Isolated Singularities of Nonlinear Polyharmonic Inequalities

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Abstract

We obtain results for the following question where $m \geq 1$ and $n \geq 2$ are integers.

Question. For which continuous functions $f\colon [0,\infty) \to [0,\infty)$ does there exist a continuous function $\varphi\colon (0,1) \to (0,\infty)$ such that every C^{2m} nonnegative solution $u(x)$ of

$$0 \leq -\Delta^m u \leq f(u) \quad \text{in} \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n$$

satisfies

$$u(x) = O(\varphi(|x|)) \quad \text{as} \quad x \to 0$$

and what is the optimal such φ when one exists?

Keywords: Isolated singularity; Polyharmonic

1 Introduction and results

In this paper we consider the following question where $m \geq 1$ and $n \geq 2$ are integers.

Question 1. For which continuous functions $f\colon [0,\infty) \to [0,\infty)$ does there exist a continuous function $\varphi\colon (0,1) \to (0,\infty)$ such that every C^{2m} nonnegative solution $u(x)$ of

$$0 \leq -\Delta^m u \leq f(u) \quad \text{in} \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n \tag{1.1}$$

satisfies

$$u(x) = O(\varphi(|x|)) \quad \text{as} \quad x \to 0 \tag{1.2}$$

and what is the optimal such φ when one exists?

We call a function φ with the above properties a pointwise a priori bound (as $x \to 0$) for C^{2m} nonnegative solutions $u(x)$ of (1.1).

As we shall see, when φ in Question 1 is optimal, the estimate (1.2) can sometimes be sharpened to

$$u(x) = o(\varphi(|x|)) \quad \text{as} \quad x \to 0.$$

Remark 1.1. Let

$$\Gamma(r) = \begin{cases} r^{-(n-2)}, & \text{if } n \geq 3; \\ \log \frac{5}{r}, & \text{if } n = 2. \end{cases} \tag{1.3}$$

Since $u(x) = \Gamma(|x|)$ is a positive solution of $-\Delta^m u = 0$ in $B_2(0)\backslash\{0\}$, and hence a positive solution of (1.1), any pointwise a priori bound φ for C^{2m} nonnegative solutions $u(x)$ of (1.1) must be at least as large as Γ, and whenever $\varphi = \Gamma$ is such a bound it is necessarily an optimal bound.

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If $m \geq 1$ and $n \geq 2$ are integers then m and n satisfy one of the following five conditions.

(i) either m is even or $2m > n$;

(ii) $m = 1$ and $n \geq 3$;

(iii) $m = 1$ and $n = 2$;

(iv) $m \geq 3$ is odd and $2m < n$;

(v) $m \geq 3$ is odd and $2m = n$.

The following three theorems, which we proved in [7], [15], and [14], completely answer Question 1 when m and n satisfy either (i), (ii), or (iii). Consequently, in this paper, we will only prove results dealing with the case that m and n satisfy either (iv) or (v).

Theorem 1.1. *Suppose $m \geq 1$ and $n \geq 2$ are integers satisfying (i) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function. Let $u(x)$ be a C^{2m} nonnegative solution of* (1.1) *or, more generally, of*

$$-\Delta^m u \geq 0 \quad in \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n. \tag{1.4}$$

Then

$$u(x) = O(\Gamma(|x|)) \quad as \quad x \to 0, \tag{1.5}$$

where Γ is given by (1.3)*.*

Theorem 1.2. *Let $u(x)$ be a C^2 nonnegative solution of* (1.1) *where the integers m and n satisfy (ii), (resp. (iii)), and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$f(t) = O(t^{n/(n-2)}), \quad (resp.\ \log(1 + f(t)) = O(t)) \quad as \quad t \to \infty. \tag{1.6}$$

Then u satisfies (1.5)*.*

By Remark 1.1 the bound (1.5) for u in Theorems 1.1 and 1.2 is optimal.

By the following theorem, the condition (1.6) on f in Theorem 1.2 for the existence of a pointwise bound for u is essentially optimal.

Theorem 1.3. *Suppose m and n are integers satisfying (ii), (resp. (iii)), and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$\lim_{t\to\infty} \frac{f(t)}{t^{n/(n-2)}} = \infty, \quad \left(resp.\ \lim_{t\to\infty} \frac{\log(1 + f(t))}{t} = \infty \right). \tag{1.7}$$

Then for each continuous function $\varphi\colon (0,1) \to (0,\infty)$ there exists a C^2 positive solution $u(x)$ of (1.1) *such that*

$$u(x) \neq O(\varphi(|x|)) \quad as \quad x \to 0.$$

If m and n satisfy (i), (ii), or (iii), then according to Theorems 1.1, 1.2, and 1.3, either the optimal pointwise bound for u is given by (1.5) or there does not exists a pointwise bound for u, (provided we don't allow the rather uninteresting and pathological possibility when m and n satisfy (ii), (resp. (iii)), that f satisfies neither (1.6) nor (1.7)).

The situation is very different and more interesting when m and n satisfy (iv) or (v). In this case, according to the following results, there are an infinite number of different optimal pointwise bounds for u depending on f.

The following three theorems deal with Question 1 when m and n satisfy (iv).

Theorem 1.4. *Let $u(x)$ be a C^{2m} nonnegative solution of (1.1) where the integers m and n satisfy (iv) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$f(t) = O(t^\lambda) \quad as \quad t \to \infty$$

where

$$0 \le \lambda \le \frac{2m+n-2}{n-2}, \qquad \left(resp. \ \frac{2m+n-2}{n-2} < \lambda < \frac{n}{n-2m} \right).$$

Then as $x \to 0$,

$$u(x) = O(|x|^{-(n-2)}), \tag{1.8}$$

$$\left(resp. \ u(x) = o(|x|^{-a}) \quad where \ a = \frac{4m(m-1)}{n-\lambda(n-2m)} \right). \tag{1.9}$$

Since a in (1.9) is also given by

$$a = n-2 + \frac{\lambda(n-2)-(2m+n-2)}{n-\lambda(n-2m)}(n-2m) \tag{1.10}$$

we see that a increases from $n-2$ to infinity as λ increases from $\frac{2m+n-2}{n-2}$ to $\frac{n}{n-2m}$.

By Remark 1.1, the bound (1.8) is optimal and by the following theorem so is the bound (1.9).

Theorem 1.5. *Suppose m and n are integers satisfying (iv) and λ and a are constants satisfying*

$$\frac{2m+n-2}{n-2} < \lambda < \frac{n}{n-2m} \quad and \quad a = \frac{4m(m-1)}{n-\lambda(n-2m)}. \tag{1.11}$$

Let $\varphi\colon (0,1) \to (0,1)$ be a continuous function satisfying $\lim_{r\to 0^+} \varphi(r) = 0$. Then there exists a C^∞ positive solution $u(x)$ of

$$0 \le -\Delta^m u \le u^\lambda \quad in \quad \mathbb{R}^n \backslash \{0\} \tag{1.12}$$

such that

$$u(x) \ne O\left(\varphi(|x|)|x|^{-a}\right) \quad as \quad x \to 0. \tag{1.13}$$

With regard to Theorem 1.4, it is natural to ask what happens when $\lambda \ge \frac{n}{n-2m}$. The answer, given by the following theorem, is that the solutions u can be arbitrarily large as $x \to 0$.

Theorem 1.6. *Suppose m and n are integers satisfying (iv) and $\lambda \ge \frac{n}{n-2m}$ is a constant. Let $\varphi\colon (0,1) \to (0,\infty)$ be a continuous function satisfying $\lim_{r\to 0^+} \varphi(r) = \infty$. Then there exists a C^∞ positive solution $u(x)$ of*

$$0 \le -\Delta^m u \le u^\lambda \quad in \quad \mathbb{R}^n \backslash \{0\} \tag{1.14}$$

such that

$$u(x) \ne O(\varphi(|x|)) \quad as \quad x \to 0.$$

The following five theorems deal with Question 1 when m and n satisfy (v). This is the most interesting case.

Theorem 1.7. *Let $u(x)$ be a C^{2m} nonnegative solution of (1.1) where the integers m and n satisfy (v) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$f(t) = O(t^\lambda) \quad as \quad t \to \infty$$

where

$$0 \le \lambda \le \frac{2n-2}{n-2}, \quad \left(\textit{resp. } \lambda > \frac{2n-2}{n-2}\right).$$

Then as $x \to 0$,

$$u(x) = O(|x|^{-(n-2)}), \tag{1.15}$$

$$\left(\textit{resp. } u(x) = o\Big(|x|^{-(n-2)} \log \frac{5}{|x|}\Big)\right). \tag{1.16}$$

By Remark 1.1, the bound (1.15) is optimal and by the following theorem so is the bound (1.16).

Theorem 1.8. *Suppose m and n are integers satisfying (v) and λ is a constant satisfying*

$$\lambda > \frac{2n-2}{n-2}. \tag{1.17}$$

Let $\varphi\colon (0,1) \to (0,1)$ be a continuous function satisfying $\lim_{r\to 0^+} \varphi(r) = 0$. Then there exists a C^∞ positive solution $u(x)$ of

$$0 \le -\Delta^m u \le u^\lambda \quad \textit{in} \quad \mathbb{R}^n \backslash \{0\} \tag{1.18}$$

such that

$$u(x) \ne O\left(\varphi(|x|)|x|^{-(n-2)} \log \frac{5}{|x|}\right) \quad \textit{as} \quad x \to 0. \tag{1.19}$$

By the following theorem $u(x)$ may satisfy a pointwise a priori bound even when $f(t)$ grows, as $t \to \infty$, faster than any power of t.

Theorem 1.9. *Let $u(x)$ be a C^{2m} nonnegative solution of* (1.1) *where the integers m and n satisfy (v) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$\log(1 + f(t)) = O(t^\lambda) \quad \textit{as} \quad t \to \infty$$

where

$$0 < \lambda < 1. \tag{1.20}$$

Then

$$u(x) = o\left(|x|^{\frac{-(n-2)}{1-\lambda}}\right) \quad \textit{as} \quad x \to 0. \tag{1.21}$$

By the following theorem, the estimate (1.21) in Theorem 1.9 is optimal.

Theorem 1.10. *Suppose m and n are integers satisfying (v) and λ is a constant satisfying* (1.20). *Let $\varphi\colon (0,1) \to (0,1)$ be a continuous function satisfying $\lim_{r\to 0^+} \varphi(r) = 0$. Then there exists a C^∞ positive solution $u(x)$ of*

$$0 \le -\Delta^m u \le e^{u^\lambda} \quad \textit{in} \quad \mathbb{R}^n \backslash \{0\} \tag{1.22}$$

such that

$$u(x) \ne O\left(\varphi(|x|)|x|^{\frac{-(n-2)}{1-\lambda}}\right) \quad \textit{as} \quad x \to 0. \tag{1.23}$$

With regard to Theorem 1.9, it is natural to ask what happens when $\lambda \ge 1$. The answer, given by the following theorem, is that the solutions u can be arbitrarily large as $x \to 0$.

Theorem 1.11. *Suppose m and n are integers satisfying (v) and $\lambda \geq 1$ is a constant. Let $\varphi\colon (0,1) \to (0,\infty)$ be a continuous function satisfying $\lim_{r\to 0^+} \varphi(r) = \infty$. Then there exists a C^∞ positive solution of*

$$0 \leq -\Delta^m u \leq e^{u^\lambda} \quad \text{in} \quad \mathbb{R}^n\backslash\{0\} \tag{1.24}$$

such that

$$u(x) \neq O(\varphi(|x|)) \quad \text{as} \quad x \to 0. \tag{1.25}$$

Theorems 1.3–1.11 are "nonradial". By this we mean that if one requires the solutions $u(x)$ in Question 1 to be radial then, according to the following theorem, which follows immediately from [7, Lemma 2.4], the complete answer to Question 1 is very different.

Theorem 1.12. *Suppose $m \geq 1$ and $n \geq 2$ are integers and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function. Let $u(x)$ be a C^{2m} nonnegative radial solution of (1.1) or, more generally, of*

$$-\Delta^m u \geq 0 \quad \text{in} \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n.$$

Then u satisfies

$$u(x) = O(\Gamma(|x|)) \quad \text{as} \quad x \to 0 \tag{1.26}$$

where Γ is given by (1.3).

