=\;&(x_1(-q_{12}^{(N-1)(\frac{a_1-3}{2}N+1)}q_{21}^{-(N-1)\frac{a_2}{2}N}q_{21}^{\frac{a_1-3}{2}N+1})(k_1x_1^{N-2}x_2^{N-2}+\cdots\\&+k_{N-2}y^{N-3}x_1x_2+k_{N-1}y^{N-2})\\&+q_{12}^{(N-1)\frac{a_1-1}{2}N}q_{21}^{-(N-1)\frac{a_2}{2}N}\bar{q}q_{21}^{-(N-2)}q_{21}^{\frac{a_1-1}{2}N}x_2^{N-2}x_1^{N-1})\Phi(a_1-2,a_2+1,a_3)\\=\;&q_{12}^{(N-1)\frac{a_1-1}{2}N}q_{21}^{-(N-1)\frac{a_2}{2}N}\bar{q}q_{21}^{-(N-2)}q_{21}^{\frac{a_1-1}{2}N}\\&(-q_{12}^{-N^2+2N}x_1(k_1x_1^{N-2}x_2^{N-2}+\cdots+k_{N-2}y^{N-3}x_1x_2+k_{N-1}y^{N-2})\\&+x_2^{N-2}x_1^{N-1})\Phi(a_1-2,a_2+1,a_3)\\=\;&0,\end{aligned}$$

since $q_{12}^{-N^2+2N}x_1[x_1^{N-1},x_2^{N-1}]_c=x_2^{N-2}x_1^{N-1}y$.

Case (iii) a_1 is even and a_3 is odd,

$$\begin{aligned}&(\tilde{\delta}_2\delta_1+\delta_1\tilde{\delta}_2)\Phi(a_1,a_2,a_3)\\=\;&-q_{21}^{-\frac{a_2}{2}N}x_1^{N-1}+q_{21}^{-\frac{a_2}{2}N}x_1^{N-1}\Phi(a_1-2,a_2+1,a_3)\\=\;&0.\end{aligned}$$

Case (iv) a_1 and a_3 are both even,

$$\begin{aligned}&(\tilde{\delta}_2d_1+d_1\tilde{d}_2)\Phi(a_1,a_2,a_3)\\=\;&x_1^{N-1}(q_{12}^{(N-1)\frac{a_1-2}{2}N}q_{21}^{-(N-1)\frac{a_2}{2}N}\bar{q}q_{21}^{-(N-2)}q_{21}^{\frac{a_1-2}{2}N}x_2^{N-2})\\&+(-q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)}q_{21}^{-(N-1)\frac{a_2}{2}N}q_{21}^{\frac{a_1-2}{2}N+1})(k_1x_1^{N-2}x_2^{N-2}+\cdots\\&+k_{N-2}y^{N-3}x_1x_2+k_{N-1}y^{N-2})x_1\Phi(a_1-2,a_2+1,a_3)\\=\;&(q_{12}^{(N-1)(\frac{a_1-2}{2}N+1)}q_{21}^{-(N-1)\frac{a_2}{2}N}q_{21}^{\frac{a_1-2}{2}N+1})(x_1^{N-1}x_2^{N-2}\\&-(k_1x_1^{N-2}x_2^{N-2}+\cdots+k_{N-2}y^{N-3}x_1x_2+k_{N-1}y^{N-2})x_1)\Phi(a_1-2,a_2+1,a_3)\\=\;&0,\end{aligned}$$

since $[x_1^{N-1},x_2^{N-1}]_cx_1=yx_1^{N-1}x_2^{N-2}$.

Similarly, we can prove that $(\tilde{\delta}_2\delta_3+\delta_3\tilde{\delta}_2)\Phi(a_1,a_2,a_3)=0$.

In conclusion, we have $\partial^2=0$.

3.2. In this subsection, we give the necessary commutative diagrams to check the relations in Theorems 2.12 and 2.13.

Set

$$X_1 = q_{12}^{-(N-1)(N-3)} x_1^{N-3} x_2^{N-3} + k_1 y x_1^{N-4} x_2^{N-4} + \cdots + k_{N-3} y^{N-3},$$

where $k_i \in \Bbbk$, $1 \leqslant i \leqslant N-3$, such that $x_2^{N-1} x_1^{N-3} = X_1 x_2^2$, and

$$X_2 = q_{12}^{(N-3)(N-1)} x_2^{N-3} x_1^{N-3} + l_1 y x_2^{N-4} x_1^{N-4} + l_2 y^2 x_2^{N-5} x_1^{N-5} + \cdots + l_{N-3} y^{N-3},$$

where $l_i \in \Bbbk$, $1 \leqslant i \leqslant N-3$, such that $x_1^{N-1} x_2^{N-3} = X_2 x_1^2$.

