

# UNIQUENESS OF COMPACT TANGENT FLOWS IN MEAN CURVATURE FLOW

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ABSTRACT. We show, for mean curvature flows in Euclidean space, that if one of the tangent flows at a given spacetime point consists of a closed, multiplicity-one, smoothly embedded self-similar shrinker, then it is the unique tangent flow at that point. That is the limit of the parabolic rescalings does not depend on the chosen sequence of rescalings. Furthermore, given such a closed, multiplicity-one, smoothly embedded self-similarly shrinker  $\Sigma$ , we show that any solution of the rescaled flow, which is sufficiently close to  $\Sigma$ , with Gaussian density ratios greater or equal to that of  $\Sigma$ , stays for all time close to  $\Sigma$  and converges to a possibly different self-similarly shrinking solution  $\Sigma'$ . The central point in the argument is a direct application of the Simon-Lojasiewicz inequality to Huisken's monotone Gaussian integral for Mean Curvature Flow.

## 1. INTRODUCTION

In this paper we study Mean Curvature Flow (MCF) of  $n$ -surfaces of codimension  $k \geq 1$  in  $\mathbb{R}^{n+k}$ , which are close to self-similarly shrinking solutions. In the smooth case we consider a family of embeddings  $F : M^n \times [t_1, t_2] \rightarrow \mathbb{R}^{n+k}$ , for  $M^n$  closed, such that

$$(1.1) \quad \frac{d}{dt}F(p, t) = \vec{H}(p, t) ,$$

where  $\vec{H}(p, t)$  is the mean curvature vector of  $M_t := F(M, t)$  at  $F(p, t)$ . We denote with  $\mathcal{M} = \bigcup_{t \in [t_1, t_2]} (M_t \times \{t\}) \subset \mathbb{R}^{n+k} \times \mathbb{R}$  its space-time track.

In the following, let  $\Sigma^n$  be a smooth, closed, embedded  $n$ -surface in  $\mathbb{R}^{n+k}$  where the mean curvature vector satisfies

$$\vec{H} = -\frac{x^\perp}{2}.$$

Here  $x$  is the position vector at a point on  $\Sigma$  and  $^\perp$  the projection to the normal space of  $\Sigma$  at that point. Such a surface gives rise to a self-similarly shrinking solution  $\mathcal{M}_\Sigma$ , where the evolving surfaces are given by

$$\Sigma_t = \sqrt{-t} \cdot \Sigma, \quad t \in (-\infty, 0).$$

We denote its space-time track by  $\mathcal{M}_\Sigma$ .

We also want to study the case that the flow is allowed to be non-smooth. Following [8], we say that a family of Radon measures  $(\mu_t)_{t \in [t_1, t_2]}$  on  $\mathbb{R}^{n+k}$  is an integer  $n$ -Brakke flow, if for almost every  $t$  the measure  $\mu_t$  comes from a  $n$ -rectifiable varifold

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with integer densities. Furthermore, we require that given any  $\varphi \in C_c^2(\mathbb{R}^{n+k}; \mathbb{R}^+)$  the following inequality holds for every  $t > 0$

$$(1.2) \quad \bar{D}_t \mu_t(\varphi) \leq \int -\varphi |\bar{H}|^2 + \langle \nabla \varphi, \bar{H} \rangle d\mu_t,$$

where  $\bar{D}_t$  denotes the upper derivative at time  $t$  and we take the left hand side to be  $-\infty$ , if  $\mu_t$  is not  $n$ -rectifiable, or does not carry a weak mean curvature. Note that if  $M_t$  is moving smoothly by mean curvature flow, then  $\bar{D}_t$  is just the usual derivative and we have equality in (1.2).

We restrict to  $n$ -rectifiable Brakke flows with unit density which are close to such a self-similarly shrinking solution. The assumption that the flow has unit density implies that for almost all  $t$  the corresponding Radon measures can be written as

$$\mu_t = \mathcal{H}^n \llcorner M_t.$$

Here  $M_t$  is a  $n$ -rectifiable subset of  $\mathbb{R}^{n+k}$  and  $\mathcal{H}^n$  is the  $n$ -dimensional Hausdorff-measure on  $\mathbb{R}^{n+k}$ . With some abuse of notation we denote such a flow again by  $\mathcal{M}$ . If the flow is (locally) smooth, then  $M_t$  can be (locally) represented by a smooth  $n$ -surface evolving by MCF. Conversely, if  $M_t$  moves smoothly by MCF, then  $\mu_t := \mathcal{H}^n \llcorner M_t$  defines a unit density  $n$ -Brakke flow.