By Remark 1.1, the bound (1.26) for u in Theorem 1.12 is optimal.

Theorems 1.4 and 1.7 are special cases of much more general results, in which, instead of obtaining pointwise upper bounds (when they exist) for u where u is a nonnegative solution of

$$0 \leq -\Delta^m u \leq (u+1)^\lambda \quad \text{in} \quad B_2(0)\backslash\{0\},$$

we obtain pointwise upper bounds (when they exist) for $|D^i u|$, $i = 0, 1, 2, \ldots, 2m-1$, where u is a nonnegative solution of

$$0 \leq -\Delta^m u \leq \sum_{k=0}^{2m-1} (|D^k u| + g_k(x))^{\lambda_k} \quad \text{in} \quad B_2(0)\backslash\{0\},$$

where the functions $g_k(x)$ tend to infinity as $x \to 0$. See Theorems 3.1 and 3.2 in Section 3 for the precise statements of these more general results.

Also estimates for some derivatives of solutions of (1.1) when m and n satisfy (i) were obtained in [7].

We next consider the following analog of Question 1 when the singularity is at ∞.

Question 2. For which continuous functions $f\colon [0,\infty) \to [0,\infty)$ does there exist a continuous function $\varphi\colon (1,\infty) \to (0,\infty)$ such that every C^{2m} nonnegative solution $v(y)$ of

$$0 \leq -\Delta^m v \leq f(v) \quad \text{in} \quad \mathbb{R}^n\backslash B_{1/2}(0) \tag{1.27}$$

satisfies

$$v(y) = O(\varphi(|y|)) \quad \text{as} \quad |y| \to \infty$$

and what is the optimal such φ when one exists?

The m-Kelvin transform of a function $u(x)$, $x \in \Omega \subset \mathbb{R}^n\backslash\{0\}$, is defined by

$$v(y) = |x|^{n-2m} u(x) \quad \text{where} \quad x = y/|y|^2. \tag{1.28}$$

By direct computation, $v(y)$ satisfies

$$\Delta^m v(y) = |x|^{n+2m} \Delta^m u(x). \tag{1.29}$$

See [16, p. 221] or [17, p. 660].

As noted in [7], an immediate consequence of this fact and Theorem 1.1 is the following result concerning Question 2 when m and n satisfy (i).

Theorem 1.13. *Suppose $m \geq 1$ and $n \geq 2$ are integers satisfying (i) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function. Let $v(y)$ be a C^{2m} nonnegative solution of (1.27) or, more generally, of*

$$-\Delta^m v \geq 0 \quad in \quad \mathbb{R}^n \backslash B_{1/2}(0).$$

Then

$$v(y) = O(\Gamma_\infty(|y|)) \quad as \quad |y| \to \infty, \tag{1.30}$$

where

$$\Gamma_\infty(r) = \begin{cases} r^{2m-2}, & \text{if } n \geq 3; \\ r^{2m-2} \log 5r, & \text{if } n = 2. \end{cases}$$

The estimate (1.30) is optimal because $\Delta^m \Gamma_\infty(|y|) = 0$ in $\mathbb{R}^n \backslash \{0\}$.

Using the m-Kelvin transform and Theorems 3.1, 3.2, and 3.3 in Section 3 we will prove in Section 4 the following three theorems dealing with Question 2, the first of which deals with the case that m and n satisfy (iv).

Theorem 1.14. *Let $v(y)$ be a C^{2m} nonnegative solution of (1.27) where the integers m and n satisfy (iv) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$f(t) = O(t^\sigma) \quad as \quad t \to \infty$$

where

$$0 < \sigma < \frac{n}{n-2m}.$$

Then

$$v(y) = O(|y|^b) \quad as \quad |y| \to \infty \tag{1.31}$$

where

$$b = \frac{2m(n-2)}{n - \sigma(n-2m)} = 2m - 2 + \frac{2(n-2m)(1+\sigma(m-1))}{n-\sigma(n-2m)}. \tag{1.32}$$

The next two theorems deal with Question 2 when m and n satisfy (v).

Theorem 1.15. *Let $v(y)$ be a C^{2m} nonnegative solution of (1.27) where the integers m and n satisfy (v) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$f(t) = O(t^\sigma) \quad as \quad t \to \infty$$

where $\sigma > 0$. Then

$$v(y) = o(|y|^{n-2} \log 5|y|) \quad as \quad |y| \to \infty.$$

Theorem 1.16. *Let $v(y)$ be a C^{2m} nonnegative solution of (1.27) where the integers m on n satisfy (v) and $f\colon [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$\log(1 + f(t)) = O(t^\lambda) \quad as \quad t \to \infty$$

where $0 < \lambda < 1$. Then

$$v(y) = o(|y|^{\frac{n-2}{1-\lambda}}) \quad as \quad |y| \to \infty.$$

Theorems 1.14–1.16 are optimal for Question 2 in the same way that Theorems 1.4, 1.7, and 1.9 are optimal for Question 1. For example, according to the following theorem, the bound (1.31) in Theorem 1.14 is optimal. We will omit the precise statements and proofs of the other optimality results for Theorems 1.14–1.16.

Theorem 1.17. *Suppose m and n are integers satisfying (iv) and λ and b are constants satisfying*

$$0 < \lambda < \frac{n}{n-2m} \quad \text{and} \quad b = \frac{2m(n-2)}{n-\lambda(n-2m)}. \tag{1.33}$$

Let $\varphi\colon (1,\infty) \to (0,1)$ be a continuous function satisfying $\lim_{r\to\infty} \varphi(r) = 0$. Then there exists a C^∞ positive solution $v(y)$ of

$$0 \le -\Delta^m v \le v^\lambda \quad \text{in} \quad \mathbb{R}^n\backslash\{0\} \tag{1.34}$$

such that

$$v(y) \ne O(\varphi(|y|)|y|^b) \quad \text{as} \quad |y| \to \infty. \tag{1.35}$$

Nonnegative solutions in a punctured neighborhood of the origin in $\mathbb{R}^n$—or near $x = \infty$ via the m-Kelvin transform—of problems of the form

$$-\Delta^m u = f(x,u) \quad \text{or} \quad 0 \le -\Delta^m u \le f(x,u) \tag{1.36}$$

when f is a nonnegative function have been studied in [2, 3, 9, 10, 11, 16, 17] and elsewhere.

Pointwise estimates at $x = \infty$ of solutions u of problems (1.36) can be crucial for proving existence results for entire solutions of (1.36) which in turn can be used to obtain, via scaling methods, existence and estimates of solutions of boundary value problems associated with (1.36), see e.g. [12, 13]. An excellent reference for polyharmonic boundary value problems is [6].

Also, weak solutions of $\Delta^m u = \mu$, where μ is a measure on a subset of $\mathbb{R}^n$, have been studied in [1, 4, 5], and removable isolated singularities of $\Delta^m u = 0$ have been studied in [10].

Our proofs rely on a representation formula for C^{2m} nonnegative solutions of (1.4) which we state in Lemma 2.1 and which we proved in [7]. Our proofs also require Riesz potential estimates as stated, for example, in [8, Lemma 7.12].

2 Preliminary Results

A fundamental solution of Δ^m in $\mathbb{R}^n$, where $m \ge 1$ and $n \ge 2$ are integers, is given by

$$\Phi(x) := A \begin{cases} (-1)^m |x|^{2m-n}, & \text{if } 2 \le 2m < n; \quad (2.1) \\ (-1)^{\frac{n-1}{2}} |x|^{2m-n}, & \text{if } 3 \le n < 2m \text{ and } n \text{ is odd}; \quad (2.2) \\ (-1)^{\frac{n}{2}} |x|^{2m-n} \log \dfrac{5}{|x|}, & \text{if } 2 \le n \le 2m \text{ and } n \text{ is even}; \quad (2.3) \end{cases}$$

where $A = A(m,n)$ is a *positive* constant whose value may change from line to line throughout this entire paper. In the sense of distributions, $\Delta^m \Phi = \delta$, where δ is the Dirac mass at the origin in $\mathbb{R}^n$. For $x \ne 0$ and $y \ne x$, let

$$\Psi(x,y) = \Phi(x-y) - \sum_{|\alpha|\le 2m-3} \frac{(-y)^\alpha}{\alpha!} D^\alpha \Phi(x) \tag{2.4}$$

be the error in approximating $\Phi(x-y)$ with the partial sum of degree $2m-3$ of the Taylor series of Φ at x.

The following lemma, which we proved in [7], gives representation formula (2.6) for nonnegative solutions of inequality (2.5).

Lemma 2.1. *Let $u(x)$ be a C^{2m} nonnegative solution of*

$$-\Delta^m u \geq 0 \quad in \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n, \tag{2.5}$$

where $m \geq 1$ and $n \geq 2$ are integers. Then $\int\limits_{|y|<1} |y|^{2m-2}(-\Delta^m u(y))\, dy < \infty$ and

$$u = N + h + \sum_{|\alpha|\leq 2m-2} a_\alpha D^\alpha \Phi \quad in \quad B_1(0)\backslash\{0\} \tag{2.6}$$

where $a_\alpha, |\alpha| \leq 2m-2$, are constants, $h \in C^\infty(B_1(0))$ is a solution of

$$\Delta^m h = 0 \quad in \quad B_1(0),$$

and

$$N(x) = \int\limits_{|y|\leq 1} \Psi(x,y)\Delta^m u(y)\, dy \quad for \quad x \neq 0. \tag{2.7}$$

Lemma 2.2. *Suppose f is locally bounded, nonnegative, and measurable in $\overline{B_1(0)}\backslash\{0\} \subseteq \mathbb{R}^n$ and*

$$\int\limits_{|y|<1} |y|^{2m-2} f(y)\, dy < \infty \tag{2.8}$$

where $m \geq 2$ and $n \geq 2$ are integers, m is odd, and $2m \leq n$. Let

$$N(x) = \int\limits_{|y|<1} -\Psi(x,y) f(y)\, dy \quad for \quad x \in \mathbb{R}^n\backslash\{0\} \tag{2.9}$$

where Ψ is given by (2.4). Then $N \in C^{2m-1}(\mathbb{R}^n\backslash\{0\})$. Moreover when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have

$$(D^\beta N)(x) = \int\limits_{\substack{|y-x|<|x|/2 \\ |y|<1}} -(D^\beta \Phi)(x-y) f(y)\, dy + O(|x|^{2-n-|\beta|}) \quad for \quad x \neq 0 \tag{2.10}$$

and when $2m = n$ we have

$$N(x) = A \int\limits_{\substack{|y-x|<|x|/2 \\ |y|<1}} \left(\log \frac{|x|}{|x-y|}\right) f(y)\, dy + O(|x|^{2-n}) \quad for \quad x \neq 0. \tag{2.11}$$

Proof. Differentiating (2.4) with respect to x we get

$$D_x^\beta \Psi(x,y) = (D^\beta \Phi)(x-y) - \sum_{|\alpha|\leq 2m-3} \frac{(-y)^\alpha}{\alpha!}(D^{\alpha+\beta}\Phi)(x) \quad \text{for} \quad x \neq 0 \quad \text{and} \quad y \neq x$$

and so by Taylor's theorem applied to $D^\beta \Phi$ we have

$$|D_x^\beta \Psi(x,y)| \leq C|y|^{2m-2}|x|^{2-n-|\beta|} \quad \text{for} \quad |y| < \frac{|x|}{2} \tag{2.12}$$

where in this proof $C = C(m, n, \beta)$ is a positive constant whose value may change from line to line.