Let f_1^i, f_2^i , f_3^i and g_1^j, g_2^j , g_3^j, $1 \leqslant i \leqslant 5$ and $j = 1, 2$ be the morphisms described by the following matrices:

f_1^i is the 5×1 matrix with 1 in the i-th position and 0 elsewhere,

$$f_2^1 = \begin{pmatrix} 1 & 0 \\ 0 & q_{12}^N \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, f_2^2 = \begin{pmatrix} 0 & 0 \\ x_1^{N-3} & 0 \\ 0 & 0 \\ 0 & 1 \\ q_{12} q_{21}^{1-N} y^{N-2} x_2 & -q_{21}^{1-N} y^{N-2} x_1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$f_2^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, f_2^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -q_{12}^2 q_{21}^{N-1} y^{N-2} x_2 & q_{12} q_{21}^{N-1} y^{N-2} x_1 \\ 1 & 0 \\ 0 & 0 \\ 0 & q_{12}^N x_2^{N-3} \\ 0 & 0 \end{pmatrix}, f_2^5 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$f_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & q_{12}^N & 0 & 0 & 0 \\ 0 & 0 & q_{12}^N & 0 & 0 \\ 0 & 0 & 0 & q_{12}^N & 0 \\ 0 & 0 & 0 & 0 & q_{12}^N \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, f_3^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{12}^{-N^2+N} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^{N^2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ q_{21}^{-1} y^{N-2} x_2 & 0 & 0 & 0 & 0 \\ 0 & q_{12}^{-N} x_1^{N-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & X_1 & 0 \\ 0 & q_{21}^{1-N} q_{12}^2 y^{N-2} x_2 & q_{12}^N & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{12} q_{21}^{N-3} y^{N-2} x_2 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^{2N} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_3^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{12}^{-N} & 0 & 0 & 0 & 0 \\ 0 & q_{12}^{-N^2+2N} X_2 & 0 & 0 & 0 \\ 0 & q_{12} q_{21}^{-N+3} y^{N-2} x_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{21}^N & q_{21}^{N-1} y^{N-2} x_1 & 0 \\ 0 & 0 & 0 & q_{12}^{2N} x_2^{N-3} & 0 \\ 0 & 0 & 0 & 0 & q_{21}^{-N+1} q_{12}^2 y^{N-2} x_1 \\ 0 & 0 & 0 & 0 & q_{12}^N \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f_3^5 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ q_{12}^{-N^2+N} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$g_1^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, g_1^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$g_2^1 = \begin{pmatrix} x_1^{N-2} & 0 \\ \bar{q} q_{12}^2 x_2 & (-q_{12} - \bar{q} q_{12}) x_1 \\ 0 & -q_{12} y^{N-1} \\ 0 & x_2 \\ 0 & 0 \end{pmatrix}, g_3^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & (-q_{12} - \bar{q} q_{12}) x_1^{N-3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{q} q_{12}^2 & 0 \\ 0 & 0 & q_{12}^N & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{12}^N \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$g_2^2 = \begin{pmatrix} 0 & 0 \\ x_1 & 0 \\ y^{N-1} & 0 \\ (-q_{21} - \bar{q} q_{21}) x_2 & \bar{q} q_{21}^2 x_1 \\ 0 & x_2^{N-2} \end{pmatrix}, g_3^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_{21}^N & 0 & 0 \\ 0 & \bar{q} q_{21}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_{12}^N (-q_{21} - \bar{q} q_{21}) x_2^{N-3} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then we have the following commutative diagrams

$$\begin{array}{ccccccccc} R^{12} & \longrightarrow & R^7 & \longrightarrow & R^5 & \longrightarrow & R^2 & \longrightarrow & R \ , \\ \downarrow{\scriptstyle f_3^i} & & \downarrow{\scriptstyle f_2^i} & & \downarrow{\scriptstyle f_1^i} & \searrow & & & \\ R^5 & \longrightarrow & R^2 & \longrightarrow & R & \longrightarrow & \Bbbk & & \end{array} \tag{13}$$

(14)
$$\begin{array}{ccccccccc}
R^7 & \longrightarrow & R^5 & \longrightarrow & R^2 & \longrightarrow & R & \longrightarrow & \Bbbk\ . \\
{\scriptstyle g_3^i}\downarrow & & {\scriptstyle g_2^i}\downarrow & & {\scriptstyle g_1^i}\downarrow & \searrow & & & \\
R^5 & \longrightarrow & R^2 & \longrightarrow & R & \longrightarrow & \Bbbk & &
\end{array}$$