**Theorem 1.1.** *Let  $\mathcal{M} = (\mu_t)_{t \in (t_1, 0)}$  with  $t_1 < 0$  be a unit density  $n$ -Brakke flow such that*

- i)  $(\mu_t)_{t \in (t_1, t_2)}$  is sufficiently close in measure to  $\mathcal{M}_\Sigma$  for some  $t_1 < t_2 < 0$ .
- ii)  $\Theta_{(0,0)}(\mathcal{M}) \geq \Theta_{(0,0)}(\mathcal{M}_\Sigma)$ , where  $\Theta_{(0,0)}(\cdot)$  is the respective Gaussian density at the point  $(0, 0)$  in space-time.

Then  $\mathcal{M}$  is a smooth flow for  $t \in [(t_1 + t_2)/2, 0)$ , and the rescaled surfaces  $\tilde{M}_t := \sqrt{-t} \cdot M_t$  can be written as normal graphs over  $\Sigma$ , given by smooth sections  $v(t)$  of the normal bundle  $T^\perp \Sigma$ , with  $|v(t)|_{C^k(T^\perp \Sigma)}$  uniformly bounded for all  $t \in [(t_1 + t_2)/2, 0)$  and all  $k \in \mathbb{N}$ . Furthermore, there exists a self-similarly shrinking surface  $\Sigma'$  with

$$\Sigma' = \text{graph}_\Sigma(v')$$

and

$$|v(t) - v'|_{C^m} \leq c_m (\log(-1/t))^{-\alpha_m}$$

for some constants  $c_m > 0$  and exponents  $\alpha_m > 0$  for all  $m \in \mathbb{N}$ .

For the definition of the Gaussian density ratios and Gaussian density we refer the reader to section 2. The above theorem implies uniqueness of compact tangent flows as follows. Let the parabolic rescaling with a factor  $\lambda > 0$  be given by

$$\mathcal{D}_\lambda : \mathbb{R}^{n+k} \times \mathbb{R} \rightarrow \mathbb{R}^{n+k} \times \mathbb{R}, (x, t) \mapsto (\lambda x, \lambda^2 t).$$

Note that any Brakke flow  $\mathcal{M}$  (smooth MCF) is mapped to a Brakke flow (smooth MCF), i.e.  $\mathcal{D}_\lambda(\mathcal{M})$  is again a Brakke flow (smooth MCF).

Let  $(x_0, t_0)$  be a point in space-time and  $(\lambda_i)_{i \in \mathbb{N}}, \lambda_i \rightarrow \infty$ , be a sequence of positive numbers. If  $\mathcal{M}$  is a Brakke flow with bounded area ratios, then the compactness theorem for Brakke flows (see [8, 7.1]) ensures that

$$(1.3) \quad \mathcal{D}_{\lambda_i}(\mathcal{M} - (x_0, t_0)) \rightarrow \mathcal{M}',$$

where  $\mathcal{M}'$  is again a Brakke flow. Such a flow is called a *tangent flow* of  $\mathcal{M}$  at  $(x_0, t_0)$ . Huisken's monotonicity formula ensures that  $\mathcal{M}'$  is self-similarly shrinking, i.e. it is invariant under parabolic rescaling.

**Corollary 1.2.** *Let  $\mathcal{M}$  be a unit density  $n$ -Brakke flow with bounded area ratios, and assume that at  $(x_0, t) \in \mathbb{R}^{m+k} \times \mathbb{R}$  a tangent flow of  $\mathcal{M}$  is  $\mathcal{M}_\Sigma$ . Then this tangent flow is unique, i.e. for any sequence  $(\lambda_i)_{i \in \mathbb{N}}$  of positive numbers,  $\lambda_i \rightarrow \infty$  it holds*

$$\mathcal{D}_{\lambda_i}(\mathcal{M} - (x_0, t_0)) \rightarrow \mathcal{M}_\Sigma.$$

Other than the shrinking sphere and the Angenent torus [2] no further examples of compact self-similarly shrinking solutions in codimension one are known so far. However, several numerical solutions of D. Chopp [3] suggest that there are a whole variety of such solutions. In higher codimensions this class of solutions should be even bigger.

