

CONTINUITY OF THE CONE SPECTRAL RADIUS

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ABSTRACT. This paper concerns the question whether the cone spectral radius $r_C(f)$ of a continuous compact order-preserving homogenous map $f: C \rightarrow C$ on a closed cone C in Banach space X depends continuously on the map. Using the fixed point index we show that if there exists $0 < a_1 < a_2 < a_3 < \dots$ **not** in the cone spectrum, $\sigma_C(f)$, and $\lim_{k \rightarrow \infty} a_k = r_C(f)$, then the cone spectral radius is continuous. An example is presented showing that if such a sequence $(a_k)_k$ does not exist, continuity may fail. We also analyze the cone spectrum of continuous order-preserving homogeneous maps on finite dimensional closed cones. In particular, we prove that if C is a polyhedral cone with m faces, then $\sigma_C(f)$ contains at most $m - 1$ elements, and this upper bound is sharp for each polyhedral cone. Moreover, for each non-polyhedral cone there exist maps whose cone spectrum is infinite.

1. INTRODUCTION

In their seminal paper [5] Krein and Rutman showed, besides the now famous Krein-Rutman Theorem, that some of the spectral properties of compact linear maps leaving invariant a closed cone C in Banach space X , can be generalized to certain classes of nonlinear maps. In particular, they considered continuous compact maps $f: C \rightarrow C$ which are homogeneous of degree one and which preserve the partial ordering induced by C .

If $f: C \rightarrow C$ is a continuous, homogeneous of degree one map which preserves the partial ordering induced by C , one can define the *Bonsall cone spectral radius* of f by

$$\tilde{r}_C(f) = \lim_{k \rightarrow \infty} \|f^k\|_C^{1/k} = \inf_{k \geq 1} \|f^k\|_C^{1/k}$$

where

$$\|f^k\|_C = \sup\{\|f^k(x)\| : x \in C \text{ and } \|x\| \leq 1\}.$$

This definition of the cone spectral radius was introduced by Bonsall [1] in the context of linear maps on cones. An alternative notion of the cone spectral radius was considered by Mallet-Paret and Nussbaum in [8, Section 2], and is defined as follows. For $x \in C$, let

$$\mu(x) = \limsup_{k \rightarrow \infty} \|f^k(x)\|^{1/k}$$

and define $r_C(f) = \sup\{\mu(x) : x \in C\}$. In many important cases it is known that $\tilde{r}_C(f) = r_C(f)$: see [8, Theorems 2.2 and 2.3] and [9, Theorem 3.4]. In particular,

Date: July 2011.

2010 Mathematics Subject Classification. Primary 47H07; Secondary 47H10, 47H14.

Key words and phrases. Cone spectral radius, continuity, cone spectrum, nonlinear cone maps, fixed point index.

The second author was partially supported by NSF DMS-0701171.

if f^m is compact for some integer $m \geq 1$, then $\tilde{r}_C(f) = r_C(f)$. It remains an open question whether the equality holds for general nonlinear maps $f: C \rightarrow C$ as above.

If C is a closed cone in a Banach space X and $f: C \rightarrow C$ is a continuous, compact, homogeneous (degree one) map preserving the partial ordering induced by C and $r_C(f) > 0$, then it follows from [14, Theorem 2.1] that there exists $u \in C$ with $\|u\| = 1$ such that

$$f(u) = r_C(f)u.$$

More precisely, if one applies [14, Theorem 2.1] to the map $x \mapsto (1/t)f(x)$, where $0 < t < r_C(f)$, one deduces as a special case that f has an eigenvector of norm one in C with eigenvalue greater than or equal to t . A simple limiting argument implies that f has an eigenvector in C with eigenvalue $r_C(f)$. More general results concerning the existence of an eigenvector with eigenvalue $r_C(f)$ can be found in [8, 9].

In this paper we study the continuity of $r_C(f)$. More precisely we discuss the following problem. Suppose that $f: C \rightarrow C$ and $f_k: C \rightarrow C$, for $k \geq 1$, are continuous compact homogeneous order-preserving maps such that

$$(1.1) \quad \lim_{k \rightarrow \infty} \|f - f_k\|_C = 0.$$

Under what further conditions on f does the equality,

$$(1.2) \quad \lim_{k \rightarrow \infty} r_C(f_k) = r_C(f),$$

hold? A similar problem was considered by Kakutani [18, pp.282–283], who gave an example of a bounded linear map on ℓ_2 whose spectral radius is discontinuous in the C^* -algebra of bounded linear maps on ℓ_2 . The problem was further studied in the setting of Banach algebras in [3, 10, 11, 12, 13].

By using the fixed point index, we prove that if there exist $0 < a_1 < a_2 < a_3 < \dots$ such that

$$\lim_{k \rightarrow \infty} a_k = r_C(f)$$

and each a_k is **not** in the cone spectrum,

$$\sigma_C(f) = \{\rho \geq 0: f(x) = \rho x \text{ for some } x \in C \text{ with } x \neq 0\},$$

of f , then (1.2) holds. In addition, we provide an example of a linear map $T: C \rightarrow C$ on a closed total cone C in an infinite dimensional Banach space X , such that T is compact on C , but not compact as a linear map on the whole of X , and for which $(0, r_C(T)] = \sigma_C(T)$, $r_C(T) > 0$, and (1.2) fails. We will also give an example of a continuous order-preserving homogeneous map f on a finite dimensional closed cone C with $\sigma_C(f) = [0, 1]$. So far, however, no finite dimensional example is known for which (1.2) fails. If C is a polyhedral cone with m faces, we will prove that $\sigma_C(f)$ contains at most $m - 1$ points, and show that this upper bound is sharp for every polyhedral cone. Thus, the cone spectral radius is continuous for maps on polyhedral cones. We will also show that on each finite dimensional closed non-polyhedral cone C , there exists a continuous order-preserving homogeneous map $f: C \rightarrow C$ such that $\sigma_C(f)$ contains a countably infinite number of distinct points.