Let $\varepsilon \in (0,1)$ be fixed. Then $N = N_1 + N_2$ in $\mathbb{R}^n \backslash \{0\}$ where

$$N_1(x) = \int_{|y|<\varepsilon} -\Psi(x,y) f(y)\, dy \quad \text{and} \quad N_2(x) = \int_{\varepsilon<|y|<1} -\Psi(x,y) f(y)\, dy.$$

It follows from (2.8) and (2.12) that $N_1 \in C^\infty(\mathbb{R}^n \backslash \overline{B_{2\varepsilon}(0)})$ and

$$(D^\beta N_1)(x) = \int_{|y|<\varepsilon} -D^\beta \Psi(x,y) f(y)\, dy \quad \text{for} \quad |x| > 2\varepsilon.$$

Also, by the boundedness of f in $B_1(0) \backslash B_\varepsilon(0)$, $N_2 \in C^{2m-1}(\mathbb{R}^n \backslash \overline{B_{2\varepsilon}(0)})$ and for $|\beta| < 2m$ we have

$$(D^\beta N_2)(x) = \int_{\varepsilon<|y|<1} -D^\beta \Psi(x,y) f(y)\, dy \quad \text{for} \quad |x| > 2\varepsilon.$$

Thus since $\varepsilon \in (0,1)$ was arbitrary, we have $N \in C^{2m-1}(\mathbb{R}^n \backslash \{0\})$ and for $|\beta| < 2m$ we have

$$(D^\beta N)(x) = \int_{|y|<1} -D_x^\beta \Psi(x,y) f(y)\, dy \quad \text{for} \quad x \neq 0. \tag{2.13}$$

Case 1. Suppose $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$. Then for $0 < |x|/2 < |y|$ we have

$$\left| \sum_{|\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^{\alpha+\beta} \Phi(x) \right| \leq C \sum_{|\alpha| \leq 2m-3} |y|^{|\alpha|} |x|^{2m-n-|\alpha|-|\beta|} \leq C|y|^{2m-2} |x|^{2-n-|\beta|}$$

and for $0 < |x|/2 < |y|$ and $|y - x| > |x|/2$ we have

$$|(D^\beta \Phi)(x-y)| \leq C|x-y|^{2m-n-|\beta|} \leq C|x|^{2m-n-|\beta|} \leq C|y|^{2m-2}|x|^{2-n-|\beta|}.$$

Thus (2.8), (2.12) and (2.13) imply (2.10).

Case 2. Suppose $2m = n$. Then for $0 < |x|/2 < |y|$ we have

$$\left| \sum_{1 \leq |\alpha| \leq 2m-3} \frac{(-y)^\alpha}{\alpha!} D^\alpha \Phi(x) \right| \leq C \sum_{1 \leq |\alpha| \leq 2m-3} |y|^{|\alpha|} |x|^{2m-n-|\alpha|} \leq C|y|^{2m-2}|x|^{2-n}$$

and if $0 < |x|/2 < |y|$ and $|y - x| > |x|/2$ then using the fact that $|\log z| \leq \log 4z$ for $z \geq 1/2$ we have

$$\begin{aligned} |-\Phi(x-y) + \Phi(x)| &= A \left| \log \frac{|x-y|}{|x|} \right| \leq A \log 4 \frac{|x-y|}{|x|} \\ &\leq A \frac{|y|^{n-2}}{|x|^{n-2}} \left(\frac{|x|}{|y|} \right)^{n-2} \log 4 \left(1 + \frac{|y|}{|x|} \right) \\ &\leq A \frac{|y|^{n-2}}{|x|^{n-2}} \max_{r \geq 1/2} r^{2-n} \log 4(1+r). \end{aligned}$$

Thus (2.11) follows from (2.8), (2.9), and (2.12). □

Lemma 2.3. *Suppose $u(x)$ is a C^{2m} nonnegative solution of*

$$-\Delta^m u \geq 0 \quad in \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n$$

where $m \geq 2$ and $n \geq 2$ are integers, m is odd, and $2m \leq n$. Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ and $\{r_j\}_{j=1}^\infty \subset \mathbb{R}$ be sequences such that

$$0 < 4|x_{j+1}| \leq |x_j| \leq 1/2 \quad and \quad 0 < r_j \leq |x_j|/4. \tag{2.14}$$

Define $f_j \colon B_2(0) \to [0,\infty)$ by

$$f_j(\eta) = |x_j|^{2m-2} r_j^n f(y) \quad where \quad y = x_j + r_j\eta \quad and \quad f = -\Delta^m u. \tag{2.15}$$

Then

$$\int\limits_{|\eta|<2} f_j(\eta)\, d\eta \to 0 \quad as \quad j \to \infty \tag{2.16}$$

and when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have for $|\xi| < 1$ that

$$\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} |x_j|^{n-2+|\beta|} |(D^\beta u)(x_j + r_j\xi)| \leq C \left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} + \varepsilon_j + \int\limits_{|\eta|<2} \frac{A f_j(\eta)\, d\eta}{|\xi - \eta|^{n-2m+|\beta|}} \tag{2.17}$$

and when $2m = n$ we have for $|\xi| < 1$ that

$$\frac{|x_j|^{n-2}}{\log \frac{|x_j|}{r_j}} u(x_j + r_j\xi) \leq \frac{C}{\log \frac{|x_j|}{r_j}} + \varepsilon_j + \frac{1}{\log \frac{|x_j|}{r_j}} \int\limits_{|\eta|<2} A \left(\log \frac{5}{|\xi - \eta|}\right) f_j(\eta)\, d\eta \tag{2.18}$$

where in (2.17) *and* (2.18) *the constant A depends only on m and n, the constant C is independent of ξ and j, the constants ε_j are independent of ξ, and $\varepsilon_j \to 0$ as $j \to \infty$.*

Proof. By Lemma 2.1, f satisfies (2.8) and for $|\beta| < 2m$ we have

$$(D^\beta u)(x) = (D^\beta N)(x) + O(|x|^{2-n-|\beta|}) \quad \text{for} \quad 0 < |x| \leq 3/4 \tag{2.19}$$

where N is given by (2.9).

If

$$|y - x| < |x|/2, \quad |y - x_j| > 2r_j, \quad \text{and} \quad |x - x_j| < r_j$$

then

$$|x - y| > r_j \quad \text{and} \quad 2|y| > |x| > |x_j| - r_j > |x_j|/2$$

and thus when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have

$$|(D^\beta \Phi)(x - y)| \leq \frac{A}{|x - y|^{n-2m+|\beta|}} \leq \frac{A}{r_j^{n-2m+|\beta|}} \leq \frac{A|y|^{2m-2}}{r_j^{n-2m+|\beta|}|x_j|^{2m-2}}$$

and when $2m = n$ we have

$$\log \frac{|x|}{|x - y|} \leq \log \frac{\frac{5}{4}|x_j|}{r_j} \leq 2 \cdot 4^{n-2} \frac{|y|^{n-2}}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j}.$$

Thus by (2.8) and Lemma 2.2, when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have

$$\begin{aligned} |(D^\beta N)(x)| &\le \int\limits_{|y-x_j|<2r_j} \frac{Af(y)\,dy}{|x-y|^{n-2m+|\beta|}} + A\frac{\int\limits_{|y-x|<|x|/2} |y|^{2m-2} f(y)\,dy}{r_j^{n-2m+|\beta|}|x_j|^{2m-2}} + \frac{C}{|x_j|^{n-2+|\beta|}} \\ &\le \int\limits_{|y-x_j|<2r_j} \frac{Af(y)\,dy}{|x-y|^{n-2m+|\beta|}} + \frac{\varepsilon_j}{r_j^{n-2m+|\beta|}|x_j|^{2m-2}} + \frac{C}{|x_j|^{n-2+|\beta|}} \quad \text{for} \quad |x-x_j| < r_j \end{aligned} \tag{2.20}$$

and when $2m = n$ we have

$$\begin{aligned} N(x) &\le A \int\limits_{|y-x_j|<2r_j} \left(\log\frac{|x|}{|x-y|}\right) f(y)\,dy + 2A4^{n-2}\left(\int\limits_{|y-x|<|x|/2} |y|^{n-2} f(y)\,dy\right) \frac{\log\frac{|x_j|}{r_j}}{|x_j|^{n-2}} + \frac{C}{|x_j|^{n-2}} \\ &\le A \int\limits_{|y-x_j|<2r_j} \left(\log\frac{|x|}{|x-y|}\right) f(y)\,dy + \varepsilon_j \frac{\log\frac{|x_j|}{r_j}}{|x_j|^{n-2}} + \frac{C}{|x_j|^{n-2}} \quad \text{for} \quad |x-x_j| < r_j \end{aligned} \tag{2.21}$$

where in (2.20) and (2.21) the constant A depends only on m and n, the constant C is independent of x and j, the constants ε_j are independent of x, and $\varepsilon_j \to 0$ as $j \to \infty$.

For $|\eta| < 2$ and y given by (2.15) we have $|x_j| < 2|y|$. Thus

$$\begin{aligned} \int\limits_{|\eta|<2} f_j(\eta)\,d\eta &= \int\limits_{|y-x_j|<2r_j} |x_j|^{2m-2} f(y)\,dy \\ &\le 2^{2m-2} \int\limits_{|y-x_j|<|x_j|/2} |y|^{2m-2} f(y)\,dy \to 0 \quad \text{as} \quad j \to \infty \end{aligned} \tag{2.22}$$

because f satisfies (2.8).

If $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ then by (2.20) and (2.15) we have for $|\xi| < 1$ that

$$\begin{aligned} &\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} |x_j|^{n-2+|\beta|} |(D^\beta N)(x_j + r_j\xi)| \\ &\qquad \le C\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} + \varepsilon_j + r_j^{n-2m+|\beta|}|x_j|^{2m-2} \int\limits_{|\eta|<2} \frac{Af(y) r_j^n\,d\eta}{r_j^{n-2m+|\beta|}|\xi-\eta|^{n-2m+|\beta|}} \\ &\qquad = C\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} + \varepsilon_j + \int\limits_{|\eta|<2} \frac{Af_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+|\beta|}}. \end{aligned} \tag{2.23}$$

If $2m = n$ and $|\xi| < 1$ then by (2.21), (2.15), and (2.22) we have

$$\begin{aligned} \frac{|x_j|^{n-2}}{\log\frac{|x_j|}{r_j}} N(x_j + r_j\xi) &\le \frac{C}{\log\frac{|x_j|}{r_j}} + \varepsilon_j + \frac{|x_j|^{n-2}}{\log\frac{|x_j|}{r_j}} A \int\limits_{|\eta|<2} \left(\log\frac{5|x_j|}{r_j|\xi-\eta|}\right) |x_j|^{2-n} f_j(\eta)\,d\eta \\ &\le \frac{C}{\log\frac{|x_j|}{r_j}} + \varepsilon_j + \frac{1}{\log\frac{|x_j|}{r_j}} \int\limits_{|\eta|<2} A\left(\log\frac{5}{|\xi-\eta|}\right) f_j(\eta)\,d\eta. \end{aligned} \tag{2.24}$$

Inequalities (2.17) and (2.18) now follow from (2.23), (2.24), and (2.19). ∎

Lemma 2.4. *Suppose $m \geq 2$ and $n \geq 2$ are integers, m is odd, and $2m \leq n$. Let $\psi\colon (0,1) \to (0,1)$ be a continuous function such that $\lim_{r\to 0^+} \psi(r) = 0$. Let $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ be a sequence such that*

$$0 < 4|x_{j+1}| \leq |x_j| \leq 1/2 \tag{2.25}$$

and

$$\sum_{j=1}^{\infty} \varepsilon_j < \infty \quad \textit{where} \quad \varepsilon_j = \psi(|x_j|). \tag{2.26}$$

Let $\{r_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be a sequence satisfying

$$0 < r_j \leq |x_j|/5. \tag{2.27}$$

Then there exists a positive function $u \in C^\infty(\mathbb{R}^n\backslash\{0\})$ and a positive constant $A = A(m,n)$ such that

$$0 \leq -\Delta^m u \leq \frac{\varepsilon_j}{|x_j|^{2m-2} r_j^n} \quad \textit{in} \quad B_{r_j}(x_j), \tag{2.28}$$

$$-\Delta^m u(x) = 0 \qquad \textit{in} \quad \mathbb{R}^n \Big\backslash \left(\{0\} \cup \bigcup_{j=1}^{\infty} B_{r_j}(x_j) \right), \tag{2.29}$$

and

$$u \geq \begin{cases} \frac{A\varepsilon_j}{|x_j|^{2m-2} r_j^{n-2m}} & \textit{in } B_{r_j}(x_j) \textit{ if } 2m < n \\ \frac{A\varepsilon_j}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j} & \textit{in } B_{r_j}(x_j) \textit{ if } 2m = n. \end{cases} \tag{2.30}$$