By a result of B. White the assumption in Theorem 1.1 and Corollary 1.2 that  $\mathcal{M}$  has unit density can be dropped, provided the Brakke flow  $\mathcal{M}$  has integer densities. See the introduction in [15] and the reference to [W5] there. The results imply that  $\mathcal{M}$  is actually smooth with unit density, provided it is close enough in measure to  $\mathcal{M}_\Sigma$ .

In a recent work of Kapouleas/Kleene/Møller [9] and X.H. Nguyen [10] non-trivial, non-compact, self-similarly shrinking solutions were constructed. In [6], G. Huisken showed that, under the assumption that the second fundamental form is bounded, the only solutions in the mean convex case are shrinking spheres and cylinders. The assumption on the second fundamental form was recently removed by T.H. Colding and W.P. Minicozzi in [4]. They also proved a smooth compactness theorem for closed self-similarly shrinking surfaces of fixed genus in  $\mathbb{R}^3$ , see [5].

The analogous problem for minimal surfaces is the uniqueness of tangent cones. This was studied in [13, 1, 14], and, in the case of multiplicity one tangent cones with isolated singularities, completely settled by L. Simon in [11]. One of the main tools in the analysis therein is the generalisation of an inequality due to Łojasiewicz for real analytic functions to the infinite dimensional setting.

Also in the present argument, this Simon-Łojasiewicz inequality for “convex” energy functionals on closed surfaces, plays a central role. We adapt several ideas from [11, 12]. In section 2 we recall Huisken's monotonicity formula and prove an smooth extension lemma for unit density Brakke flows close to  $\mathcal{M}_\Sigma$ . We also introduce the rescaled flow. In section 3 we treat the Gaussian integral of Huisken's monotonicity formula for the rescaled flow as an appropriate “energy functional” on  $\Sigma$  and use the Simon-Łojasiewicz inequality to prove a closeness lemma. This lemma and the extension lemma are then applied to prove the main theorem and its corollary.

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## 2. GRAPHICAL REPRESENTATION AND RESCALING

As in the introduction, let  $\mathcal{M}$  be a smooth mean curvature flow of embedded  $n$ -dimensional surfaces in  $\mathbb{R}^{n+k}$ . Let

$$\rho_{x_0, t_0}(x, t) = \frac{1}{(4\pi(t_0 - t))^{n/2}} \cdot \exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right), \quad t < t_0$$

be the backward heat kernel centered at  $(x_0, t_0)$ . Huisken's monotonicity formula states that for  $t < t_0$

$$(2.1) \quad \frac{d}{dt} \int_{M_t} \rho_{x_0, t_0}(x, t) d\mathcal{H}^n = - \int_{M_t} \left| \vec{H} + \frac{x^\perp}{2(t_0 - t)} \right|^2 \rho_{x_0, t_0}(x, t) d\mathcal{H}^n.$$

Thus the Gaussian density ratio

$$\Theta_{x_0, t_0}(\mathcal{M}, t) := \int_{M_t} \rho_{x_0, t_0}(x, t) d\mathcal{H}^n$$

is a decreasing function in  $t$ . If  $t_0 \in (t_1, t_2]$  then the limit

$$\Theta_{x_0, t_0}(\mathcal{M}) = \lim_{t \nearrow t_0} \Theta_{x_0, t_0}(t)$$

exists for all  $x_0 \in \mathbb{R}^{n+k}$  and is called the Gaussian density at  $(x_0, t_0)$ . The corresponding result also holds for Brakke flows, see [7, Lemma 7]. In that case the equality in (2.1) is replaced by the appropriate inequality. It can be deduced from (2.1) that any limit of parabolic rescalings as in (1.3) is self-similarly shrinking. This is also true if the convergence and limit are not smooth, see [7, Lemma 8]. Self-similarly shrinking solutions  $\mathcal{M}_\Sigma$  can be characterized by the fact that the Gaussian density ratio  $\Theta_{(0,0)}(\mathcal{M}_\Sigma, t)$  is constant in time.