2. A CONDITION ON THE CONE SPECTRUM

Throughout the exposition X will be a real Banach space and C will be a closed cone in X . Recall that $C \subseteq X$ is a *cone* if C is convex, $\lambda C \subseteq C$ for all $\lambda \geq 0$, and $C \cap (-C) = \{0\}$. A cone C in X is said to be *total* if $X = \text{cl}(C - C)$. A cone C

induces a partial ordering, \leq , on X by $x \leq y$ if $y - x \in C$. A map $f: C \rightarrow C$ is said to be *homogeneous* (of degree one) if $\lambda f(x) = f(\lambda x)$ for all $\lambda \geq 0$ and $x \in C$. It is said to be *order-preserving* if $f(x) \leq f(y)$ whenever $x \leq y$. Furthermore $f: C \rightarrow C$ is called *compact* if for each $D \subseteq C$ bounded, $\text{cl}(f(D))$ is compact.

The following lemma is elementary, but useful in the sequel.

Lemma 2.1. *If S is compact subset of a complete metric space (M, d) and $(x_k)_k$ is a sequence in M with*

$$(2.1) \quad \lim_{k \rightarrow \infty} d(x_k, S) = 0,$$

then $(x_k)_k$ has a convergent subsequence with limit in S .

Proof. Recall that S is compact in (M, d) if and only if every sequence in S has a convergent subsequence. Let $a_k = d(x_k, S)$ and let $b_k = a_k + 1/k$, so $b_k \rightarrow 0$ as $k \rightarrow \infty$. From the assumption we know that there exists $s_k \in S$ such that $d(x_k, s_k) < b_k$. As S is compact, the sequence $(s_k)_k$ has a convergent subsequence with limit $s \in S$. The corresponding subsequence of $(x_k)_k$ also converges to s . \square

Another useful basic lemma is the following.

Lemma 2.2. *Let C be a closed cone in X and $f: C \rightarrow C$ be a continuous homogeneous order-preserving map. If $av \leq f(v)$ for some $v \in C \setminus \{0\}$, then $a \leq r_C(f)$.*

Proof. Assume, for the sake of contradiction, that $a > r_C(f)$. Select $\epsilon > 0$ such that $a > r_C(f) + \epsilon$. A simple induction argument shows that $v \leq (1/a)^k f^k(v)$ for all $k \geq 1$. The definition of $r_C(f)$, however, implies that $\|f^k(v)\| \leq (r_C(f) + \epsilon)^k$ for all k sufficiently large. Since $a > r_C(f) + \epsilon$, we deduce that $(1/a)^k f^k(v) \rightarrow 0$ as $k \rightarrow \infty$. But $(1/a)^k f^k(v) - v \in C$ for all k ; so, letting $k \rightarrow \infty$ we find that $-v \in C$, which is impossible, as $v \in C \setminus \{0\}$ and $C \cap (-C) = \{0\}$. \square

The next lemma shows that the cone spectral radius is upper-semicontinuous.

Lemma 2.3. *Suppose that C is a closed cone in X and $f: C \rightarrow C$ is a continuous compact homogeneous order-preserving map. If for each $k \geq 1$, $f_k: C \rightarrow C$ is a continuous compact homogeneous order-preserving map such that*

$$\lim_{k \rightarrow \infty} \|f - f_k\|_C = 0,$$

then $\limsup_{k \rightarrow \infty} r_C(f_k) \leq r_C(f)$.

Proof. Let $\lambda = \limsup_{k \rightarrow \infty} r_C(f_k)$. Note that the case $\lambda = 0$ is trivial. So, assume that $\lambda > 0$ and write $\lambda_k = r_C(f_k)$ for all $k \geq 1$. By [14, Theorem 2.1] and the fact that $r_C(f_k) = \tilde{r}_C(f_k)$, see [8, Theorems 2.2 and 2.3], there exist $x_k \in C$ with $\|x_k\| = 1$ and $f_k(x_k) = \lambda_k x_k$ for all $k \geq 1$. By taking a subsequence we may assume that $\lim_{k \rightarrow \infty} \lambda_k = \lambda$.

Let $\Sigma = \{x \in C: \|x\| = 1\}$ and note that $\text{cl}(f(\Sigma))$ is compact. Moreover, as $\lim_{k \rightarrow \infty} \|f - f_k\|_C = 0$,

$$\lim_{k \rightarrow \infty} d(f_k(x_k), \text{cl}(f(\Sigma))) = 0.$$

By Lemma 2.1 $(f_k(x_k))_k$ has a convergent subsequence $(f_{k_i}(x_{k_i}))_i$ with limit $v \in C$. Put $u = v/\lambda$. As $f_{k_i}(x_{k_i}) = \lambda_{k_i} x_{k_i}$ and $\lim_{i \rightarrow \infty} \lambda_{k_i} = \lambda$, we see that

$$\lim_{i \rightarrow \infty} \|x_{k_i} - u\| = \lim_{i \rightarrow \infty} \left\| \frac{\lambda_{k_i}}{\lambda} x_{k_i} - u \right\| = \lim_{i \rightarrow \infty} \frac{1}{\lambda} \|\lambda_{k_i} x_{k_i} - v\| = 0.$$

As f is continuous and $\lim_{i \rightarrow \infty} \|f(x_{k_i}) - f_{k_i}(x_{k_i})\| = 0$, we deduce that

$$\lim_{i \rightarrow \infty} \|f(x_{k_i}) - \lambda_{k_i} x_{k_i}\| = 0 = \|f(u) - \lambda u\|.$$

Thus, $f(u) = \lambda u$ and hence $\lambda \leq r_C(f)$ by Lemma 2.2. \square

The following theorem gives a condition on $\sigma_C(f)$ that insures that the cone spectral radius is lower-semicontinuous. The proof relies on ideas from topological degree theory. More to the point, it uses basic properties of the fixed point index, $i_W(g, V)$, where W is a closed convex subset of X , V is a relatively open subset of W , and $g: V \rightarrow W$ is compact. A discussion of the fixed point index in this setting along with further references to the literature can be found in [4] and in [15]. In our case W will be equal to C .

Theorem 2.4. *Suppose that C is a closed cone in X and $f: C \rightarrow C$ is a continuous compact homogeneous order-preserving map. If for each $k \geq 1$, $f_k: C \rightarrow C$ is a continuous compact homogeneous order-preserving map such that*

$$\lim_{k \rightarrow \infty} \|f - f_k\|_C = 0,$$

then for each $0 < \lambda < r_C(f)$ with $\lambda \notin \sigma_C(f)$ we have that $\liminf_{k \rightarrow \infty} r_C(f_k) \geq \lambda$.