It is also routine to check the commutativity of the diagrams (13) and (14). But we need to mention that the following equations hold

$$X_1(\bar{q}q_{21}^2x_1x_2 - (q_{21} + \bar{q}q_{21})x_2x_1) = -q_{12}^{-N^2+2N}\overline{D},$$
$$X_2(\bar{q}q_{12}^2x_2x_1 - (q_{12} + \bar{q}q_{12})x_1x_2) = -\overline{D},$$

which follow from Lemma 2.5 and the following two equations

$$\begin{aligned}
q_{12}^{-N^2+2N}\overline{D}x_2 &= x_2^{N-1}x_1^{N-2} \\
&= X_1x_2^2x_1 \\
&= X_1(-\bar{q}q_{21}^2x_1x_2 + (q_{21} + \bar{q}q_{21})x_2x_1)x_2,
\end{aligned}$$

$$\begin{aligned}
\overline{D}x_1 &= x_1^{N-1}x_2^{N-2} \\
&= X_2x_1^2x_2 \\
&= X_2(-\bar{q}q_{12}^2x_2x_1 + (q_{12} + \bar{q}q_{12})x_1x_2)x_1.
\end{aligned}$$

References

[1] N. Andruskiewitsch, *Some remarks on Nichols algebras*, Hopf algebras, Lecture Notes in Pure and Appl. Math., **237**, 35-45, Dekker, New York, 2004.

[2] N. Andruskiewitsch and S. Dăscălescu, *On Finite Quantum Groups at -1*, Algebr. Represent. Theory, **8** (2005), no. 1, 11-34.

[3] N. Andruskiewitsch, I. Heckenberger and H.-J. Schneider, *Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132** (2010), no. 6, 1493-1547.

[4] N. Andruskiewitsch and H.-J. Schneider, *Pointed Hopf algebras*, New Directions in Hopf Algebras, MSRI Publications **43**, 1-68, Cambridge Univ. Press, 2002.

[5] N. Andruskiewitsch and H.-J. Schneider, *On the classification of finite-dimensional pointed Hopf algebras*, Ann. Math. **171** (2010), 375-417.

[6] P. A. Bergh and S. Oppermann, *Cohomology of twisted tensor products*, J. Algebra, **320** (2008), no. 8, 3327–3338.

[7] J.E. Bjork, *Rings of differential operators*, North Holland Mathematical Library, 1979.

[8] T. Cassidy and B. Shelton, *Generalizing the notion of Koszul algebra*, Math. Z., **260** (2008), no. 1, 93-114.

[9] K. Erdmann, M. Holloway, R. Taillefer and N. Snashall, *Support varieties for selfinjective algebras*, K-Theory, **33** (2004), no. 1, 67–87.

[10] C. De Concini and C. Procesi, *Quantum groups*, D-Modules, Representation Theory, and Quantum Groups, Lecture Notes in Math. **1565**, 31-140, Springer-Verlag, Berlin, 1993.

[11] F. Fantino and G. A. Garcia, *On pointed Hopf algebras over dihedral groups*, arXiv: math.QA/1007.0227.

[12] J. Feldvoss and S. Witherspoon, *Support varieties and represetation type of small quantum groups*, Int. Math. Res. Not. IMRN 2010, no. 7, 1346-1362.

[13] D. Fischman, S. Montgomery and H.-J. Schneider, *Frobenius extensions of subalgebras of Hopf algebras*, Trans. Amer. Math. Soc., **349** (1997), 4857-4895.

[14] V. Ginzburg, *Calabi-Yau algebras*, arXiv:math.AG/0612139.

[15] V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J., **69** (1993), no. 1, 179-198.

[16] D.-M. Lu, J. H. Palmieri, Q.-S. Wu and J. J. Zhang, *Regular algebras of dimension 4 and their A_∞-Ext-algebras*, Duke Math. J., **137** (2007), 537-584.

[17] M. Mastnak and S. Witherspoon, *Bialgebra cohomology, pointed Hopf algebras, and deformations*, J. Pure and Applied Algebra, **213** (2009), no. 7, 1399-1417.

[18] J. Pevtsova and S. Witherspoon, *Varieties for Modules of Quantum Elementary Abelian Groups*, Algebr. Represent. Theory, **12** (2009), no. 6, 567-595.

[19] M. Mastnak, J. Pevtsova, P. Schauenburg and S. Witherspoon, *Cohomology of finite dimensional pointed Hopf algebras*, Proc. London Math. Soc., **100** (2010), no. 2, 377-404.

[20] S. Scherotzke and M. Towers, *Rank varieties for Hopf algebras*, arXiv: math.RT/0906.4213, to appear in J. Pure Appl. Algebra.

[21] N. Snashall and Ø. Solberg, *Support Varieties and Hochschild Cohomology Rings*, Proc. London Math. Soc. **88** (2004), no. 3, 705-732.

Xiaolan YU
Department WNI, University of Hasselt, Universitaire Campus, 3590 Diepeenbeek,Belgium

E-mail address: xiaolan.yu@uhasselt.be

Yinhuo ZHANG
Department WNI, University of Hasselt, Universitaire Campus, 3590 Diepeenbeek,Belgium

E-mail address: yinhuo.zhang@uhasselt.be