Now let us assume that  $\mathcal{M}$  is smooth for  $t \in [\tau - 1, \tau] \subset (-\infty, 0)$ . Even more we assume that  $\mathcal{M}$  can be written on this time interval as a normal graph over  $M_\Sigma$ , i.e.

$$M_t = \text{graph}_{\Sigma_t}(u(\cdot, t))$$

where  $u(\cdot, t)$  is a smooth section of the normal bundle  $T^\perp \Sigma_t$ , with sufficiently small  $C^1$ -norm. Note that any  $n$ -surface which is sufficiently close in  $C^1$  to  $\Sigma_t$  can be written as such a normal graph. Since the evolution of  $\Sigma$  is self-similarly shrinking we can also write

$$M_t = \sqrt{-t} \cdot \text{graph}_\Sigma(v(\cdot, t)),$$

which implies that  $u(p, t) = \sqrt{-t} \cdot v(\tilde{p}, t)$ , where  $\tilde{p} \in \Sigma$  is given by  $\sqrt{-t} \cdot \tilde{p} = p$ . We have the following extension lemma:

**Lemma 2.1.** *Let  $\beta > 1$ ,  $\tau < 0$ . For every  $\sigma > 0$  there exists a  $\delta > 0$ , depending only on  $\sigma, \beta, \Sigma$ , such that if  $\mathcal{M}$  is a unit density Brakke flow with  $\Theta_{(0,0)}(\mathcal{M}) = \Theta_{0,0}(\mathcal{M}_\Sigma)$ , which is a smooth graph over  $\mathcal{M}_\Sigma$  for  $t \in [\beta\tau, \tau] \subset (-\infty, 0)$  such that*

$$(2.2) \quad \|v\|_{C^{2,\alpha}(\Sigma \times [\beta\tau, \tau])} \leq \sigma$$

and

$$(2.3) \quad \sup_{t \in [\beta\tau, \tau]} \|v(\cdot, t)\|_{L^2(\Sigma)} \leq \delta,$$

then  $\mathcal{M}$  is a smooth graph over  $\mathcal{M}_\Sigma$  on  $[\beta\tau, \tau/\beta]$  for an extension  $\tilde{v}$  of  $v$  with

$$(2.4) \quad \|\tilde{v}\|_{C^{2,\alpha}(\Sigma \times [\beta\tau, \tau/\beta])} \leq \sigma.$$

*Proof.* By changing scale, we can assume that  $\tau = -1$ . If the statement were false we could find a sequence of Brakke flows  $\mathcal{M}^k$  with  $\Theta_{(0,0)}(\mathcal{M}^k) = \Theta_{0,0}(\mathcal{M}_\Sigma)$ , which are smooth graphs over  $\mathcal{M}_\Sigma$  for  $t \in [-\beta, -1]$ . Furthermore we can assume that (2.2) holds for all  $k$  and

$$(2.5) \quad \sup_{t \in [-\beta, -1]} \|v(\cdot, t)\|_{L^2(\Sigma)} \leq \frac{1}{k},$$

but  $\mathcal{M}^k$  is not a smooth graph over  $\mathcal{M}_\Sigma$  for  $t \in [-1, -1/\beta]$  satisfying (2.4). By the compactness theorem for Brakke flows there exists a subsequence  $\mathcal{M}^{k'}$  such that

$$\mathcal{M}^{k'} \rightarrow \mathcal{M}',$$

where  $\mathcal{M}'$  is again a Brakke flow. Since the Gaussian density is upper semi-continuous we have

$$\Theta_{(0,0)}(\mathcal{M}') = \Theta_{0,0}(\mathcal{M}_\Sigma)$$

and (2.4), (2.5) implies that

$$\mathcal{M}' = \mathcal{M}_\Sigma \quad \text{for } t \in [-\beta, -1].$$

But this implies that  $\mathcal{M}'$  is self similarly shrinking for  $t \in (-\beta, 0)$  and coincides with  $\mathcal{M}_\Sigma$  for  $t \in (-\beta, 0)$  (with unit density). By Brakke's regularity theorem, see [8, 12.1], the convergence

$$\mathcal{M}^{k'} \rightarrow \mathcal{M}_\Sigma$$

is smooth on any compact subset of  $\mathbb{R}^{n+k} \times (-\beta, 0)$ , which gives the desired contradiction.  $\square$