Proof. Let $\varphi\colon \mathbb{R}^n \to [0,1]$ be a C^∞ function whose support is $\overline{B_1(0)}$. Define $\varphi_j\colon \mathbb{R}^n \to [0,1]$ by

$$\varphi_j(y) = \varphi(\eta) \quad \text{where} \quad y = x_j + r_j \eta. \tag{2.31}$$

Then

$$\int_{\mathbb{R}^n} \varphi_j(y)\, dy = \int_{\mathbb{R}^n} \varphi(\eta) r_j^n \, d\eta = r_j^n I \tag{2.32}$$

where $I = \int_{\mathbb{R}^n} \varphi(\eta)\, d\eta > 0$. Let

$$f = \sum_{j=1}^{\infty} M_j \varphi_j \quad \text{where} \quad M_j = \frac{\varepsilon_j}{|x_j|^{2m-2} r_j^n}. \tag{2.33}$$

Since the functions φ_j have disjoint supports, $f \in C^\infty(\mathbb{R}^n\backslash\{0\})$ and by (2.27), (2.32), (2.33), and (2.26) we have

$$\begin{aligned} \int_{\mathbb{R}^n} |y|^{2m-2} f(y)\, dy &= \sum_{j=1}^{\infty} M_j \int_{|y-x_j|<r_j} |y|^{2m-2} \varphi_j(y)\, dy \\ &\leq 2^{2m-2} I \sum_{j=1}^{\infty} M_j |x_j|^{2m-2} r_j^n \\ &= 2^{2m-2} I \sum_{j=1}^{\infty} \varepsilon_j < \infty. \end{aligned} \tag{2.34}$$

Using the fact that

$$|x - x_j| < r_j \le |x_j|/5 \quad \text{implies} \quad B_{r_j}(x_j) \subset B_{\frac{|x|}{2}}(x), \tag{2.35}$$

we have for $2m < n$, $x = x_j + r_j\xi$, and $|\xi| < 1$ that

$$\begin{aligned}
\int\limits_{|y-x|<|x|/2} \frac{1}{|x-y|^{n-2m}} f(y)\,dy &\ge \int\limits_{|y-x_j|<r_j} \frac{1}{|x-y|^{n-2m}} M_j\varphi_j(y)\,dy \\
&= \int\limits_{|\eta|<1} \frac{1}{r_j^{n-2m}} \frac{M_j}{|\xi-\eta|^{n-2m}} \varphi(\eta) r_j^n \, d\eta \\
&= \frac{\varepsilon_j}{|x_j|^{2m-2} r_j^{n-2m}} \int\limits_{|\eta|<1} \frac{\varphi(\eta)}{|\xi-\eta|^{n-2m}} d\eta \\
&\ge \frac{J\varepsilon_j}{|x_j|^{2m-2} r_j^{n-2m}} \quad \text{where} \quad J = \min_{|\xi|\le 1} \int\limits_{|\eta|<1} \frac{\varphi(\eta)\,d\eta}{|\xi-\eta|^{n-2m}}.
\end{aligned}$$

Similarly, using (2.35) we have for $2m = n$, $x = x_j + r_j\xi$, and $|\xi| < 1$ that

$$\begin{aligned}
\int\limits_{|y-x|<|x|/2} \left(\log \frac{|x|}{|x-y|}\right) f(y)\,dy &\ge \int\limits_{|y-x_j|<r_j} \left(\log \frac{|x|}{|x-y|}\right) M_j\varphi_j(y)\,dy \\
&\ge \int\limits_{|\eta|<1} \left(\log \frac{\frac{4}{5}|x_j|}{r_j|\xi-\eta|}\right) M_j\varphi(\eta) r_j^n \, d\eta \\
&= \frac{\varepsilon_j}{|x_j|^{n-2}} \int\limits_{|\eta|<1} \left(\log \frac{2}{|\xi-\eta|} + \log \frac{|x_j|}{r_j} - \log \frac{5}{2}\right) \varphi(\eta)\,d\eta \\
&\ge \frac{I\varepsilon_j}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j} - \frac{I}{|x_j|^{n-2}} \log \frac{5}{2}.
\end{aligned}$$

Thus defining N by (2.9), where f is given by (2.33), it follows from (2.34) and Lemma 2.2 that there exists a positive constant C independent of ξ and j such that if we define $u\colon \mathbb{R}^n\backslash\{0\} \to \mathbb{R}$ by

$$u(x) = N(x) + C|x|^{-(n-2)}$$

then u is a C^∞ positive solution of

$$-\Delta^m u = f \quad \text{in} \quad \mathbb{R}^n\backslash\{0\} \tag{2.36}$$

and for some positive constant $A = A(m,n)$, u satisfies (2.30).

Also, (2.36) and (2.33) imply that u satisfies (2.28) and (2.29). □

Remark 2.1. Suppose the hypotheses of Lemma 2.4 hold and u is as in Lemma 2.4.

Case 1. Suppose $2m < n$. Then it follows from (2.28), (2.29), and (2.30) that u is a C^∞ positive solution of

$$0 \le -\Delta^m u \le |x|^\tau u^\lambda \quad \text{in} \quad \mathbb{R}^n\backslash\{0\}, \quad \lambda > 0, \quad \tau \in \mathbb{R},$$

provided

$$\frac{\psi(|x_j|)}{|x_j|^{2m-2} r_j^n} \le 2^{-|\tau|} |x_j|^\tau \left(\frac{A\psi(|x_j|)}{|x_j|^{2m-2} r_j^{n-2m}} \right)^\lambda$$

which holds if and only if

$$r_j^{n-\lambda(n-2m)} \ge \frac{2^{|\tau|}}{A^\lambda} \frac{|x_j|^{(\lambda-1)(2m-2)-\tau}}{\psi(|x_j|)^{\lambda-1}}. \tag{2.37}$$

Case 2. Suppose $2m = n$. Then it follows from (2.28), (2.29), and (2.30) that u is a C^∞ positive solution of

$$0 \le -\Delta^m u \le f(u) \quad \text{in} \quad \mathbb{R}^n \backslash \{0\},$$

where $f\colon [0,\infty) \to [0,\infty)$ is a nondecreasing continuous function, provided

$$\frac{\psi(|x_j|)}{|x_j|^{n-2} r_j^n} \le f\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j} \right). \tag{2.38}$$

If $f(u) = u^\lambda, \lambda > 1$, then (2.38) holds if and only if

$$\log \frac{|x_j|}{r_j} \ge \left(\frac{|x_j|}{r_j} \right)^{\frac{n}{\lambda}} \frac{|x_j|^a}{A\psi(|x_j|)^{\frac{\lambda-1}{\lambda}}} \quad \text{where} \quad a = \frac{(n-2)(\lambda-1)-n}{\lambda}.$$

If $f(u) = e^{u^\lambda}$, $\lambda > 0$, then (2.38) holds if and only if

$$\log \frac{\psi(|x_j|)}{|x_j|^{2n-2}} + n \log \frac{|x_j|}{r_j} \le \left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j} \right)^\lambda.$$

Lemma 2.5. *Suppose $p > 1$ and $R \in (0,2)$ are constants and $g\colon \mathbb{R}^n \to \mathbb{R}$ is defined by*

$$g(\xi) = \int_{|\eta|<R} \left(\log \frac{5}{|\xi - \eta|} \right) f(\eta)\, d\eta$$

where $f \in L^1(B_R(0))$, (resp. $f \in L^p(B_R(0))$) . Then

$$\|g\|_{L^p(B_R(0))} \le C\|f\|_{L^1(B_R(0))}, \quad (\textit{resp. } \|g\|_{L^\infty(B_R(0))} \le C\|f\|_{L^p(B_R(0))}),$$

where $C = C(n,p,R)$ is a positive constant.

Proof. Define p' by $\frac{1}{p} + \frac{1}{p'} = 1$. Then by Hölder's inequality we have

$$\begin{aligned}
\int_{|\xi|<R} |g(\xi)|^p\, d\xi &\le \int_{|\xi|<R} \left[\int_{|\eta|<R} \left(\log \frac{5}{|\xi-\eta|} \right) |f(\eta)|^{1/p} |f(\eta)|^{1/p'}\, d\eta \right]^p d\xi \\
&\le \int_{|\xi|<R} \left[\left(\int_{|\eta|<R} \left(\log \frac{5}{|\xi-\eta|} \right)^p |f(\eta)|\, d\eta \right)^{1/p} \left(\int_{|\eta|<R} |f(\eta)|\, d\eta \right)^{1/p'} \right]^p d\xi \\
&= \left(\int_{|\eta|<R} |f(\eta)|\, d\eta \right)^{p/p'} \int_{|\eta|<R} \left(\int_{|\xi|<R} \left(\log \frac{5}{|\xi-\eta|} \right)^p d\xi \right) |f(\eta)|\, d\eta \\
&\le C(n,p,R) \left(\int_{|\eta|<R} |f(\eta)|\, d\eta \right)^p.
\end{aligned}$$

The parenthetical part follows from Hölder's inequality. ☐

3 Proofs when the singularity is at the origin

In this section we prove Theorems 1.4–1.11 which deal with the case that the singularity is at the origin. By scaling and translating u in Theorem 1.4 and using for a in Theorem 1.4 the expression (1.10), we see that the following theorem implies Theorem 1.4.

Theorem 3.1. *Suppose $u(x)$ is a C^{2m} nonnegative solution of*

$$0 \le -\Delta^m u \le \sum_{k=0}^{2m-1} (|D^k u| + g_k)^{\lambda_k} \quad in \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n \tag{3.1}$$

where $m \ge 2$ and $n \ge 2$ are integers, m is odd, $2m < n$,

$$\lambda_k < \frac{n}{n-2m+k} \tag{3.2}$$

and $g_k \colon B_2(0)\backslash\{0\} \to [1,\infty)$ is a continuous function. Let

$$b = \max\left\{0, \max_{0\le k\le 2m-1} \frac{\lambda_k(n-2+k)-(2m+n-2)}{n-\lambda_k(n-2m+k)}\right\}. \tag{3.3}$$

(i) *If $b = 0$ (i.e. $\lambda_k - 1 \le \frac{2m-k}{n-2+k}$ for all $k \in \{0,1,2,\dots,2m-1\}$) and*

$$g_k(x) = O(|x|^{-(n-2+k)}) \quad as \quad x \to 0 \tag{3.4}$$

then for $i = 0,1,\dots,2m-1$ we have

$$|D^i u(x)| = O(|x|^{-(n-2+i)}) \quad as \quad x \to 0.$$

(ii) *If $b > 0$ (i.e. $\frac{2m-k_0}{n-2+k_0} < \lambda_{k_0} - 1 < \frac{2m-k_0}{n-2m+k_0}$ for some $k_0 \in \{0,1,\dots,2m-1\}$) and*

$$g_k(x) = o(|x|^{-a(k)}) \quad as \quad x \to 0 \tag{3.5}$$

where

$$a(i) = (n-2m+i)b + (n-2+i)$$

then for $i = 0,1,\dots,2m-1$ we have

$$|D^i u(x)| = o(|x|^{-a(i)}) \quad as \quad x \to 0. \tag{3.6}$$

Remark 3.1. By making only very minor changes in the proof of Theorem 3.1 below, one can easily verify that part (ii) of Theorem 3.1 remains true if one replaces "little oh" in (3.5) and (3.6) with "big oh".