Proof. For simplicity write $r = r_C(f)$ and $r_k = r_C(f_k)$. Define $g: C \rightarrow C$ by $g(x) = \lambda^{-1}f(x)$ for $x \in C$. Similarly, let $g_k: C \rightarrow C$ be given by $g_k(x) = \lambda^{-1}f_k(x)$ for all $x \in C$. Clearly

$$r_C(g) = \lambda^{-1}r > 1 \quad \text{and} \quad r_C(g_k) = \lambda^{-1}r_k.$$

Write $\rho = \lambda^{-1}r$ and $\rho_k = \lambda^{-1}r_k$. We wish to show that $\liminf_{k \rightarrow \infty} \rho_k \geq 1$.

Let $\Sigma = \{x \in C: \|x\| = 1\}$ and denote $V = \{x \in C: \|x\| < 1\}$. Remark that as $\lambda \notin \sigma_C(f)$,

$$(2.2) \quad g(x) \neq x \quad \text{for all } x \in \Sigma.$$

Furthermore there exists $\delta > 0$ such that

$$(2.3) \quad \|g(x) - x\| > \delta \quad \text{for all } x \in \Sigma.$$

Indeed, if $(z_k)_k$ in Σ and $\lim_{k \rightarrow \infty} \|g(z_k) - z_k\| = 0$, then it follows from Lemma 2.1 that $(z_k)_k$ has a convergent subsequence with limit $\xi \in \Sigma$. Continuity of g implies that $g(\xi) = \xi$, which is impossible

We claim that for all k sufficiently large,

$$(2.4) \quad (1-t)g(x) + tg_k(x) \neq x \quad \text{for all } x \in \Sigma \text{ and } 0 \leq t \leq 1.$$

For the sake of contradiction suppose that there exist a sequence $(x_k)_k$ of distinct points in Σ and a sequence $(t_k)_k$ in $[0, 1]$ such that $(1-t_k)g(x_k) + t_k g_k(x_k) = x_k$. As $\lim_{k \rightarrow \infty} \|g - g_k\|_C = 0$,

$$\lim_{k \rightarrow \infty} d(x_k, \text{cl}(g(\Sigma))) = 0.$$

Using the compactness of g we see that $(x_k)_k$ has a convergent subsequence $(x_{k_i})_i$ with limit $\zeta \in \Sigma$ by Lemma 2.1. The continuity of g now implies that

$$g(\zeta) = \lim_{i \rightarrow \infty} g(x_{k_i}) = \lim_{i \rightarrow \infty} (1-t_{k_i})g(x_{k_i}) + t_{k_i}g_{k_i}(x_{k_i}) = \zeta,$$

which contradicts (2.2).

We now utilize the fixed point index as defined in [15]. In particular, if we apply the homotopy property of the fixed point index we find that

$$(2.5) \quad i_C(g, V) = i_C(g_k, V) \quad \text{for all } k \text{ sufficiently large.}$$

It was proved in [14] as a special case of Theorem 2.1 that $i_C(g, V) = 0$. Using this fact the proof can be completed as follows. From (2.5) we deduce that $i_C(g_k, V) = 0$ for all k sufficiently large. Now suppose that for some large k , $r_C(g_k) \leq 1$. By (2.4) there exists no fixed point $x \in \Sigma$ of g_k ; so, $r_C(g_k) < 1$. Consider the homotopy $tg_k(x)$ for $0 \leq t \leq 1$. By assumption $tg_k(x) \neq x$ for all $0 \leq t \leq 1$ and $x \in \Sigma$. It follows that $i_C(g_k, V) = i_C(h, V)$ where $h(x) = 0$ for all $x \in V$. If $D := \{0\}$, $h(V)$ is contained in D , and D is contained in V , so the commutativity property of the fixed point index (see [4] or [15]) implies that $i_C(h, V) = i_D(h, D)$. The normalization property of the fixed point index (see [4] or [15]) implies that $i_D(h, D)$ is the Lefschetz number of the map $h: D \rightarrow D$, which equals 1. It follows that $i_C(h, V) = i_D(h, D) = 1$, which is a contradiction. So, $r_C(g_k) > 1$ for all k sufficiently large, and hence $\liminf_{k \rightarrow \infty} r_C(g_k) \geq 1$. \square

The fact that the maps f_k in Theorem 2.4 are homogeneous and order-preserving plays a limited role in the proof. Indeed, suppose that f , C , V and λ are as in Theorem 2.4. For $k \geq 1$ suppose that $f_k: \text{cl}(V) \rightarrow C$ is continuous and compact, and $\sup\{\|f_k(x) - f(x)\|: x \in C \text{ and } \|x\| = 1\} \rightarrow 0$ as $k \rightarrow \infty$. Then the proof of Theorem 2.4 shows that for all k large, there exist $x_k \in C$, with $\|x_k\| = 1$, and $\lambda_k > 0$ such that $f_k(x_k) = \lambda_k x_k$ and $\liminf_{k \rightarrow \infty} \lambda_k \geq \lambda$.

A continuous compact order-preserving homogeneous map $f: C \rightarrow C$ on a closed cone C in X is said to have a *continuous cone spectral radius* if for each sequence $f_k: C \rightarrow C$ of continuous compact order-preserving homogeneous maps with $\lim_{k \rightarrow \infty} \|f - f_k\|_C = 0$ we have that

$$\lim_{k \rightarrow \infty} r_C(f_k) = r_C(f).$$

Thus, a combination of Lemma 2.3 and Theorem 2.4 yields the following result.

Theorem 2.5. *If $f: C \rightarrow C$ is a continuous compact order-preserving homogeneous map and either, $r_C(f) = 0$, or, there exist $0 < a_1 < a_2 < a_3 < \dots$ not in $\sigma_C(f)$ with $\lim_{k \rightarrow \infty} a_k = r_C(f)$, then f has a continuous cone spectral radius.*

In particular we see that every compact linear map $T: X \rightarrow X$ leaving invariant a closed cone C , has a continuous cone spectral radius, as the cone spectrum is totally disconnected.