To study mean curvature flows close to the evolution of  $\Sigma$  it is convenient to consider the rescaled flow, i.e. if  $F : M^n \times [t_1, t_2] \rightarrow \mathbb{R}^{n+k}$ ,  $t_1 < t_2 \leq 0$  is a smooth mean curvature flow, the rescaled embeddings

$$\tilde{F}(\cdot, \tau) := \frac{1}{\sqrt{-t}} F(\cdot, t)$$

with  $\tau = -\log(-t)$  have normal speed

$$(2.6) \quad \left( \frac{\partial}{\partial \tau} \tilde{F} \right)^\perp = \tilde{H} + \frac{x^\perp}{2}.$$

The monotonicity formula (2.1), centered at  $(0, 0)$ , in the rescaled setting reads

$$(2.7) \quad \frac{d}{d\tau} \int_{\tilde{M}_\tau} \rho(x) d\mathcal{H}^n = - \int_{\tilde{M}_\tau} \left| \tilde{H} + \frac{x^\perp}{2} \right|^2 \rho(x) d\mathcal{H}^n,$$

with  $\rho(x) = (4\pi)^{-n/2} \exp(-|x|^2/4)$ . If the surfaces  $\tilde{M}_\tau := 1/\sqrt{-t} \cdot M_t$  can be written as normal graphs over  $\Sigma$ , i.e.

$$\tilde{M}_\tau = \text{graph}_\Sigma(v(\cdot, \tau)),$$

equation (2.6) implies

$$(2.8) \quad \left( \frac{\partial}{\partial \tau} v \right)^\perp = \tilde{H} + \frac{x^\perp}{2},$$

where, as before,  $^\perp$  denotes the projection onto the normal space of  $\tilde{M}_\tau$ .

## 3. CONVERGENCE

To show that the flow  $\tilde{M}_\tau$  stays close to  $\Sigma$  we treat the Gaussian integral of the monotonicity formula as an “energy functional” for surfaces which can be written as normal graphs over  $\Sigma$ . Let  $v$  be a smooth section of the normal bundle  $T^\perp\Sigma$  with sufficiently small  $C^1$ -norm, i.e. such that  $M := \text{graph}_\Sigma(v)$  is a smooth  $n$ -surface, which is close in  $C^1$  to  $\Sigma$ . The energy  $\mathcal{E}$  is then given by

$$(3.1) \quad \mathcal{E}(v) = \int_M \rho(x) d\mathcal{H}^n(x) = \int_\Sigma \rho(y + v(y)) J(y, v, \nabla^\Sigma v) d\mathcal{H}^n(y),$$

where by the area formula  $J$  is a smooth function with analytic dependence on its arguments. Furthermore it is uniformly convex in  $\nabla^\Sigma v$ , if the  $C^1$ -norm of  $v$  is sufficiently small. The first variation of  $\mathcal{E}$  at  $v$  is given by

$$(3.2) \quad \begin{aligned} \frac{\partial}{\partial s} \Big|_{s=0} \mathcal{E}(v + sf) &= - \int_M \left\langle \vec{H} + x^\perp/2, f \right\rangle_{\mathbb{R}^{n+k}} \rho(x) d\mathcal{H}^n(x) \\ &= - \int_\Sigma h \left( \Pi(\vec{H} + x^\perp/2), f \right) \rho(y + v(y)) J(y, v, \nabla^\Sigma v) d\mathcal{H}^n(y), \end{aligned}$$

where  $h$  is the induced metric on the normal bundle to  $\Sigma$ , and  $\Pi$  the projection to the normal space of  $\Sigma$  at  $y$ . Thus the  $L^2$ -gradient operator of  $\mathcal{E}$  is given by

$$(3.3) \quad \text{grad } \mathcal{E}(v) = -\Pi \left( \vec{H} + \frac{(y + v(y))^\perp}{2} \right) \rho(y + v(y)) J(y, v, \nabla^\Sigma v)$$

where  $\vec{H}$  is the mean curvature operator of  $\text{graph}_\Sigma(v)$  at  $y + v(y)$ .