Proof of Theorem 3.1. Since increasing to one those λ_k which are less than 1 will not change the value of b but will increase the right side of the second inequality in (3.1), we can, without loss of generality, assume, instead of (3.2), the stronger condition that

$$1 \le \lambda_k < \frac{n}{n-2m+k}. \tag{3.7}$$

Let b and g_k be as in part (i) (resp. part (ii)) of Theorem 3.1. Suppose for contradiction that part (i) (resp. part (ii)) is false. Then there exist $i \in \{0, 1, 2, \ldots, 2m-1\}$ and a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ such that

$$0 < 4|x_{j+1}| < |x_j| < 1/2,$$

and

$$|x_j|^{n-2+i}|D^i u(x_j)| \to \infty \quad \text{as} \quad j \to \infty, \tag{3.8}$$

$$\text{(resp.} \ \liminf_{j\to\infty} |x_j|^{a(i)}|D^i u(x_j)| > 0). \tag{3.9}$$

Let

$$r_j = \frac{|x_j|^{b+1}}{4}.$$

Then x_j and r_j satisfy (2.14). Let f_j be as in Lemma 2.3. Since

$$\frac{r_j}{|x_j|} = \frac{|x_j|^b}{4} \tag{3.10}$$

it follows from (2.17) with $|\beta| = i$ and $\xi = 0$ that

$$\frac{|x_j|^{(n-2m+i)b+(n-2+i)}}{4^{n-2m+i}}|D^i u(x_j)| \le C|x_j|^{(n-2m+i)b} + \varepsilon_j + \int\limits_{|\eta|<2} \frac{Af_j(\eta)d\eta}{|\eta|^{n-2m+i}}.$$

Hence (3.8) (resp. (3.9)) implies

$$\int\limits_{|\eta|<2} \frac{f_j(\eta)d\eta}{|\eta|^{n-2m+i}} \to \infty \quad \text{as} \quad j \to \infty \tag{3.11}$$

$$\left(\text{resp.} \ \liminf_{j\to\infty} \int\limits_{|\eta|<2} \frac{f_j(\eta)d\eta}{|\eta|^{n-2m+i}} > 0\right). \tag{3.12}$$

On the other hand, (2.15), (3.1), and (2.17) imply for $|\xi| < 1$ that

$$\begin{aligned} f_j(\xi) &\le |x_j|^{2m+n-2}\left(\frac{r_j}{|x_j|}\right)^n \sum_{k=0}^{2m-1}(g_k(x_j + r_j\xi) + |D^k u(x_j + r_j\xi)|)^{\lambda_k} \\ &\le \sum_{k=0}^{2m-1} \frac{|x_j|^{2m+n-2}\left(\frac{r_j}{|x_j|}\right)^n}{\left(|x_j|^{n-2+k}\left(\frac{r_j}{|x_j|}\right)^{n-2m+k}\right)^{\lambda_k}}\left(C\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} + \varepsilon_j + \int\limits_{|\eta|<2} \frac{Af_j(\eta)d\eta}{|\xi-\eta|^{n-2m+k}}\right. \\ &\quad \left. + |x_j|^{n-2+k}\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} g_k(x_j + r_j\xi)\right)^{\lambda_k}. \end{aligned} \tag{3.13}$$

But (3.10) and (3.3) imply

$$\begin{aligned}
\frac{(|x_j|^{2m+n-2}\left(\frac{r_j}{|x_j|}\right)^{n}}{\left(|x_j|^{n-2+k}\left(\frac{r_j}{|x_j|}\right)^{n-2m+k}\right)^{\lambda_k}} &= |x_j|^{(2m+n-2)-\lambda_k(n-2+k)}\left(\frac{r_j}{|x_j|}\right)^{n-(n-2m+k)\lambda_k} \\
&\le |x_j|^{(2m+n-2)-\lambda_k(n-2+k)+(n-(n-2m+k)\lambda_k)b} \\
&\le 1, \\
|x_j|^{n-2+k}\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} &\le |x_j|^{(n-2m+k)b+(n-2+k)} \\
&= |x_j|^{n-2+k}, \quad (\text{resp. } |x_j|^{a(k)}),
\end{aligned}$$

and

$$\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} \le |x_j|^{(n-2m+k)b}.$$

Hence by (3.4), (resp. (3.5)) and (3.13) we have

$$f_j(\xi) \le \sum_{k=0}^{2m-1}\left(C + \int_{|\eta|<2}\frac{Af_j(\eta)d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \quad \text{for} \quad |\xi|<1, \tag{3.14}$$

$$\left(\text{resp. } f_j(\xi) \le \sum_{k=0}^{2m-1}\left(\varepsilon_j + \int_{|\eta|<2}\frac{Af_j(\eta)d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \quad \text{for} \quad |\xi|<1\right). \tag{3.15}$$

Since

$$\int_{2R\le|\eta|<2}\frac{Af_j(\eta)d\eta}{|\xi-\eta|^{n-2m+k}} \le \frac{A}{R^{n-2m+k}}\int_{|\eta|<2} f_j(\eta)d\eta \quad \text{for} \quad |\xi|<R<1$$

we have by (3.14), (resp. (3.15)), and (2.16) that

$$f_j(\xi) \le \sum_{k=0}^{2m-1}\left(\frac{C}{R^{n-2m+k}} + \int_{|\eta|<2R}\frac{Af_j(\eta)d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \quad \text{for} \quad |\xi|<R\le 1$$

where C is independent of ξ, j, and R, (resp.

$$f_j(\xi) \le \sum_{k=0}^{2m-1}\left(\frac{\varepsilon_j}{R^{n-2m+k}} + \int_{|\eta|<2R}\frac{Af_j(\eta)d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \quad \text{for} \quad |\xi|<R\le 1$$

where ε_j is independent of ξ and R and $\varepsilon_j \to 0$ as $j\to\infty$).

It therefore follows from Riesz potential estimates (see [8, Lemma 7.12] that if the functions f_j are bounded (resp. tend to zero) in $L^p(B_{2R}(0))$ for some $p\ge 1$ and $R\in(0,1]$ then the functions f_j are bounded (resp. tend to zero) in $L^q(B_R(0))$ for $1\le q\le\infty$ and

$$\frac{1}{p}-\frac{1}{q} < \min_{0\le k\le 2m-1}\frac{n-\lambda_k(n-2m+k)}{n} > 0 \quad \text{by (3.7)}.$$

So starting with (2.16) and iterating this fact a finite number of times we see that there exists $R_0 \in (0,1)$ such that the functions f_j are bounded (resp. tend to zero) in $L^\infty(B_{R_0}(0))$ which together with (2.16) contradicts (3.11) (resp. (3.12)) and thereby completes the proof of Theorem 3.1. $\square$

Proof of Theorem 1.5. Define $\psi\colon (0,1) \to (0,1)$ by

$$\psi(r) = \max\left\{\varphi(r)^p, r^{\frac{n-\lambda(n-2m)}{\lambda-1}\frac{b}{2}}\right\} \tag{3.16}$$

where

$$b := \frac{\lambda(n-2)-(2m+n-2)}{n-\lambda(n-2m)} \quad \text{and} \quad p := \frac{n-\lambda(n-2m)}{4m}$$

are positive by (1.11). Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ be a sequence satisfying (2.25) and (2.26). Define $r_j > 0$ by (2.37) with the greater than sign replaced with an equal sign and with $\tau = 0$. Then by (3.16)

$$\begin{aligned} r_j &= A^{\frac{-\lambda}{n-\lambda(n-2m)}} \frac{|x_j|^{1+b}}{\psi(|x_j|)^{\frac{\lambda-1}{n-\lambda(n-2m)}}} \\ &\le A^{\frac{-\lambda}{n-\lambda(n-2m)}} |x_j|^{1+b/2}. \end{aligned} \tag{3.17}$$

Thus by taking a subsequence of j, r_j will satisfy (2.27).

Let u be as in Lemma 2.4. Then by Case 1 of Remark 2.1, u is a C^∞ positive solution of (1.12) and by (2.30), (3.17), (3.16), and (1.10) we have

$$\begin{aligned} u(x_j) &\ge \frac{A\psi(|x_j|)}{|x_j|^{2m-2}} \frac{A^{\frac{\lambda(n-2m)}{n-\lambda(n-2m)}} \psi(|x_j|)^{\frac{(\lambda-1)(n-2m)}{n-\lambda(n-2m)}}}{|x_j|^{(n-2m)(1+b)}} \\ &= C(m,n,\lambda) \frac{\psi(|x_j|)^{\frac{2m}{n-\lambda(n-2m)}}}{|x_j|^{n-2+(n-2m)b}} \\ &\ge C(m,n,\lambda) \frac{\varphi(|x_j|)^{1/2}}{|x_j|^a} \end{aligned}$$

which implies (1.13). $\square$

Proof of Theorem 1.6. Define $\psi\colon (0,1) \to (0,1)$ by $\psi(r) = r^{m-1}$. Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ be a sequence satisfying (2.25), (2.26), and

$$\frac{1}{A^\lambda} \frac{|x_j|^{(\lambda-1)(2m-2)}}{\psi(|x_j|)^{\lambda-1}} = A^{-\lambda}|x_j|^{(\lambda-1)(m-1)} < 1 \tag{3.18}$$

where $A = A(m,n)$ is as in Lemma 2.4. Let $\{r_j\}_{j=1}^\infty \subset \mathbb{R}$ be a sequence satisfying (2.27) and

$$\frac{A\psi(|x_j|)}{|x_j|^{2m-2} r_j^{n-2m}} > \varphi(|x_j|)^2. \tag{3.19}$$

Since $r_j < 1$ we see that (3.18) implies (2.37) with $\tau = 0$. Let u be as in Lemma 2.4. Then by (2.30) and (3.19)

$$\frac{u(x_j)}{\varphi(|x_j|)} \ge \varphi(|x_j|) \to \infty \quad \text{as} \quad j \to \infty$$

and by Case 1 of Remark 2.1, u is a C^∞ positive solution of (1.14). $\square$

By scaling and translating u in Theorem 1.7, the following theorem implies Theorem 1.7.

Theorem 3.2. *Suppose u is a C^{2m} nonnegative solution of*

$$0 \leq -\Delta^m u \leq \sum_{k=0}^{2m-1} (|D^k u| + g_k)^{\lambda_k} \quad in \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n \tag{3.20}$$

where $m \geq 2$ and $n \geq 2$ are integers, m is odd, $2m = n$,

$$\lambda_0 \in \mathbb{R}, \quad \lambda_k < n/k \quad for \quad k = 1, 2, \ldots, n-1, \tag{3.21}$$

and $g_k \colon B_2(0)\backslash\{0\} \to [1, \infty)$ is a continuous function. Let

$$b = \max\{0, b_0, b_1, \ldots, b_{n-1}\} \quad where \quad b_k = \frac{\lambda_k(n-2+k) - (2n-2)}{n - k\lambda_k}.$$

(i) *If $b = 0$ (i.e. $\lambda_k \leq \frac{2n-2}{n-2+k}$ for all $k \in \{0, 1, \ldots, n-1\}$) and for $k = 0, 1, \ldots, n-1$ we have*

$$g_k(x) = O(|x|^{-(n-2+k)}) \quad as \quad x \to 0 \tag{3.22}$$

then for $i = 0, 1, \ldots, n-1$ we have

$$|D^i u(x)| = O(|x|^{-(n-2+i)}) \quad as \quad x \to 0.$$

(ii) *If $b > 0$ and as $x \to 0$ we have*

$$g_0(x) = o\left(|x|^{-(n-2)} \log \frac{5}{|x|}\right) \tag{3.23}$$

and

$$g_k(x) = o(|x|^{-(n-2+k)} a(x)^{-k}) \quad for \quad k = 1, 2, \ldots, n-1 \tag{3.24}$$

where

$$a(x) = \min\left\{\frac{|x|^{b_0}}{\left(\log \frac{5}{|x|}\right)^{\lambda_0/n}}, |x|^{b_1}, \ldots, |x|^{b_{n-1}}\right\} \tag{3.25}$$

then as $x \to 0$ we have

$$u(x) = o\left(|x|^{-(n-2)} \log \frac{5}{|x|}\right)$$

and

$$|D^i u(x)| = o(|x|^{-(n-2+i)} a(x)^{-i}) \quad for \quad i = 1, 2, \ldots, n-1.$$