3. AN EXAMPLE

In this section we give an example of a linear map $T: C \rightarrow C$ on a closed, total cone in a Banach space X , which does not have a continuous cone spectral radius. The map $T: C \rightarrow C$ is compact, but does not have a continuous compact linear extension as a map from X to X ; so, we do not contradict the final remark in the previous section. We follow Bonsall [1, §2] and define Y to be the Banach space of all continuous functions on $[0, 1]$ with $f(0) = 0$, and uniform-norm $\|f\| = \max\{|f(t)|: 0 \leq t \leq 1\}$. Let $K = \{f \in Y: f \text{ is nonnegative and convex on } [0, 1]\}$. As Bonsall remarks, K is a closed total cone in Y , because the subspace U of twice continuously differentiable functions on $[0, 1]$ is dense in Y and contained in $K - K$.

Let $T: K \rightarrow K$ be defined by

$$Tf(t) = f(t/2) \quad \text{for all } f \in Y \text{ and } 0 \leq t \leq 1.$$

Bonsall [1, §4] shows that T is a compact linear map on K , but its extension to Y is not compact. Let us recall the argument that shows that T is compact on K . Because $f \in K$ is convex, nonnegative and $f(0) = 0$, f is an increasing function on $[0, 1]$, so that $\|f\| = f(1)$ for all $f \in K$. Convexity of f implies for $0 \leq s < t \leq 1$ that

$$0 \leq \frac{Tf(t) - Tf(s)}{t/2 - s/2} \leq \frac{f(t/2) - f(s/2)}{t/2 - s/2} \leq \frac{f(1) - f(1/2)}{1 - 1/2} \leq 2f(1).$$

It follows that $|Tf(t) - Tf(s)| \leq |t - s|$ for all $s, t \in [0, 1]$ and $f \in K$ with $\|f\| \leq 1$. This shows that $\{Tf: f \in K \text{ and } \|f\| \leq 1\}$ is bounded and equicontinuous, and hence T is compact on K .

To define the approximating maps $T_k: K \rightarrow K$, we let for $k \geq 3$,

$$\phi_k(t) = \begin{cases} t^2 & \text{for } 0 \leq t \leq (1/2)^k \\ (1/2)^{2k} + \epsilon_k(t - (1/2)^k) & \text{for } (1/2)^k \leq t \leq 1, \end{cases}$$

where $\epsilon_k = \frac{1}{2}(1 - \frac{1}{2^{k-1}})$. It is easy to verify that $\phi_k(0) = 0$, and ϕ_k is nonnegative and convex on $[0, 1]$. Furthermore $\phi_k(t) \leq t/2$ for $t \in [0, 1]$, and $\lim_{k \rightarrow \infty} \phi_k(t) = t/2$ uniformly on $[0, 1]$. For $k \geq 3$ define $T_k: K \rightarrow K$ by

$$T_k f(t) = f(\phi_k(t)) \quad \text{for } 0 \leq t \leq 1.$$

The linear maps T_k have the following properties:

- (1) T_k is continuous and compact on K ,
- (2) $T_k(K) \subseteq K$, and
- (3) $\lim_{k \rightarrow \infty} \|T - T_k\|_K = 0$.

To show the first assertion we first note that T_k is clearly continuous. Using convexity of f and ϕ_k we find for $0 \leq s < t \leq 1$ that

$$\begin{aligned} \frac{f(\phi_k(t)) - f(\phi_k(s))}{t - s} &= \left(\frac{f(\phi_k(t)) - f(\phi_k(s))}{\phi_k(t) - \phi_k(s)} \right) \left(\frac{\phi_k(t) - \phi_k(s)}{t - s} \right) \\ &\leq \left(\frac{f(1) - f(1/2)}{1 - 1/2} \right) \left(\frac{\phi_k(1) - \phi_k(s)}{1 - s} \right) \leq 2\epsilon_k f(1) \leq f(1). \end{aligned}$$

It follows that $|T_k f(t) - T_k f(s)| \leq |t - s|$ for $s, t \in [0, 1]$ and $f \in K$ with $\|f\| \leq 1$. So, $\{T_k f: f \in K \text{ and } \|f\| \leq 1\}$ is bounded and equicontinuous, which shows that T_k is compact on K .

Note that the second assertion is clear, as each $f \in K$ is convex and increasing on $[0, 1]$. To show the third assertion we again use convexity of f to deduce for $0 \leq t \leq 1$ that

$$\frac{f(t/2) - f(\phi_k(t))}{t/2 - \phi_k(t)} \leq \frac{f(1) - f(1/2)}{1 - 1/2} \leq 2f(1) \leq 2$$

for all $f \in K$ with $\|f\| \leq 1$, which gives $\lim_{k \rightarrow \infty} \|T - T_k\|_K = 0$.

Remark that for $g(t) = t$, then $g \in K$ and $Tg = \frac{1}{2}g$, so that $r_K(T) \geq 1/2$. In fact, Bonsall [1] shows that $r_K(T) = 1/2$, which is not difficult.

For integers $m \geq 1$ let ϕ_k^m denote the m -fold composition of ϕ_k . Since $\|T_k^m\|_K = T_k^m g(1) = \phi_k^m(1)$,

$$r_K(T_k) = \lim_{m \rightarrow \infty} \|T_k^m\|_K^{1/m} = \lim_{m \rightarrow \infty} \phi_k^m(1)^{1/m}.$$

As $\phi_k(t) \leq t/2$ for $t \in [0, 1]$, we have that $\phi_k^k(1) \leq (1/2)^k$. Recall that $\phi_k(t) = t^2$ for all $0 \leq t \leq (1/2)^k$. So, for $m \geq k$ we have that

$$\phi_k^m(1) \leq (1/2)^{2^{m-k}k},$$

which implies that

$$\lim_{m \rightarrow \infty} \|T_k^m\|_K^{1/m} \leq ((1/2)^{2^{m-k}k})^{1/m} = 0.$$

Thus, $\lim_{k \rightarrow \infty} r_K(T_k) \neq r_K(T)$.

Of course this example does not contradict Theorem 2.5. Simply observe that the function $f(t) = t^\alpha \in K$ is an eigenvector of T with eigenvalue $(1/2)^\alpha$ for all $\alpha \geq 1$; so, $\sigma_K(T) = (0, 1/2]$.

