A Blow-up Criterion for Two Dimensional Compressible Viscous Heat-Conductive Flows*

Daoyuan Fang[†], Ruizhao Zi [‡] and Ting Zhang[§]

Department of Mathematics, Zhejiang University, Hangzhou 310027, China

Abstract

We establish a blow-up criterion in terms of the upper bound of the density and temperature for the strong solution to 2D compressible viscous heat-conductive flows. The initial vacuum is allowed.

1 Introduction

This paper is concerned with a blow-up criterion for the two dimensional compressible viscous heat-conductive flows in $(0,T)\times\Omega$

$$\begin{cases} \rho_t+\mathrm{div}(\rho u)=0, \\ (\rho u)_t+\mathrm{div}(\rho u\otimes u)+\nabla P(\rho,\theta)=\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u, \\ c_\nu((\rho\theta)_t+\mathrm{div}(\rho\theta u))-\kappa\Delta\theta+P(\rho,\theta)\mathrm{div}u=\frac{\mu}{2}|\nabla u+\nabla u^T|^2+\lambda(\mathrm{div}u)^2, \end{cases} \tag{1.1}$$

together with the initial-boundary conditions

$$(\rho,u,\theta)=(\rho_0,u_0,\theta_0) \quad \text{in} \quad \Omega, \tag{1.2}$$
$$(u,\theta)=(0,0) \quad \text{on} \quad (0,T)\times\partial\Omega. \tag{1.3}$$

Here Ω is a bounded smooth domain in $\mathbb{R}^2$, and (ρ,u,θ) are the density, velocity and temperature of the fluid, respectively. For a perfect gas, the pressure is given by

$$P(\rho,\theta)=R\rho\theta, \tag{1.4}$$

where $R>0$ is a generic gas constant. The viscous coefficients μ and λ are constants satisfying

$$\mu>0, \quad \lambda+\mu\geq 0. \tag{1.5}$$

*Supported by NSFC 10871175, 10931007, 10901137, Zhejiang Provincial Natural Science Foundation of China (Z6100217), and SRFDP 20090101120005.

[†]E-mail: dyf@zju.edu.cn
[‡]E-mail: ruizhao3805@163.com
[§]E-mail: zhangting79@zju.edu.cn

Finally, $c_\nu > 0$ and $\kappa > 0$ are the specific heat at a constant volume and thermal conductivity coefficient, respectively.

Some of the previous works in this direction can be summarized as follows. In the absence of vacuum, Matsumura and Nishida [25] proved the global well-posedness of the classical solution to (1.1)-(1.5) with the initial data close to an equilibrium state and Danchin [6, 7] considered similar problems in the framework of critical spaces. With the aid of the effective viscous flux F, Hoff [13] proved the global existence of weak solutions with less restrictions on the initial data. If vacuum is taken into account, the problem becomes more complicated. Recently, Wen and Zhu [30] obtained a unique global classical solution to the 1D model with large initial data and vacuum. For spherically symmetric flow in the exterior domain without the origin, the global existence of a strong solution was obtained by Jiang [19]. To our best knowledge, the first attempt towards the existence of weak solutions to the full compressible Navier-Stokes equations in dimension $N \geq 2$ is given by Feireisl [11] where he proved the global existence of the so-called variational solutions in the case of real gases. In addition, it is worth mentioning that, by using a new mathematical entropy equality, Bresch and Desjardins [2] got the global weak solutions to the Navier-Stokes equations for heat conducting fluids with density and temperature dependent viscosity. Certainly, for the isentropic case, the results are more satisfactory, see [10, 12, 17, 21, 22, 24] and references therein.

On the other hand, when the initial density is compactly supported, Xin [31] proved that a smooth solution will blow-up in finite time in the whole space, see also [26] for a more general blow-up result. Thus, it is interesting to investigate the mechanism of blow-up and the structure of possible singularities. Some progress for the isentropic flow can be found in [15, 16, 18, 27, 28] and references therein. For the non-isentropic case, Fan and Jiang [8] proved the following blow-up criteria for the local strong solutions to (1.1)-(1.5) in the case of two dimensions

$$\lim_{T\to T^*}\left(\sup_{0\le t\le T}\{\|\rho\|_{L^\infty}, \|\rho^{-1}\|_{L^\infty}, \|\theta\|_{L^\infty}\} + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4 + \|u\|_{L^{r,\infty}}^{\frac{2r}{r-2}})dt\right) = \infty,$$

or

$$\lim_{T\to T^*}\left(\sup_{0\le t\le T}\{\|\rho\|_{L^\infty}, \|\rho^{-1}\|_{L^\infty}, \|\theta\|_{L^\infty}\} + \int_0^T (\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4)dt\right) = \infty,$$