Proof. Since increasing to one those λ_k which are less that one will change neither the value of b nor, when $b > 0$, the value of $a(x)$ for x small, but will increase the right side of the second inequality in (3.20) we can, without loss of generality, assume, instead of (3.21), the stronger condition that

$$\lambda_0 \geq 1 \quad \text{and} \quad 1 \leq \lambda_k < n/k \quad \text{for} \quad k = 1, 2, \ldots, n-1. \tag{3.26}$$

Let b and g_k be as in part (i) (resp. part (ii)) of Theorem 3.2. For $0 < |x| < 1$ let $a(x)$ be defined by $a(x) \equiv \frac{1}{4}$, (resp. by (3.25)). Then

$$\log \frac{1}{a(x)} \equiv \log 4, \qquad \left(\text{resp.} \quad \frac{\log \frac{1}{a(x)}}{\log \frac{5}{|x|}} \to b > 0 \quad \text{as} \quad x \to 0 \right). \tag{3.27}$$

Suppose for contradiction that part (i) (resp. part (ii)) is false. Then there exists $i \in \{0, 1, 2, \ldots, 2m-1\}$ and a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ satisfying

$$0 < 4|x_{j+1}| < |x_j| < 1/2 \quad \text{and} \quad a(x_j) \leq \frac{1}{4}$$

such that

$$\liminf_{j \to \infty} \frac{|x_j|^{n-2}}{\log \frac{1}{a(x_j)}} u(x_j) = \infty, \quad (\text{resp.} \ > 0) \quad \text{if} \quad i = 0 \tag{3.28}$$

$$\liminf_{j \to \infty} |x_j|^{n-2+i} a(x_j)^i D^i u(x_j) = \infty, \quad (\text{resp.} \ > 0) \quad \text{if} \quad i \in \{1, 2, \ldots, n-1\}. \tag{3.29}$$

Let $r_j = |x_j| a(x_j)$. Then x_j and r_j satisfy (2.14). Let f_j be as in Lemma 2.3. Since

$$\frac{r_j}{|x_j|} = a(x_j) \equiv \frac{1}{4}, \quad (\text{resp.} \ \to 0 \quad \text{as} \quad j \to \infty),$$

and by (2.18) and (2.17) with $\xi = 0$,

$$\frac{|x_j|^{n-2}}{\log \frac{|x_j|}{r_j}} u(x_j) \leq \frac{C}{\log \frac{|x_j|}{r_j}} + \varepsilon_j + \frac{1}{\log \frac{|x_j|}{r_j}} \int_{|\eta|<2} A \left(\log \frac{5}{|\eta|} \right) f_j(\eta)\, d\eta$$

and for $i \in \{1, 2, \ldots, n-1\}$

$$\left(\frac{r_j}{|x_j|} \right)^i |x_j|^{n-2+i} |D^i u(x_j)| \leq C \left(\frac{r_j}{|x_j|} \right)^i + \varepsilon_j + \int_{|\eta|<2} \frac{A f_j(\eta)}{|\eta|^i} d\eta,$$

it follows from (3.28) and (3.29) that

$$\liminf_{j \to \infty} \int_{|\eta|<2} \left(\log \frac{5}{|\eta|} \right) f_j(\eta)\, d\eta = \infty, \quad (\text{resp.} \ > 0) \quad \text{if} \quad i = 0 \tag{3.30}$$

and

$$\liminf_{j \to \infty} \int_{|\eta|<2} \frac{f_j(\eta)}{|\eta|^i} d\eta = \infty, \quad (\text{resp.} \ > 0) \quad \text{if} \quad i \in \{1, 2, \ldots, n-1\}. \tag{3.31}$$

On the other hand, (2.15), (3.20), (2.18), and (2.17) imply for $|\xi| < 1$ that

$$f_j(\xi) \le |x_j|^{2n-2} \left(\frac{r_j}{|x_j|}\right)^n \sum_{k=0}^{n-1} (|D^k u(x_j + r_j\xi)| + g_k(x_j + r_j\xi))^{\lambda_k}$$

$$\le B_{0j} \left(\frac{C}{\log \frac{|x_j|}{r_j}} + \varepsilon_j + \frac{1}{\log \frac{|x_j|}{r_j}} \int\limits_{|\eta|<2} A \left(\log \frac{5}{|\xi - \eta|} \right) f_j(\eta)\, d\eta + G_{0j}(\xi) \right)^{\lambda_0}$$

$$+ \sum_{k=1}^{n-1} B_{kj} \left(C \left(\frac{r_j}{|x_j|} \right)^k + \varepsilon_j + \int\limits_{|\eta|<2} \frac{A f_j(\eta)}{|\xi - \eta|^k} d\eta + G_{kj}(\xi) \right)^{\lambda_k}$$

where

$$B_{0j} := \frac{|x_j|^{2n-2} \left(\frac{r_j}{|x_j|}\right)^n}{\left(\frac{|x_j|^{n-2}}{\log \frac{|x_j|}{r_j}} \right)^{\lambda_0}} = \left(\frac{a(x_j) \left(\log \frac{1}{a(x_j)} \right)^{\lambda_0/n}}{|x_j|^{b_0}} \right)^n$$

$$\le \frac{1}{4^n} (\log 4)^{\lambda_0}, \qquad \left(\text{resp. } \left(\frac{\log \frac{1}{a(x_j)}}{\log \frac{5}{|x_j|}} \right)^{\lambda_0} \right),$$

$$B_{kj} := \frac{|x_j|^{2n-2} \left(\frac{r_j}{|x_j|}\right)^n}{\left(\left(\frac{r_j}{|x_j|} \right)^k |x_j|^{n-2+k} \right)^{\lambda_k}} = \left(\frac{a(x_j)}{|x_j|^{b_k}} \right)^{n-k\lambda_k} \le 1,$$

$$G_{0j}(\xi) := \frac{|x_j|^{n-2}}{\log \frac{|x_j|}{r_j}} g_0(x_j + r_j\xi) = \frac{|x_j|^{n-2}}{\log \frac{1}{a(x_j)}} g_0(x_j + r_j\xi),$$

$$G_{kj}(\xi) := \left(\frac{r_j}{|x_j|} \right)^k |x_j|^{n-2+k} g_k(x_j + r_j\xi) = a(x_j)^k |x_j|^{n-2+k} g_k(x_j + r_j\xi).$$

It follows therefore from (3.27), (3.22), (3.23), and (3.24) that for $|\xi| < 1$ we have

$$f_j(\xi) \le C \left[\left(1 + \int\limits_{|\eta|<2} \left(\log \frac{5}{|\xi - \eta|} \right) f_j(\eta)\, d\eta \right)^{\lambda_0} + \sum_{k=1}^{n-1} \left(1 + \int\limits_{|\eta|<2} \frac{f_j(\eta)}{|\xi - \eta|^k} d\eta \right)^{\lambda_k} \right], \tag{3.32}$$

$$\left(\text{resp. } f_j(\xi) \le C \left[\left(\varepsilon_j + \int\limits_{|\eta|<2} \left(\log \frac{5}{|\xi - \eta|} \right) f_j(\eta)\, d\eta \right)^{\lambda_0} + \sum_{k=1}^{n-1} \left(\varepsilon_j + \int\limits_{|\eta|<2} \frac{f_j(\eta)}{|\xi - \eta|^k} d\eta \right)^{\lambda_k} \right] \right), \tag{3.33}$$

where C is independent of ξ and j, ε_j is independent of ξ, and $\varepsilon_j \to 0$ as $j \to \infty$.

Using an argument very similar to the one used at the end of the proof of Theorem 3.1 to show that (3.14), (resp. (3.15)), leads to a contradiction of (3.11), (resp. (3.12)), one can show that (3.32), (resp. (3.33)), leads to a contradiction of (3.28), (resp. (3.29))—the only significant difference being that where we used Riesz potential estimates in the proof of Theorem 3.1, we must now use Riesz potential estimates *and* Lemma 2.5. □

Proof of Theorem 1.8. It follows from (1.17) that

$$a := \frac{(n-2)(\lambda-1)-n}{\lambda} > 0. \tag{3.34}$$

Define $\psi\colon (0,1) \to (0,1)$ and $\rho\colon (0,1) \to (0,\infty)$ by

$$\psi(r) = \max\{\sqrt{\varphi(r)}, r^{\frac{a\lambda}{2(\lambda-1)}}\} \tag{3.35}$$

and

$$\rho(r) = \frac{n}{\lambda A}\frac{r^a}{\psi(r)^{\frac{\lambda-1}{\lambda}}} \tag{3.36}$$

where $A = A(m,n)$ is as in Lemma 2.4. By (3.35)

$$\rho(r) \le \frac{n}{\lambda A} r^{a/2}. \tag{3.37}$$

Thus there exists a sequence $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ satisfying (2.25), (2.26), and

$$e^{-1} > \rho_j := \rho(|x_j|) \to 0 \quad \text{as} \quad j \to \infty \tag{3.38}$$

such that if we define the sequence $\{r_j\}_{j=1}^\infty$ by

$$\left(\frac{|x_j|}{r_j}\right)^{n/\lambda} = \frac{1}{\rho_j}\log\frac{1}{\rho_j} \tag{3.39}$$

then r_j satisfies (2.27). By (3.39), (3.38), and (3.36) we have

$$\begin{aligned}
\log\frac{|x_j|}{r_j} &= \frac{\lambda}{n}\log\left(\frac{1}{\rho_j}\log\frac{1}{\rho_j}\right) \\
&\ge \frac{\lambda}{n}\log\frac{1}{\rho_j} && (3.40)\\
&= \frac{\lambda}{n}\rho_j\left(\frac{|x_j|}{r_j}\right)^{n/\lambda} \\
&= \frac{1}{A}\frac{|x_j|^a}{\psi(|x_j|)^{\frac{\lambda-1}{\lambda}}}\left(\frac{|x_j|}{r_j}\right)^{n/\lambda}. && (3.41)
\end{aligned}$$

Let u be as in Lemma 2.4. Then by (3.41) and Case 2 of Remark 2.1, u is a C^∞ positive solution of (1.18) and by Lemma 2.4 we have

$$u(x_j) \ge \frac{A\psi(|x_j|)}{|x_j|^{n-2}}\log\frac{|x_j|}{r_j}$$

and by (3.40) and (3.37),

$$\begin{aligned}
\log\frac{|x_j|}{r_j} &\ge \frac{\lambda}{n}\log\frac{1}{\rho_j} \ge \frac{\lambda}{n}\log\left(\frac{\lambda A}{n}|x_j|^{-a/2}\right) \\
&= \frac{\lambda}{n}\left(\frac{a}{2}\log\frac{1}{|x_j|} + \log\frac{\lambda A}{n}\right).
\end{aligned}$$

Thus by (3.35) we have

$$\liminf_{j\to\infty}\frac{u(x_j)}{\sqrt{\varphi(|x_j|)}|x_j|^{-(n-2)}\log\frac{1}{|x_j|}} \ge A\frac{\lambda a}{2n} > 0$$

from which we obtain (1.19). □

By scaling u in Theorem 1.9, the following theorem implies Theorem 1.9.

Theorem 3.3. *Let $u(x)$ be a C^{2m} nonnegative solution of*

$$0 \le -\Delta^m u \le e^{u^\lambda + g^\lambda} \quad in \quad B_2(0)\backslash\{0\} \subset \mathbb{R}^n \tag{3.42}$$

where $n \ge 2$ and $m \ge 2$ are integers, m is odd, $2m = n$,

$$0 < \lambda < 1, \tag{3.43}$$

and $g\colon B_2(0)\backslash\{0\} \to [0,\infty)$ is a continuous function such that

$$g(x) = o\left(|x|^{\frac{-(n-2)}{1-\lambda}}\right) \quad as \quad x \to 0. \tag{3.44}$$

Then

$$u(x) = o\left(|x|^{\frac{-(n-2)}{1-\lambda}}\right) \quad as \quad x \to 0. \tag{3.45}$$

Proof. Suppose for contradiction that (3.45) does not hold. Then there exists a sequence $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ such that

$$0 < 4|x_{j+1}| < |x_j| < 1/2$$

and

$$\liminf_{j\to\infty} |x_j|^{\frac{n-2}{1-\lambda}} u(x_j) > 0. \tag{3.46}$$

Define $r_j > 0$ by

$$\log \frac{1}{r_j} = |x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}. \tag{3.47}$$

Then

$$\begin{aligned} \log \frac{|x_j|}{r_j} &= \log \frac{1}{r_j} - \log \frac{1}{|x_j|} = |x_j|^{\frac{-(n-2)\lambda}{1-\lambda}} \left[1 - |x_j|^{\frac{(n-2)\lambda}{1-\lambda}} \log \frac{1}{|x_j|}\right] \\ &= |x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}(1 + o(1)) \quad \text{as} \quad j \to \infty. \end{aligned} \tag{3.48}$$

So, by taking a subsequence of j if necessary, we can assume $r_j < |x_j|/4$.