We aim to apply the Simon-Lojasiewicz inequality as proven in [11]. One can easily check that all condition to apply Theorem 3 therein are met, and we thus get that there are constants  $\sigma_0 > 0$  and  $\theta \in (0, \frac{1}{2})$  such that if  $v$  is a  $C^{2,\alpha}$  section of  $T^\perp\Sigma$  with  $|v|_{2,\alpha} < \sigma_0$  then

$$(3.4) \quad \|\text{grad } \mathcal{E}(v)\|_{L^2(\Sigma)} \geq |\mathcal{E}(v) - \mathcal{E}(0)|^{1-\theta}.$$

This estimate allows us to show that solutions of (2.8) stay close to  $\Sigma$ , provided they are bounded in  $C^{2,\alpha}$ .

**Lemma 3.1.** *Let  $v : \Sigma \times [\tau_1, \tau_2] \rightarrow T^\perp\Sigma$  be a smooth solution of (2.8) with  $|v(\cdot, \tau)|_{2,\alpha} < \sigma_0$  for all  $\tau \in [\tau_1, \tau_2]$  and  $\mathcal{E}(v(\tau_2)) \geq \Theta_{(0,0)}(\mathcal{M}_\Sigma)$ . Then there exists  $\gamma > 0$  such that*

$$(3.5) \quad \int_{\tau_1}^{\tau_2} \left\| \frac{\partial v}{\partial \tau} \right\|_{L^2(\Sigma)} d\tau \leq \frac{1}{\gamma\theta} (\mathcal{E}(v(\tau_1)) - \mathcal{E}(0))^\theta$$

and thus

$$(3.6) \quad \sup_{\tau \in [\tau_1, \tau_2]} \|v(\tau) - v(\tau_1)\|_{L^2(\Sigma)} \leq \frac{1}{\gamma\theta} (\mathcal{E}(v(\tau_1)) - \mathcal{E}(0))^\theta.$$

*Proof.* We have

$$\begin{aligned}
\frac{d}{d\tau}\mathcal{E}(v(\tau)) &= - \int_{M_\tau} \left| \vec{H} + \frac{x^\perp}{2} \right|^2 \rho(x) d\mathcal{H}^n \\
&= - \left( \int_{M_\tau} \left| \vec{H} + \frac{x^\perp}{2} \right|^2 \rho(x) d\mathcal{H}^n \right)^{1/2} \cdot \left( \int_\Sigma \left| \vec{H} + \frac{x^\perp}{2} \right|^2 \rho(x) J(y) d\mathcal{H}^n(y) \right)^{1/2} \\
&\leq - \left( \int_\Sigma |\text{grad } \mathcal{E}(v(\tau))|^2 (\rho(x) J(y))^{-1} d\mathcal{H}^n(y) \right)^{1/2} \\
&\quad \cdot \left( \int_\Sigma \left| \left( \frac{\partial}{\partial \tau} v \right)^\perp \right|^2 \rho(y + v(y)) J(y) d\mathcal{H}^n(y) \right)^{1/2} \\
&\leq -\gamma \|\text{grad } \mathcal{E}(v)\|_{L^2(\Sigma)} \cdot \left\| \frac{\partial v}{\partial \tau} \right\|_{L^2(\Sigma)},
\end{aligned}$$

where  $\gamma > 0$  depends only on  $\sigma_0$  and  $\Sigma$ . Since  $\mathcal{E}(v(\tau))$  is a decreasing function of  $\tau$  and  $\mathcal{E}(0) = \Theta_{(0,0)}(\mathcal{M}_\Sigma)$  this yields together with (3.4),

$$\begin{aligned}
-\frac{d}{d\tau}(\mathcal{E}(v(\tau)) - \mathcal{E}(0))^\theta &\geq \gamma \theta (\mathcal{E}(v(\tau)) - \mathcal{E}(0))^{\theta-1} \|\text{grad } \mathcal{E}(v)\|_{L^2(\Sigma)} \cdot \left\| \frac{\partial v}{\partial \tau} \right\|_{L^2(\Sigma)} \\
&\geq \gamma \theta \left\| \frac{\partial v}{\partial \tau} \right\|_{L^2(\Sigma)}.
\end{aligned}$$

Integrating this inequality yields (3.5), which implies (3.6).  $\square$

*Proof of Theorem 1.1.* Choose  $0 < \sigma < \sigma_0$  such that for any normal section  $v$  with  $|v|_{C^{2,\alpha}} < \sigma$  the normal graph,  $\text{graph}_\Sigma(v)$ , is a smooth  $n$ -surface. Let  $\beta = (t_1 + t_2)/(2t_2)$  and  $\delta > 0$  be given by lemma 2.1. Choose  $0 < \tilde{\sigma} < \sigma$  such that that  $|v|_{C^{2,\alpha}} < \tilde{\sigma}$  implies

- i)  $\|v\|_{L^2(\Sigma)} \leq \delta/3$ ,
- ii)  $\frac{1}{\gamma\theta} |\mathcal{E}(v) - \mathcal{E}(0)|^\theta \leq \frac{2\delta}{3}$ ,

where  $\gamma, \delta > 0$  are the constants given in Lemma 3.1.