provided $2\mu > \lambda$, where $T^* < \infty$ is the maximal time of existence of a strong solution, $q_0 > 3$ is a certain number, $3 < r \leq \infty$ with $2/s + 3/r = 1$, and $L^{r,\infty} \equiv L^{r,\infty}(\Omega)$ is the Lorentz space. If the domain is a periodic or unit square domain in $\mathbb{R}^2$, the blow-up criterion is refined by Jiang and Ou [20] to be

$$\lim_{T\to T^*}\int_0^T \|\nabla u\|_{L^\infty}dt = \infty, \tag{1.6}$$

which coincides with the Beale-Kato-Majda criterion [3] for ideal incompressible flows. For the case that Ω is a bounded domain in $\mathbb{R}^3$, Fan, Jiang and Ou [9] established a blow-up criterion with additional upper bound of the temperature θ

$$\lim_{T\to T^*}\left(\sup_{0\le t\le T}\|\theta\|_{L^\infty} + \int_0^T \|\nabla u\|_{L^\infty}dt\right) = \infty, \tag{1.7}$$

provided

$$7\mu > \lambda. \tag{1.8}$$

Very recently, under the same condition (1.8), Sun, Wang and Zhang [29] obtained a blow-up criterion in terms of the upper bound of $(\rho, \rho^{-1}, \theta)$

$$\lim_{T\to T^*}\left(\sup_{0\le t\le T}\{\|\rho\|_{L^\infty}+\|\rho^{-1}\|_{L^\infty}+\|\theta\|_{L^\infty}\}\right)=\infty. \tag{1.9}$$

However, from the physical point of view, it is natural to expect that the solution does not blow-up under conditions on the non-appearance of the concentration of the temperature and the density. For this reason, the aim of the current paper is to remove the lower bound of the density ρ in (1.9). As pointed out in [29], without the lower bound of the density ρ, it is difficult to deal with the highly nonlinear terms $|\nabla u+\nabla u^\top|^2$, $(\mathrm{div}u)^2$ in the temperature equation. To overcome this difficulty, we put together the estimates of $\sup_{0\le t\le T}\|\nabla u\|_{L^2}^2$, $\sup_{0\le t\le T}\|\sqrt{\rho}\dot{u}\|_{L^2}^2+\int_0^T\|\nabla\dot{u}\|_{L^2}^2$ and $\sup_{0\le t\le T}\|\nabla\theta\|_{L^2}^2+\int_0^T\|\rho\dot{\theta}\|_{L^2}^2$ (see Lemma 3.2-3.4 below) and make full use the good term $\int_0^T\|\rho\dot{\theta}\|_{L^2}^2$. This is the main ingredient of our proof. Besides, let us emphasize that, instead of $\dot{\theta}$, we use θ_t as the test function in the proof of Lemma 3.4, which enable us to rewrite

$$\int_0^t[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2]\theta_t$$

as

$$\frac{d}{dt}\int_0^t[\frac{\mu}{2}|\nabla u+\nabla u^\top|^2+\lambda(\mathrm{div}u)^2]\theta+\text{other terms},$$

and thus we can avoid using the lower bound of the density. Finally, we remark that in the process of our proof, we have to deal with the troublesome term $\int_0^T\|\nabla u\|_{L^4}^4$. To this end, we use the decomposition of the velocity $u=v+w$ introduced by Sun, Wang and Zhang in [27]. More precisely, let v solves the elliptic system

$$\begin{cases}\mu\Delta v+(\lambda+\mu)\nabla\mathrm{div}v=\nabla P(\rho,\theta) & \text{in } \Omega,\\ v=0 & \text{on } \partial\Omega.\end{cases} \tag{1.10}$$

then from the momentum equation $(1.1)_2$ and (1.10), it is easy to see that w solves

$$\begin{cases}\mu\Delta w+(\lambda+\mu)\nabla\mathrm{div}w=\rho\dot{u} & \text{in } \Omega,\\ w=0 & \text{on } \partial\Omega.\end{cases} \tag{1.11}$$

Here w is an important quantity, whose divergence could be regarded as the effective viscous flux and thus possesses more regularity information than u. Based on this decomposition, our approach next is to use the Gagliardo-Nirenberg inequality and then close the estimates with the help of Gronwall's inequality. Unfortunately, if the dimension $N=3$, this approach does not work anymore, and that is why we only consider the two dimensional case.

We would like to give some notations which will be used throughout the paper.