4. CONDITION G

A useful condition in the analysis of the continuity of the cone spectral radius is the so-called condition G, which was introduced in [2]. A cone C in X is said to satisfy *condition G* at $x \in C$ if for each sequence $(x_k)_k$ in C with $\lim_{k \rightarrow \infty} x_k = x$ and each $0 < \lambda < 1$ there exists $m \geq 1$ such that

$$\lambda x \leq x_k \quad \text{for all } k \geq m.$$

We say that C satisfies *condition G* if condition G holds at each $x \in C$.

Theorem 4.1. *If $f: C \rightarrow C$ is a continuous compact homogeneous order-preserving map on a closed cone C in X , and there exists $u \in C$ at which condition G holds and $f(u) = r_C(f)u$, then f has a continuous cone spectral radius.*

Proof. Suppose for $k \geq 1$ that $f_k: C \rightarrow C$ is a continuous compact homogeneous order-preserving map and (1.1) holds. Let $u_k = f_k(u)$ for $k \geq 1$ and note that $\lim_{k \rightarrow \infty} u_k = r_C(f)u$ by (1.1). As condition G holds at u , there exists a sequence of positive reals $(\mu_k)_k$, with $\mu_k \rightarrow 1^-$ and

$$(4.1) \quad \mu_k r_C(f)u \leq u_k \quad \text{for all } k \geq 1.$$

So for each $k \geq 1$,

$$(4.2) \quad \mu_k r_C(f)u_k = \mu_k r_C(f)f_k(u) \leq f_k(u_k).$$

Thus, $\mu_k r_C(f) \leq r_C(f_k)$ for all $k \geq 1$ by Lemma 2.2. Letting $k \rightarrow \infty$ we conclude that

$$r_C(f) \leq \liminf_{k \rightarrow \infty} r_C(f_k).$$

It now follows from Lemma 2.3 that $\lim_{k \rightarrow \infty} r_C(f_k) = r_C(f)$. \square

It turns out that condition G always holds at points in the interior of a closed cone. To prove this, it is convenient to recall the definition of Thompson's (part) metric [22] on cones and some other related notions. Given a closed cone C in X and $x, y \in C$ we say that y dominates x if there exists $\beta > 0$ such that $x \leq \beta y$. This yields an equivalence relation \sim_C on C by $x \sim_C y$ if y dominates x , and x dominates y . In other words, $x \sim_C y$ if and only if there exist $0 < \alpha \leq \beta$ such that $\alpha y \leq x \leq \beta y$. The equivalence classes are called *parts* of the cone. For $x \sim_C y$ one can consider the function $M(x/y) = \inf\{\beta > 0: x \leq \beta y\}$. Using this function Thompson's (part) metric is defined by

$$d_T(x, y) = \log \max\{M(x/y), M(y/x)\} \quad \text{for } x, y \in C \setminus \{0\} \text{ with } x \sim_C y,$$

and $d_T(0,0) = 0$. It is known [22] that d_T is a metric on each part of a closed cone C . Moreover, if C has a non-empty interior, C° , and $x, y \in C^\circ$ with $\|x - y\| < r$, then

$$(4.3) \quad d_T(x, y) \leq \log \max \left\{ \frac{r + \|x - y\|}{r}, \frac{r}{r - \|x - y\|} \right\},$$

see [17, p.16].

Lemma 4.2. *If C is a closed cone in X with a non-empty interior, then C satisfies property **G** at each $x \in C^\circ$.*

Proof. Let $(x_k)_k$ be sequence in C° with $\lim_{k \rightarrow \infty} \|x_k - x\| = 0$ and $0 < \lambda < 1$. By (4.3) there exists $m \geq 1$ such that

$$d_T(x_k, x) < \log 1/\lambda \quad \text{for all } k \geq m.$$

This implies that $M(x/x_k) \leq 1/\lambda$, so that $x \leq x_k/\lambda$ for all $k \geq m$. Thus, $\lambda x \leq x_k$ for all $k \geq m$. \square

In the sequel we shall also need the following basic result from [16].

Lemma 4.3. *Suppose that $f: C \rightarrow C$ is an order-preserving homogeneous map on a closed cone C in X and $x, y \in C$ are such that y dominates x and $x \neq 0$. If $\lambda x \leq f(x)$ and $f(y) \leq \mu y$, then $\lambda \leq \mu$.*

Proof. The assertion is obvious if $\lambda = 0$. Suppose $\lambda > 0$ and note that, as y dominates x , there exists $\beta > 0$ such that $x \leq \beta y$. This implies that $\lambda x \leq f(x) \leq \beta f(y) \leq \beta \mu y$. It follows that $\lambda^k x \leq f^k(x) \leq \beta f^k(y) \leq \beta \mu^k y$, so that

$$\left(\frac{\mu}{\lambda}\right)^k y - \frac{1}{\beta} x \in C \quad \text{for all } k \geq 1.$$

By letting $k \rightarrow \infty$ we find that $-x/\beta \in C$, if $\mu < \lambda$. This is impossible, as $x \neq 0$, and hence $\lambda \leq \mu$. \square

Remark that if $x \in C$ and $y \in C^\circ$, then y dominates x , as $y - \delta x \in C$ for all $\delta > 0$ sufficiently small. Thus, it follows from Lemma 4.3 that if $f: C \rightarrow C$ is a continuous compact order-preserving homogeneous map on a closed cone $C \subseteq X$ with non-empty interior and $v \in C^\circ$ is an eigenvector of f with eigenvalue λ , then $\lambda = r_C(f)$. Combining this fact with Theorem 4.1 and Lemma 4.2 yields the following result.

Corollary 4.4. *If $f: C \rightarrow C$ is a continuous compact order-preserving homogeneous map on a closed cone C with non-empty interior, and f has an eigenvector in C° , then f has a continuous cone spectral radius.*

It was shown [2] that if C is a closed cone in a finite dimensional vector space X , then C satisfies condition **G** if and only if C is polyhedral. Recall that a closed cone $C \subseteq X$ is a *polyhedral* cone, if it is intersection of finitely many closed half-space, i.e., there exists $\phi_1, \dots, \phi_m \in X^*$ such that

$$C = \{x \in X : \phi_i(x) \geq 0 \text{ for all } 1 \leq i \leq m\}.$$

Thus, we have the following consequence of Theorem 4.1.