If  $\mathcal{M}$  is sufficiently close in measure to  $\mathcal{M}_\Sigma$  on  $(t_1, t_2)$ , Brakke's regularity theorem implies that  $\mathcal{M}$  is actually a smooth graph over  $\Sigma_t$  for  $t \in ((t_1 + t_2)/2, t_2)$ , such that  $M_t = \text{graph}_{\Sigma_t}(u(\cdot, t))$ . Even more, we can assume that in the rescaled setting for

$$\tilde{M}_\tau = \frac{1}{\sqrt{-t}} M_t = \text{graph}_\Sigma(v(\cdot, \tau))$$

the normal section  $v(\tau)$  satisfies  $|v(\tau)|_{C^{2,\alpha}} < \tilde{\sigma}$ , where  $\tau = -\log(-t)$  and  $\tau \in (-\tau_0 - \log(\beta), -\tau_0)$ , for  $\tau_0 = -\log(-t_2)$ . Lemma 2.1 implies that  $\tilde{M}_\tau$  is actually graphical over  $\Sigma$  for an extension  $\tilde{v}$  of  $v$  on  $(\tau_0 - \log(\beta), \tau_0 + \log(\beta))$  satisfying  $|\tilde{v}(\tau)|_{C^{2,\alpha}} < \sigma$ . But note that Lemma 3.1 then implies that for  $\tau \in [\tau_0, \tau_0 + \log(\beta))$

$$\begin{aligned}
(3.7) \quad \|\tilde{v}(\tau)\|_{L^2(\Sigma)} &\leq \|v(\tau_0)\|_{L^2(\Sigma)} + \frac{1}{\gamma\theta} (\mathcal{E}(\tilde{v}(\tau_0 + \log(\beta))) - \mathcal{E}(0))^\theta \\
&\leq \frac{\delta}{3} + \frac{1}{\gamma\theta} (\mathcal{E}(v(\tau_0)) - \mathcal{E}(0))^\theta \leq \frac{\delta}{2},
\end{aligned}$$

since  $\mathcal{E}(v(\tau))$  is decreasing in  $\tau$ . This now allows us to iterate the previous step to find that actually  $\tilde{M}_\tau$  is a smooth graph over  $\Sigma$  for an extension  $\tilde{v}(\tau)$  of  $v$  for all

$\tau \in (\tau_0, \tau_0 + 2 \log(\beta))$  such that  $|\tilde{v}(\tau)|_{C^{2,\alpha}} < \sigma$ . Again we have

$$\begin{aligned} \|\tilde{v}(\tau)\|_{L^2(\Sigma)} &\leq \|v(\tau_0)\|_{L^2(\Sigma)} + \frac{1}{\gamma\theta} (\mathcal{E}(\tilde{v}(\tau_0 + 2 \log(\beta))) - \mathcal{E}(0))^\theta \\ &\leq \frac{\delta}{3} + \frac{1}{\gamma\theta} (\mathcal{E}(v(\tau_0)) - \mathcal{E}(0))^\theta \leq \frac{\delta}{2} \end{aligned}$$

for  $\tau \in (\tau_0, \tau_0 + 2 \log(\beta))$ . This can be iterated to yield that there actually is an extension  $\tilde{v}(\tau)$  of  $v$  for all  $\tau \in (\tau_0, \infty)$  such that  $|\tilde{v}(\tau)|_{C^{2,\alpha}} < \sigma$  for all  $\tau \geq \tau_0$ . Standard estimates for graphical solutions imply the bounds on all higher derivatives.