Let f_j be as in Lemma 2.3. Multiplying (2.18) by $|x_j|^{\frac{(n-2)\lambda}{1-\lambda}} \log \frac{|x_j|}{r_j}$ and using (3.48) we get for $|\xi| < 1$ that

$$|x_j|^{\frac{n-2}{1-\lambda}} u(x_j + r_j\xi) \le \varepsilon_j + |x_j|^{\frac{(n-2)}{1-\lambda}} \int_{|\eta|<2} A\left(\log \frac{5}{|\xi - \eta|}\right) \frac{f_j(\eta)}{|x_j|^{n-2}}\, d\eta \tag{3.49}$$

where the constant A depends only on m and n, the constants ε_j are independent of ξ, and $\varepsilon_j \to 0$ as $j \to \infty$. Substituting $\xi = 0$ in (3.49) and using (3.46) and (2.16) we get

$$\liminf_{j\to\infty} |x_j|^{\frac{n-2}{1-\lambda}} \int_{|\eta|<1} \left(\log \frac{5}{|\eta|}\right) \frac{f_j(\eta)}{|x_j|^{n-2}}\, d\eta > 0. \tag{3.50}$$

By (2.15), (3.42), (3.49), and (3.44) we have

$$\frac{f_j(\xi)}{|x_j|^{n-2} r_j^n} \le e^{u_j(\xi)^\lambda + M_j^\lambda} \quad \text{for} \quad |\xi| < 1 \tag{3.51}$$

where

$$u_j(\xi) = \int\limits_{|\eta|<2} A\left(\log \frac{5}{|\xi-\eta|}\right) \frac{f_j(\eta)}{|x_j|^{n-2}}\, d\eta$$

and the positive constants M_j satisfy

$$M_j |x_j|^{\frac{n-2}{1-\lambda}} \to 0 \quad \text{and} \quad M_j \to \infty \quad \text{as} \quad j \to \infty. \tag{3.52}$$

Let $\Omega_j = \{\xi \in B_1(0)\colon u_j(\xi) > M_j\}$. Then for $\xi \in \Omega_j$ it follows from (3.51) that

$$\begin{aligned} \frac{f_j(\xi)^2}{(|x_j|^{n-2} r_j^n)^2} &\le e^{4u_j(\xi)^\lambda} \\ &\le \exp\left[\left(\int\limits_{|\eta|<2} b_j \left(\log \frac{5}{|\xi-\eta|}\right) \frac{f_j(\eta)}{\int_{B_2} f_j} d\eta\right)^\lambda\right] \end{aligned} \tag{3.53}$$

where

$$b_j = 4^{1/\lambda} A |x_j|^{-(n-2)} \max\left\{\int\limits_{B_2} f_j, |x_j|^{\frac{n-2}{2}}\right\}.$$

By (2.16),

$$b_j |x_j|^{n-2} \to 0 \quad \text{and} \quad b_j \to \infty \quad \text{as} \quad j \to \infty. \tag{3.54}$$

Hence by (3.53), Jensen's inequality, and the fact that $\exp(t^\lambda)$ is concave up for t large we have for $\xi \in \Omega_j$ that

$$\frac{f_j(\xi)^2}{(|x_j|^{n-2} r_j^n)^2} \le \int\limits_{|\eta|<2} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi-\eta|}\right)^\lambda\right) \frac{f_j(\eta)}{\int_{B_2} f_j}\, d\eta$$

and consequently

$$\begin{aligned} \int\limits_{\Omega_j} \frac{f_j(\xi)^2}{(|x_j|^{n-2} r_j^n)^2}\, d\xi &\le \int\limits_{|\eta|<2} \left(\int\limits_{|\xi|<1} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi-\eta|}\right)^\lambda\right) d\xi\right) \frac{f_j(\eta)}{\int_{B_2} f_j}\, d\eta \\ &\le \max_{|\eta|\le 2} \int\limits_{|\xi|<1} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi-\eta|}\right)^\lambda\right) d\xi \\ &= \int\limits_{|\xi|<1} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi|}\right)^\lambda\right) d\xi \\ &= I_1 + I_2 \end{aligned} \tag{3.55}$$

where

$$I_1 = \int\limits_{\substack{\log \frac{5}{|\xi|} < (b_j^\lambda \lambda)^{\frac{1}{1-\lambda}} \\ |\xi|<1}} \exp\left(b_j^\lambda (\log \frac{5}{|\xi|})^\lambda\right) d\xi \quad \text{and} \quad I_2 = \int\limits_{\log \frac{5}{|\xi|} > (b_j^\lambda \lambda)^{\frac{1}{1-\lambda}}} \exp\left(b_j^\lambda (\log \frac{5}{|\xi|})^\lambda\right) d\xi.$$

Clearly

$$\frac{I_1}{|B_1(0)|} \le \exp\left((b_j(b_j^\lambda \lambda)^{\frac{1}{1-\lambda}})^\lambda\right) = \exp\left((b_j\lambda)^{\frac{\lambda}{1-\lambda}}\right) \le \exp\left(b_j^{\frac{\lambda}{1-\lambda}}\right)$$

and using Jensen's inequality and the fact that $e^{b_j^\lambda(\log t)^\lambda}$ is concave down as a function of t for $\log t > (b_j^\lambda \lambda)^{\frac{1}{1-\lambda}}$ one can show that

$$I_2 \le \exp\left(Cb_j^{\frac{\lambda}{1-\lambda}}\right)$$

where C depends only on n. Therefore by (3.55) and (3.47),

$$\begin{aligned}\int\limits_{\Omega_j} \left(\frac{f_j(\xi)}{|x_j|^{n-2}}\right)^2 d\xi &\le \exp\left(-2n|x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}\right)\exp\left(Cb_j^{\frac{\lambda}{1-\lambda}}\right) \\ &= \exp\left(Cb_j^{\frac{\lambda}{1-\lambda}} - 2n|x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}\right) \to 0 \quad \text{as} \quad j \to \infty\end{aligned}$$

by (3.54). Thus by Hölder's inequality

$$\lim_{j\to\infty} \int\limits_{\Omega_j} \left(\log\frac{5}{|\eta|}\right)\frac{f_j(\eta)}{|x_j|^{n-2}}\, d\eta = 0.$$

Hence defining $g_j\colon B_1(0) \to [0,\infty)$ by

$$g_j(\xi) := \begin{cases} f_j(\xi) & \text{for } \xi \in B_1\backslash\Omega_j, \\ 0 & \text{for } \xi \in \Omega_j, \end{cases}$$

it follows from (3.50) and (3.52) that

$$\frac{1}{M_j^\lambda}\int\limits_{|\eta|<1}\left(\log\frac{5}{|\eta|}\right) g_j(\eta)\, d\eta \to \infty \quad \text{as} \quad j \to \infty. \tag{3.56}$$

By (2.16), we have

$$\int\limits_{|\eta|<1} g_j(\eta)\, d\eta \to 0 \quad \text{as} \quad j \to \infty \tag{3.57}$$

and by (3.51) we have

$$g_j(\xi) \le e^{2M_j^\lambda} \quad \text{in} \quad B_1(0). \tag{3.58}$$

For fixed j, think of $g_j(\eta)$ as the density of a distribution of mass in B_1 satisfying (3.56), (3.57), and (3.58). By moving small pieces of this mass nearer to the origin in such a way that the new density (which we again denote by $g_j(\eta)$) does not violate (3.58), we will not change the total mass $\int_{B_1} g_j(\eta)\, d\eta$ but $\int_{B_1}(\log 5/|\eta|)g_j(\eta)\, d\eta$ will increase. Thus for some $\rho_j \in (0,1)$ the functions

$$g_j(\eta) = \begin{cases} e^{2M_j^\lambda} & \text{for } |\eta| < \rho_j, \\ 0 & \text{for } \rho_j < |\eta| < 1 \end{cases}$$

satisfy (3.56), (3.57), and (3.58), which, as elementary and explicit calculations show, is impossible because $M_j \to \infty$ as $j \to \infty$. This contradiction proves Theorem 3.3. □

Proof of Theorem 1.10. Define $\psi\colon (0,1) \to (0,1)$ by

$$\psi(r) = \max\left\{\varphi(r)^{\frac{1-\lambda}{2}}, r^{\frac{n-2}{2}}\right\}.$$

Since $\psi(r) \geq r^{\frac{n-2}{2}}$ there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ satisfying (2.25) and (2.26) such that if we define the sequence $\{r_j\}_{j=1}^{\infty} \subset (0,\infty)$ by

$$\log \frac{|x_j|}{r_j} = \left[\frac{1}{2n}\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\right)^{\lambda}\right]^{\frac{1}{1-\lambda}}, \tag{3.59}$$

where $A = A(m,n)$ is as in Lemma 2.4, then r_j will satisfy (2.27) and

$$\log \frac{1}{|x_j|^{2n-2}} < \log \frac{|x_j|}{r_j}.$$

Thus

$$\begin{aligned} \frac{\log \frac{\psi(|x_j|)}{|x_j|^{2n-2}} + n \log \frac{|x_j|}{r_j}}{\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j}\right)^{\lambda}} &\leq \frac{2n \log \frac{|x_j|}{r_j}}{\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j}\right)^{\lambda}} \\ &= \frac{\left(\log \frac{|x_j|}{r_j}\right)^{1-\lambda}}{\frac{1}{2n}\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\right)^{\lambda}} = 1. \end{aligned} \tag{3.60}$$

Let u be as in Lemma 2.4. Then by (3.60) and Case 2 of Remark 2.1, u is a C^∞ positive solution of (1.22) and by Lemma 2.4 and (3.59) we have

$$\begin{aligned} u(x_j) &\geq \frac{A\psi(|x_j|)}{|x_j|^{n-2}}\left[\frac{1}{2n}\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\right)^{\lambda}\right]^{\frac{1}{1-\lambda}} \\ &= \left(\frac{A\psi(|x_j|)}{2n}\right)^{\frac{1}{1-\lambda}} \frac{1}{|x_j|^{\frac{n-2}{1-\lambda}}} \\ &\geq \left(\frac{A}{2n}\right)^{\frac{1}{1-\lambda}} \sqrt{\varphi(|x_j|)}\, |x_j|^{-\frac{n-2}{1-\lambda}} \end{aligned}$$

which implies (1.23). □

Proof of Theorem 1.11. Define $\psi\colon (0,1) \to (0,1)$ by $\psi(r) = r^{\frac{n-2}{2}}$. Choose a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ satisfying (2.25), (2.26), and

$$\frac{A\psi(|x_j|)}{|x_j|^{n-2}} > n+1 \tag{3.61}$$

where $A = A(m,n)$ is as in Lemma 2.4. Choose a sequence $\{r_j\}_{j=1}^{\infty} \subset \mathbb{R}$ satisfying (2.27),

$$\log \frac{1}{|x_j|^{2n-2}} < \log \frac{|x_j|}{r_j} \tag{3.62}$$

and

$$(n+1)\log \frac{|x_j|}{r_j} > \varphi(|x_j|)^2. \tag{3.63}$$

Then by (3.62) and (3.61) we have

$$\log \frac{\psi(|x_j|)}{|x_j|^{2n-2}} + n \log \frac{|x_j|}{r_j} \le (n+1) \log \frac{|x_j|}{r_j} \le \left((n+1)\log \frac{|x_j|}{r_j}\right)^\lambda \le \left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j}\right)^\lambda . \quad (3.64)$$

Let u be as in Lemma 2.4. Then by (3.64) and Case 2 of Remark 2.1, u is a C^∞ positive solution of (1.24) and by Lemma 2.4, (3.61) and (3.63) we have

$$u(x_j) \ge \varphi(x_j)^2$$

which implies (1.25). □

4 Proofs when the singularity is at infinity

In this section we prove Theorems 1.14–1.17 which deal with the case that the singularity is at infinity.