Corollary 4.5. *If $f: C \rightarrow C$ is a continuous order-preserving homogeneous map on a polyhedral cone C , then f has a continuous spectral radius.*

In the next section we will derive the same result by analyzing the cone spectrum of maps on polyhedral cones and applying Theorem 2.5.

5. THE CONE SPECTRUM FOR FINITE DIMENSIONAL CONES

Regarding Theorem 2.5 it is interesting to further analyze the following question: If $f: C \rightarrow C$ is a continuous order-preserving homogeneous map on a finite dimensional closed cone C and $r_C(f) > 0$, when do there exist $0 < a_1 < a_2 < a_3 < \dots$ **not** in $\sigma_C(f)$ such that $\lim_{k \rightarrow \infty} a_k = r_C(f)$? The next example shows that even in finite dimensional spaces such a sequence $(a_k)_k$ may not exist.

Example 5.1. Consider the cone of real positive semi-definite $n \times n$ matrices, $\text{Pos}_n(\mathbb{R})$, inside the vector space of symmetric $n \times n$ matrices. Let $n \geq 2$ and let A be the $n \times n$ diagonal matrix with diagonal $(1, 0, \dots, 0)$. Define $f: \text{Pos}_n(\mathbb{R}) \rightarrow \text{Pos}_n(\mathbb{R})$ by

$$f(X) = \left(\text{tr}(XA)X \right)^{1/2} \quad \text{for } X \in \text{Pos}_n(\mathbb{R}).$$

Clearly, if $X \leq Y$, then $\text{tr}(XA) = \text{tr}(A^T X A) \leq \text{tr}(A^T Y A) = \text{tr}(YA)$. It follows from Löwner's theory [7] of order-preserving maps on $\text{Pos}_n(\mathbb{R})$ that $M \mapsto M^{1/2}$ is order-preserving. Thus, f is a continuous order-preserving homogeneous map.

Let $0 < \alpha \leq 1$ and let I denote the $(n-2) \times (n-2)$ identity matrix. Consider the $n \times n$ matrix X_α in the interior of $\text{Pos}_n(\mathbb{R})$,

$$X_\alpha = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & I \end{array} \right).$$

Obviously, $\text{tr}(X_\alpha A) = 1$, and

$$f(X_\alpha) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & I \end{array} \right)^{1/2} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & \sqrt{\alpha} & 0 \\ 0 & 0 & I \end{array} \right).$$

So, if $f(X_\alpha) = \lambda X_\alpha$, then $\lambda = 1$ and $\sqrt{\alpha} = \alpha$, which is equivalent to $\alpha = 1$. Thus, X_1 is an eigenvector of f with eigenvalue 1 in the interior of $\text{Pos}_n(\mathbb{R})$. It follows from Lemma 4.3 that the cone spectral radius of f is equal to 1.

For $0 \leq \theta \leq 2\pi$ consider the matrix Z_θ in the boundary of $\text{Pos}_n(\mathbb{R})$ given by

$$\begin{aligned} Z_\theta &= \left(\begin{array}{cc|c} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right) \left(\begin{array}{cc|c} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 \end{array} \right) \\ &= \left(\begin{array}{cc|c} \cos^2 \theta & \cos \theta \sin \theta & 0 \\ \cos \theta \sin \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 \end{array} \right). \end{aligned}$$

Note that

$$f(Z_\theta) = \left(\text{tr}(Z_\theta A)Z_\theta \right)^{1/2} = |\cos \theta| Z_\theta^{1/2} = |\cos \theta| Z_\theta.$$

So, Z_θ is an eigenvector with eigenvalue $|\cos \theta|$, and hence the cone spectrum of f is equal to $[0, 1]$.

Note, however, that as f has an eigenvector in the interior we can apply, instead of Theorem 2.5, Corollary 4.4 to deduce that f has a continuous cone spectral radius. For $n \geq 3$ the example can easily be modified so as not to have an eigenvector

in the interior, but still have $[0, 1]$ as its cone spectrum. Indeed, let B be the $n \times n$ diagonal matrix with diagonal $(1, 1, 0, \dots, 0)$, and consider $g: \text{Pos}_n(\mathbb{R}) \rightarrow \text{Pos}_n(\mathbb{R})$ given by,

$$g(X) = B^T(\text{tr}(XA)X)^{1/2}B \quad \text{for } X \in \text{Pos}_n(\mathbb{R}).$$

The reader can verify that the cone spectrum of g is $[0, 1]$ and g has no eigenvectors in the interior of $\text{Pos}_n(\mathbb{R})$. It is, however, unclear if g has a continuous cone spectral radius.

In view of the results so far it interesting to further study the following problem.

Problem 5.2. *Which finite dimensional closed cones C admit a continuous order-preserving homogeneous map $f: C \rightarrow C$ with a continuum in its cone spectrum?*

In the remainder we will present some partial results for this problem. We will first recall some basic concepts.

Given a closed cone $C \subseteq X$ the *dual cone* $C^* \subseteq X^*$ is given by $C^* = \{\phi \in X^*: \phi(x) \geq 0 \text{ for all } x \in C\}$. In general C^* need not be a cone, but it is easy to prove that C^* is a cone if C is total. Furthermore, for $x, y \in C$ it follows from the Hahn-Banach separation theorem that

$$(5.1) \quad x \leq y \text{ if and only if } \phi(x) \leq \phi(y) \text{ for all } \phi \in C^*.$$

A *face* of a closed cone $C \subseteq X$ is a non-empty convex subset F of C such that whenever $x, y \in C$ and $(1 - \lambda)x + \lambda y \in F$ for some $0 < \lambda < 1$ it follows that $x, y \in F$. Note that C and $\{0\}$ are both faces of C . It is known [6] that if C is a closed cone in X , then the parts of C are precisely the relative interiors of the faces of C .

A face F of a polyhedral cone C is called a *facet* if $\dim F = \dim C - 1$. It is a basic result from polyhedral geometry, see e.g., [20], that if C is a polyhedral cone in X with N facets, then there exist N linear functionals $\psi_1, \dots, \psi_N \in X^*$ such that

$$C = \{x \in X: \psi_i(x) \geq 0 \text{ for } i = 1, \dots, N\} \cap \text{span } C,$$

and each linear functional ψ_i corresponds to a unique facet F_i of C , in the sense that $F_i = \{x \in C: \psi_i(x) = 0\}$.