The estimate (3.5) implies that

$$\int_{\tau_0}^{\infty} \left\| \frac{\partial v}{\partial \tau} \right\|_{L^2(\Sigma)} d\tau \leq \frac{1}{\gamma\theta} (\mathcal{E}(v(\tau_0)) - \mathcal{E}(0))^\theta \leq \frac{\delta}{3}.$$

By the bounds on all higher derivatives this gives that

$$\frac{\partial v}{\partial \tau} \rightarrow 0 \quad \text{in } C^k$$

for all  $k$  uniformly as  $\tau \rightarrow \infty$ . Thus there is a sequence of times  $\tau_i \rightarrow \infty$  such that  $v(\cdot, \tau_i + \tau)|_{\tau \in (0,1)}$  converges uniformly in  $C^k(\Sigma \times (0,1))$ , for all  $k$ , to a time independent solution  $v'$  of (2.8). In other words

$$\Sigma' := \text{graph}_\Sigma(v')$$

gives rise to a self-similarly shrinking solution of MCF. Furthermore note that

$$\Theta_{(0,0)}(\mathcal{M}) = \Theta_{(0,0)}(\mathcal{M}_{\Sigma'}) = \mathcal{E}_{\Sigma'}(0).$$

Thus we can repeat the whole argument before, writing now  $M_\tau$  as normal graphs over  $\Sigma'$ , given by a time-dependent normal section  $w(\tau)$ . Since now  $\mathcal{E}_{\Sigma'}(w(\tau)) \rightarrow \mathcal{E}_{\Sigma'}(0)$  the estimate (3.6) already implies that

$$w(\tau) \rightarrow 0 \quad \text{uniformly as } \tau \rightarrow \infty.$$

To get the claimed decay rate we follow ideas in [12]. Note that from a calculation as in the proof of Lemma 3.1 and (3.4), we get for some  $\tilde{\gamma} > 0$

$$\begin{aligned} (3.8) \quad \frac{d}{d\tau} (\mathcal{E}_{\Sigma'}(w(\tau)) - \mathcal{E}_{\Sigma'}(0)) &\leq -\tilde{\gamma} \|\text{grad } \mathcal{E}_{\Sigma'}\|_{L^2(\Sigma')}^2 \\ &\leq -\tilde{\gamma} (\mathcal{E}_{\Sigma'}(w(\tau)) - \mathcal{E}_{\Sigma'}(0))^{2(1-\theta)}. \end{aligned}$$

Integrating this inequality yields

$$|\mathcal{E}_{\Sigma'}(w(\tau)) - \mathcal{E}_{\Sigma'}(0)| \leq C \frac{1}{\tau^{1+\alpha}},$$

where  $\alpha = 2\theta/(1-2\theta)$ . The estimate (3.4) gives

$$\|v(\tau)\|_{L^2(\Sigma')} \leq C \frac{1}{\tau^{\theta(1+\alpha)}}.$$

Rewriting  $\tau = -\log(-t)$  yields the claimed decay for the  $L^2$ -norm. The higher norms follow by interpolation.  $\square$

*Proof of Corollary 1.2.* We can assume w.l.o.g. that  $(x_0, t_0) = (0, 0)$ . Since  $\mathcal{M}_\Sigma$  is smooth with unit density, and  $\Sigma$  is compact, the Brakke regularity theorem implies that the convergence

$$\mathcal{D}_{\lambda_i}(\mathcal{M}) \rightarrow \mathcal{M}_\Sigma$$



is smooth on  $\mathbb{R}^{m+k} \times (t_1, t_2)$ , for any  $t_1 < t_2 < 0$ . Since  $\Theta_{(0,0)}(\mathcal{M}) = \Theta_{(0,0)}(\mathcal{M}_\Sigma)$  we can apply Theorem 1.1 to see that  $\mathcal{M}$  is a smooth graph over  $\mathcal{M}_\Sigma$  for  $t \in (t_1, 0)$ . The assumption that  $\mathcal{M}_\Sigma$  is a tangent flow of  $\mathcal{M}$  at  $(0, 0)$  implies that after rescaling, there is a sequence of times  $\tau_i \rightarrow \infty$  such that  $M_{\tau_i} \rightarrow \Sigma$ . Thus  $\Sigma' = \Sigma$  and  $M_\tau \rightarrow \Sigma$  as  $\tau \rightarrow \infty$ . This implies the statement of the corollary.  $\square$

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