By scaling and translating v in Theorems 1.14, 1.15, and 1.16, we see that Theorems 4.1, 4.2, and 4.3 below imply Theorems 1.14, 1.15, and 1.16 respectively.

Theorem 4.1. *Let $v(y)$ be a C^{2m} nonnegative solution of*

$$0 \le -\Delta^m v \le (v+g)^\sigma \quad in \quad \mathbb{R}^n \backslash B_{1/2}(0) \tag{4.1}$$

where $m \ge 2$ and $n \ge 2$ are integers, m is odd, $2m < n$,

$$0 < \sigma < \frac{n}{n-2m},$$

and $g\colon \mathbb{R}^n \backslash B_{1/2}(0) \to [1,\infty)$ is a continuous function satisfying

$$g(y) = O(|y|^b) \quad as \quad |y| \to \infty \tag{4.2}$$

where b is given by (1.32). Then

$$v(y) = O(|y|^b) \quad as \quad |y| \to \infty.$$

Proof. Let λ be the unique solution of $E = a$ where

$$E = \frac{n+2m-\sigma(n-2m)}{\lambda - \sigma}$$

and

$$a := n-2+(n-2m)\frac{\lambda(n-2)-(2m+n-2)}{n-\lambda(n-2m)} = \frac{4m(m-1)}{n-\lambda(n-2m)}. \tag{4.3}$$

Then $\sigma < \lambda < \frac{n}{n-2m}$ and thus

$$p := \frac{\lambda}{\sigma} > 1. \tag{4.4}$$

Also

$$\lambda = \frac{2m+n-2}{n-2} + \frac{8m(m-1)(1+\sigma(m-1))}{(n-2)(4m(m-1)+(n-2m)(n+2m-\sigma(n-2m)))} > \frac{2m+n-2}{n-2},$$

$$a = E = \frac{n+2m+b\sigma}{\lambda} \tag{4.5}$$

and

$$b = a - (n - 2m). \tag{4.6}$$

Let $u(x)$ be defined by (1.28). Then by (1.29) and (4.1) we have

$$0 \le -|x|^{n+2m}\Delta^m u(x) \le \left(|x|^{n-2m}u(x) + g\left(\frac{x}{|x|^2}\right)\right)^\sigma$$

and thus letting q be conjugate Hölder exponent of p and using (4.2), (4.4), and (4.5) we obtain

$$\begin{aligned}
0 \le -\Delta^m u(x) &\le \left(\left(\frac{1}{|x|}\right)^{\frac{n+2m-\sigma(n-2m)}{\sigma}} u(x) + \left(\frac{1}{|x|}\right)^{\frac{n+2m}{\sigma}} g\left(\frac{x}{|x|^2}\right)\right)^\sigma \\
&\le \left(u(x)^p + \left(\frac{1}{|x|}\right)^{\frac{q(n+2m-\sigma(n-2m))}{\sigma}} + O\left(\left(\frac{1}{|x|}\right)^{\frac{n+2m+b\sigma}{\sigma}}\right)\right)^\sigma \\
&\le \left[u(x) + \left(\frac{1}{|x|}\right)^{\frac{n+2m-\sigma(n-2m)}{\sigma(p-1)}} + O\left(\left(\frac{1}{|x|}\right)^{\frac{n+2m+b\sigma}{\lambda}}\right)\right]^\lambda \\
&= [u(x) + O(|x|^{-a})]^\lambda \quad \text{in} \quad B_2(0)\backslash\{0\}.
\end{aligned}$$

Thus by (4.3) and Remark 3.1 after Theorem 3.1 we have

$$u(x) = O(|x|^{-a}) \quad \text{as} \quad x \to 0.$$

Hence

$$\begin{aligned}
v(y) = |x|^{n-2m}u(x) &= O(|x|^{n-2m-a}) = O(|y|^{a-(n-2m)}) \\
&= O(|y|^b) \quad \text{as} \quad |y| \to \infty
\end{aligned}$$

by (4.6). ∎

Theorem 4.2. *Let $v(y)$ be a C^{2m} nonnegative solution of*

$$0 \le -\Delta^m v \le (v + g(y))^\sigma \quad \textit{in} \quad \mathbb{R}^n\backslash B_{1/2}(0) \tag{4.7}$$

where $m \ge 2$ and $n \ge 2$ are integers, m is odd, $2m = n$, $\sigma > 0$, and $g\colon \mathbb{R}^n\backslash B_{1/2}(0) \to [1,\infty)$ is a continuous function satisfying

$$g(y) = o(|y|^{n-2}(\log 5|y|)^{1+\frac{2n}{\sigma(n-2)}}) \quad \textit{as} \quad |y| \to \infty. \tag{4.8}$$

Then

$$v(y) = o(|y|^{n-2}\log 5|y|) \quad \textit{as} \quad |y| \to \infty. \tag{4.9}$$

Proof. Let

$$\lambda = \frac{2n}{n-2} + \sigma \quad \text{and} \quad p = \frac{\lambda}{\sigma} = 1 + \frac{2n}{(n-2)\sigma}. \tag{4.10}$$

Then

$$\frac{2n}{\lambda-\sigma}=n-2 \quad \text{and} \quad \frac{2n}{\lambda}+\frac{n-2}{p}=n-2. \tag{4.11}$$

Let $u(x)$ be defined by (1.28). Then by (4.7) and (1.29) we have

$$0\le -|x|^{2n}\Delta^m u(x)\le \left(u(x)+g\left(\frac{x}{|x|^2}\right)\right)^\sigma$$

and thus letting q be the conjugate Hölder exponent of p and using (4.8), (4.10) and (4.11) we get

$$\begin{aligned}
0\le -\Delta^m u(x) &\le \left(\left(\frac{1}{|x|}\right)^{\frac{2n}{\sigma}} u(x)+o\left(\left(\frac{1}{|x|}\right)^{\frac{2n}{\sigma}+(n-2)}\left(\log\frac{5}{|x|}\right)^p\right)\right)^\sigma\\
&\le \left(u(x)^p+\left(\frac{1}{|x|}\right)^{\frac{2n}{\sigma}q}+o\left(\left(\frac{1}{|x|}\right)^{\frac{2n}{\sigma}+(n-2)}\left(\log\frac{5}{|x|}\right)^p\right)\right)^\sigma\\
&\le \left(u(x)+\left(\frac{1}{|x|}\right)^{\frac{2n}{\sigma(p-1)}}+o\left(\left(\frac{1}{|x|}\right)^{\frac{2n}{p\sigma}+\frac{n-2}{p}}\left(\log\frac{5}{|x|}\right)\right)\right)^\lambda\\
&= \left(u(x)+o\left(\left(\frac{1}{|x|}\right)^{n-2}\left(\log\frac{5}{|x|}\right)\right)\right)^\lambda \quad \text{in} \quad B_2(0)\backslash\{0\}.
\end{aligned}$$

Thus by Theorem 3.2 we have

$$u(x)=o\left(|x|^{-(n-2)}\log\frac{5}{|x|}\right) \quad \text{as} \quad x\to 0$$

and hence (4.9) holds. □

Theorem 4.3. *Let $v(y)$ be a C^{2m} nonnegative solution of*

$$0\le -\Delta^m v\le e^{v^\lambda+g^\lambda} \quad \textit{in} \quad \mathbb{R}^n\backslash B_{1/2}(0) \tag{4.12}$$

where $m\ge 2$ and $n\ge 2$ are integers, m is odd, $2m=n$, $0<\lambda<1$, and $g\colon \mathbb{R}^n\backslash B_{1/2}(0)\to[1,\infty)$ is a continuous function satisfying

$$g(y)=o(|y|^{\frac{n-2}{1-\lambda}}) \quad \textit{as} \quad |y|\to\infty. \tag{4.13}$$

Then

$$v(y)=o(|y|^{\frac{n-2}{1-\lambda}}) \quad \textit{as} \quad |y|\to\infty. \tag{4.14}$$

Proof. Let $u(x)$ be defined by (1.28). Then by (4.12) and (1.29) we have

$$0\le -|x|^{2n}\Delta^m u(x)\le \exp\left(u(x)^\lambda+g\left(\frac{x}{|x|^2}\right)^\lambda\right)$$

and thus by (4.13),

$$0\le -\Delta^m u(x)\le \exp\left(u(x)^\lambda+o\left(\left(\frac{1}{|x|}\right)\right)^{\frac{\lambda(n-2)}{1-\lambda}}\right) \quad \text{in} \quad B_2(0)\backslash\{0\}.$$

Hence Theorem 3.3 implies

$$u(x) = o(|x|^{-\frac{(n-2)}{1-\lambda}}) \quad \text{as} \quad x \to 0$$

and so (4.14) holds. □

Proof of Theorem 1.17. By using the m-Kelvin transform (1.28), we see that to prove Theorem 1.17 it suffices to prove that there exists a C^∞ positive solution $u(x)$ of

$$0 \le -\Delta^m u \le |x|^\tau u^\lambda \quad \text{in} \quad \mathbb{R}^n \backslash \{0\}, \tag{4.15}$$

where

$$\tau = \lambda(n-2m) - n - 2m$$

such that

$$u(x) \neq O(\varphi(|x|^{-1})|x|^{-(b+n-2m)}) \quad \text{as} \quad x \to 0. \tag{4.16}$$

Define $\psi\colon (0,1) \to (0,1)$ by

$$\psi(r) = \max\left\{\varphi(r^{-1})^p, r^{a\frac{n-\lambda(n-2m)}{\lambda}}\right\} \tag{4.17}$$

where

$$a := \frac{\lambda(m-1)+1}{n-\lambda(n-2m)} \quad \text{and} \quad p := \frac{n-\lambda(n-2m)}{2n}.$$

By (1.33), a and p are positive. Also

$$1 + 2a = \frac{\lambda(2m-2) - 2m + 2 - \tau}{n - \lambda(n-2m)} \quad \text{and} \quad b = 2m - 2 + (n-2m)2a. \tag{4.18}$$

Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ be a sequence satisfying (2.25) and (2.26). Define $r_j > 0$ by

$$r_j^{n-\lambda(n-2m)} = \frac{2^{|\tau|}}{A^\lambda} \frac{|x_j|^{\lambda(2m-2)-2m+2-\tau}}{\psi(|x_j|)^\lambda}$$

where $A = A(m,n)$ is as in Lemma 2.4. Then r_j satisfies (2.37) and by (4.17) and (4.18),

$$\begin{aligned} r_j &= C(m,n,\lambda) \frac{|x_j|^{1+2a}}{\psi(|x_j|)^{\frac{\lambda}{n-\lambda(n-2m)}}} \\ &\le C(m,n,\lambda)|x_j|^{1+a}. \end{aligned} \tag{4.19}$$

Thus by taking a subsequence of j, r_j will satisfy (2.27). Let u be as in Lemma 2.4. Then by Case I of Remark 2.1, u is a C^∞ positive solution of (4.15) and by (2.30), (4.17), (4.18), and (4.19) we have

$$\begin{aligned} u(x_j) &\ge \frac{C(m,n,\lambda)\psi(|x_j|)}{|x_j|^{2m-2}} \frac{\psi(|x_j|)^{\frac{\lambda(n-2m)}{n-\lambda(n-2m)}}}{|x_j|^{(1+2a)(n-2m)}} \\ &= \frac{C(m,n,\lambda)\psi(|x_j|)^{\frac{n}{n-\lambda(n-2m)}}}{|x_j|^{(n-2m)+(2m-2)+(n-2m)2a}} \\ &\ge C(m,n,\lambda) \frac{\varphi(|x_j|^{-1})^{1/2}}{|x_j|^{b+n-2m}} \end{aligned}$$

which implies (4.16). □

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