Given a closed cone C in X we let $\mathcal{P}(C)$ denote the set of parts of C . Note that if C is a polyhedral cone, then $\mathcal{P}(C)$ is finite. On $\mathcal{P}(C)$ we have a partial ordering \trianglelefteq given by $P \trianglelefteq Q$ if there exist $y \in Q$ and $x \in P$ such that y dominates x . Given a polyhedral cone C with facet defining functionals ψ_1, \dots, ψ_N and $P \in \mathcal{P}(C)$, we let $I(P) = \{i: \psi_i(x) > 0 \text{ for some } x \in P\}$. The next lemma is elementary.

Lemma 5.3. *If $C \subseteq V$ is a polyhedral cone with N facets, then*

- (i) *for $P \in \mathcal{P}(C)$ we have $P = \{x \in C: \psi_i(x) > 0 \text{ if and only if } i \in I(P)\}$, and*
- (ii) *$P \trianglelefteq Q$ if and only if $I(P) \subseteq I(Q)$.*

Proof. Let $\psi_1, \dots, \psi_N \in C^*$ be the facet defining functionals of C . By (5.1) we know that $x \leq y$ is equivalent to $\psi_i(x) \leq \psi_i(y)$ for all i . For $x \in C$ write $I_x = \{i: \psi_i(x) > 0\}$, and note that $I_x \subseteq I_y$ if and only if y dominates x . Therefore $x \sim_C y$ is equivalent to $I_x = I_y$. It follows that $x \in P$ if and only if $I_x = I(P)$, which proves the first assertion.

If $P \trianglelefteq Q$, there exist $x \in P$ and $y \in Q$ such that y dominates x . So, $I(P) = I_x \subseteq I_y = I(Q)$. On the other hand, if $I(P) \subseteq I(Q)$, then for each $x \in P$ and each $y \in Q$

we know that $I_x \subseteq I_y$, and hence there exists $\beta > 0$ such that $\psi_i(x) \leq \beta\psi_i(y)$ for all i . It follows from (5.1) that y dominates x , and hence $P \preceq Q$. \square

The following lemma is a direct consequence of Lemma 4.3.

Corollary 5.4. *If $f: C \rightarrow C$ is an order-preserving homogeneous map on a closed cone $C \subseteq V$, and f has eigenvectors x and y in C with $x \sim_C y$, then the eigenvalues of x and y are equal.*

As $\{0\}$ is a part of any cone C , we see that the number of distinct eigenvalues of an order-preserving homogeneous map $f: C \rightarrow C$ is bounded by $m - 1$, where m is the number of parts of C . In case $C = \mathbb{R}_+^n$ we can exploit the lattice structure on \mathbb{R}_+^n to construct an example that shows that the upper bound, $2^n - 1$, is sharp. Recall that if $x, y \in \mathbb{R}_+^n$, then $x \vee y \in \mathbb{R}_+^n$ is given by $(x \vee y)_i = \max\{x_i, y_i\}$ for all i . For $I \subseteq \{1, \dots, n\}$ non-empty, define $\chi^I \in \mathbb{R}_+^n$ by $\chi_i^I = 1$ if $i \in I$, and $\chi_i^I = 0$ otherwise. Furthermore, let $\lambda_I > 0$ be such that $\lambda_I < \lambda_J$ if $I \subseteq J$ and $I \neq J$. Consider the map $f: \mathbb{R}_+^n \rightarrow \mathbb{R}_+^n$ given by,

$$f(x) = \bigvee_{\emptyset \neq I \subseteq \{1, \dots, n\}} \lambda_I (\min_{i \in I} x_i) \chi^I \quad \text{for } x \in \mathbb{R}_+^n.$$

Obviously, f is a continuous order-preserving homogeneous map on \mathbb{R}_+^n , and for $J \subseteq \{1, \dots, n\}$ with $J \neq \emptyset$ we have that

$$f(\chi^J) = \bigvee_{\emptyset \neq I \subseteq J} \lambda_I \chi^I = \lambda_J \chi^J,$$

as $\lambda_I < \lambda_J$ for $I \subseteq J$ with $I \neq J$. So, f has $2^n - 1$ distinct eigenvalues.

We will now show that the upper bound, $m - 1$, is sharp for every polyhedral cone. To establish this result we need to introduce some more notation. Given a polyhedral cone $C \subseteq V$ with non-empty interior and facet defining functionals ψ_1, \dots, ψ_N , we define for $r < 0$ and $I \subseteq \{1, \dots, N\}$ non-empty, the function $M_r(I): C^\circ \rightarrow [0, \infty)$ by

$$M_r(I)(x) = \left(\sum_{i \in I} \psi_i(x)^r \right)^{1/r} \quad \text{for } x \in C^\circ.$$

From [2] we know that $M_r(I)$ has a continuous, order-preserving, homogeneous extension to ∂C , as C is polyhedral. Note that if $x \in \partial C$ and there exists $i \in I$ with $\psi_i(x) = 0$, then $M_r(I)(x) = 0$, since $r < 0$. For $r = -\infty$ we define $M_{-\infty}(I): C \rightarrow [0, \infty)$ by

$$M_{-\infty}(I)(x) = \min_{i \in I} \psi_i(x) \quad \text{for } x \in C.$$

Theorem 5.5. *Let $C \subseteq V$ be a polyhedral cone with non-empty interior and m faces. If $f: C \rightarrow C$ is a order-preserving homogeneous map, then $|\sigma_C(f)| \leq m - 1$. Moreover, on C there exists a continuous order-preserving homogeneous map with $m - 1$ distinct eigenvalues.*

Proof. As the parts of C coincide with the relative interiors of the faces of C we know that C has m parts, see [6]. Omitting the trivial part $\{0\}$ and using Corollary 5.4 we see that $m - 1$ is an upper bound for the size of $\sigma_C(f)$.

To construct an example with $m - 1$ distinct eigenvalues on C , we let ψ_1, \dots, ψ_N denote the facet defining functionals of C . By Lemma 5.3 we know that for each $P \in \mathcal{P}(C)$ we have that

$$P = \{x \in C : \psi_i(x) > 0 \text{ if and only if } i \in I(P)\}.$$

For each $P \in \mathcal{P}(C)$ with $P \neq \{0\}$ we select $z^P \in P$. The idea is to construct a continuous order-preserving homogeneous map which has the points z^P as eigenvectors with distinct eigenvalues. Let $r \in [-\infty, 0)$ and $M_r(I)(x)$ be defined as above for $x \in C$ and $I \subseteq \{1, \dots, N\}$ non-empty. The continuous order-preserving homogeneous map $f_r : C \rightarrow C$ will be of the form:

$$(5.2) \quad f_r(x) = \sum_{P \in \mathcal{P}(C), P \neq \{0\}} \lambda_P M_r(I(P))(x) u^P \quad \text{for } x \in C,$$

where $\lambda_P > 0$ and $u^P \in P$ are chosen appropriately.

Recall from Lemma 5.3 that $P \trianglelefteq Q$ if and only if $I(P) \subseteq I(Q)$. Thus, for each $x \in Q$ and $P \in \mathcal{P}(C)$ with $P \neq \{0\}$ we have that $M_r(I(P))(x) > 0$ if $P \trianglelefteq Q$, and $M_r(I(P))(x) = 0$ otherwise.

We define $\lambda_P > 0$ and $u^P \in P$ inductively using the height of P in the finite partially ordered set $(\mathcal{P}(C), \trianglelefteq)$. For $P \in \mathcal{P}(C)$ with height 1, or equivalently $\dim P = 1$, take $u^P = z^P$ and chose $\lambda_P > 0$ such that the positive numbers $\mu_P = \lambda_P M_r(I(P))(u^P)$ are all distinct for $P \in \mathcal{P}(C)$ with height 1. So, $f(z^P) = \mu_P z^P$ for those parts P .

Suppose that we have already selected $u^P \in P$ and $\lambda_P > 0$ for all $P \trianglelefteq Q$ with $\{0\} \neq P \neq Q$. Consider

$$f_r(z^Q) = \sum_{P \in \mathcal{P}(C), P \neq \{0\}} \lambda_P M_r(I(P))(z^Q) u^P = \sum_{P \trianglelefteq Q, P \neq \{0\}} \lambda_P M_r(I(P))(z^Q) u^P$$

and write

$$w^Q = \sum_{P \trianglelefteq Q, \{0\} \neq P \neq Q} \lambda_P M_r(I(P))(z^Q) u^P.$$

We observe that for each $\mu_Q > 0$ sufficiently large, $\mu_Q z^Q - w^Q \in Q$. Now take $\lambda_Q > 0$ and $u^Q \in Q$ such that

$$\lambda_Q M_r(I(Q))(z^Q) u^Q = \mu_Q z^Q - w^Q.$$

Recall that $M_r(I(Q))(z^Q) > 0$; so, once we have selected $\mu_Q > 0$ and $\lambda_Q > 0$ the vector $u^Q \in Q$ is fixed.

It follows from the construction that $f_r(z^Q) = w^Q + \mu_Q z^Q - w^Q = \mu_Q z^Q$. Thus, we can chose $\mu_Q > 0$ such that f_r has $|\mathcal{P}(C)| - 1 = m - 1$ distinct eigenvalues. \square

We would like to point out that for $-\infty < r < 0$, the function f constructed in the proof of Theorem 5.5 is infinitely differentiable on C° . One may wonder whether there exists a continuous order-preserving homogeneous map on C as in Theorem 5.5 which is continuously differentiable on C° and whose derivative, $Df(\cdot)$, extends continuously to 0. The answer is clearly no if $\dim C \geq 2$, because the limit L of $Df(x)$ as $x \rightarrow 0$, would necessarily have $m - 1$ distinct eigenvalues.

Our final result shows that the cone spectrum may contain a countably infinite number of distinct elements, if the cone is not polyhedral. The proof uses Straszewicz's theorem [21], see also [19, p.167], which says that the exposed points

of a closed convex set S in a finite dimensional vector space V are dense in the extreme points of S .

Theorem 5.6. *If $C \subseteq V$ is a closed non-polyhedral cone with non-empty interior, then there exists a continuous order-preserving homogeneous map $f: C \rightarrow C$ with infinitely many distinct eigenvalues.*

Proof. Let $u \in C^\circ$ and note that if C is non-polyhedral, then the dual cone C^* is also non-polyhedral. Define $\Sigma^* = \{\phi \in C^* : \phi(u) = 1\}$, which is a compact convex set in V^* . As C^* is non-polyhedral, Σ^* has infinitely many extreme points. By Straszewicz's theorem [21] the exposed points of Σ^* are dense in the extreme points of Σ^* . Thus, we can find a sequence $(\phi_k)_k$ in Σ^* of distinct exposed points such that $\phi_k \rightarrow \psi \in \Sigma^*$ as $k \rightarrow \infty$ and $\phi_k \neq \psi$ for all $k \geq 1$.

As ϕ_k is an exposed point of Σ^* and $C^{**} = C$, there exists $x^k \in C$ with $\|x^k\| = 1$ such that $\phi_k(x^k) = 0$ and $\phi(x^k) > 0$ for all $\phi \in \Sigma^* \setminus \{\phi_k\}$. In particular,

$$(5.3) \quad \psi(x^k) > 0 \quad \text{and} \quad \phi_m(x^k) > 0 \quad \text{for all } m \neq k.$$

Let $f: C \rightarrow C$ be defined as follows,

$$f(x) = \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \lambda_k \inf_{m \neq k} \phi_m(x) x^k \quad \text{for } x \in C,$$

where $(\lambda_k)_k$ is a bounded sequence of strictly positive reals. For each $k \neq q$, we have that

$$\lambda_k \inf_{m \neq k} \phi_m(x^q) = 0,$$

as $\phi_q(x^q) = 0$. On the other hand, if $k = q$, then

$$\lambda_q \inf_{m \neq q} \phi_m(x^q) > 0,$$

by (5.3) and the fact that $\phi_k \rightarrow \psi$ as $k \rightarrow \infty$, $\phi_k \neq \phi_q$ for all $k \neq q$, and $\phi_q \neq \psi$.

So, for each $q \geq 1$ we find that

$$f(x^q) = \left(\frac{1}{2}\right)^q \lambda_q \inf_{m \neq q} \phi_m(x^q) x^q,$$

which shows that x^q is an eigenvector of f . By selecting $0 < \lambda_q \leq 1$ appropriately we can insure that f has infinitely many distinct eigenvalues. \square

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