Notations:

1. Throughout this paper, we denote by Ω a bounded smooth domain in $\mathbb{R}^2$ and $L^r = L^r(\Omega)$, $\quad W^{k,r} = W^{k,r}(\Omega)$, $\quad H^k = W^{k,2}$,
   $H_0^1 = \{v \in H^1(\Omega) : \|v\|_{H^1} < \infty$ and $v|_{\partial\Omega} = 0$ in the sence of trace$\}$, by virtue of Poincaré's inequality, we redefine $\|v\|_{H_0^1} = \|\nabla v\|_{L^2}$.

2. $\dot{f} = f_t + u \cdot \nabla f$ is the material derivative of f, and $\ddot{f} = (\dot{f})^{\cdot}$ denotes the twice material derivative of f.

3. $\displaystyle\int f = \int_\Omega f dx$ and $\displaystyle\int_0^T \int f = \int_0^T \int_\Omega f dx dt.$

Before proceeding any further, we recall that Cho and Kim [5] obtained the local strong solution to (1.1)-(1.5) with initial vacuum for the spacial dimension $N = 3$ (and Ω need not be bounded). Now if Ω is a bounded smooth domain in $\mathbb{R}^2$ with (u, θ) satisfying the Dirichlet boundary condition (1.3), it is not difficult to verify that Cho and Kim's proof in [5] still works, and we state the corresponding result below.

Proposition 1.1. *Let $q \in (2, \infty)$ be a fixed constant. Assume that the initial data satisfy*

$$\rho_0 \geq 0, \quad \rho_0 \in W^{1,q}, \quad (u_0, \theta_0) \in H_0^1 \cap H^2,$$

and the compatibility conditions

$$\begin{aligned} &\mu\Delta u_0 + (\lambda + \mu)\nabla divu_0 - R\nabla(\rho_0\theta_0) = \rho_0^{\frac{1}{2}} g_1, \\ &\kappa\Delta\theta_0 + \tfrac{\mu}{2}|\nabla u_0 + \nabla u_0^\top|^2 + \lambda(divu_0)^2 = \rho_0^{\frac{1}{2}} g_2, \end{aligned} \tag{1.12}$$

for some $(g_1, g_2) \in L^2$. Then there exist a positive constant T_0 and a unique strong solution (ρ, u, θ) to (1.1)-(1.5) such that

$$\begin{aligned} &\rho \geq 0, \quad \rho \in C([0, T_0]; W^{1,q}), \\ &(u, \theta) \in C([0, T_0]; H_0^1 \cap H^2) \cap L^2(0, T_0; W^{2,q}), \\ &(u_t, \theta_t) \in L^2([0, T_0]; H_0^1), \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty([0, T_0]; L^2). \end{aligned} \tag{1.13}$$

Our main result is stated as follows.

Theorem 1.2 (**Blow-up Criterion**). *Suppose that the assumptions in Proposition 1.1 are satisfied, and assume that (ρ, u, θ) is the strong solution constructed in Proposition 1.1. Let T^* be the maximal existence time. If $T^* < \infty$, then*

$$\lim_{T \to T^*} \sup_{0 \leq t \leq T} (\|\rho\|_{L^\infty} + \|\theta\|_{L^\infty}) = \infty. \tag{1.14}$$

Remark 1.3. If either the boundary condition (1.3) on θ is replaced by the Neumann boundary condition $\frac{\partial\theta}{\partial n}|_{\partial\Omega} = 0$ or the bounded domain Ω is replaced by the whole space $\mathbb{R}^2$, we are able to obtain the same blow-up criterion as (1.14) provided the corresponding local strong solution exists as in Proposition 1.1. In these cases, we can not use Poincaré's inequality directly to get the lower order estimates of u and θ, however, these would be recovered by using Lions's (for Ω bounded) and Hoff's (for $\Omega = \mathbb{R}^2$) technics, and the readers are referred to [24] Remark 5.1 and [14] Lemma 2.3, for instance.

2 Preliminaries

Obviously, the appearance of the systems (1.10) and (1.11) make the estimates on the strongly elliptic operator $\mu\Delta + (\lambda + \mu)\nabla\text{div}$ necessary. Now let U solves

$$\begin{cases} \mu\Delta U + (\lambda + \mu)\nabla\text{div}U = F \quad \text{in} \quad \Omega, \\ U = 0 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{2.1}$$

First of all, we state some classical estimates for the above strongly elliptic systems, which will be used later frequently .

Lemma 2.1 ([27])**.** *Let $p \in (1, \infty)$ and U be a solution of (2.1), then there exists a constant C depending only on λ, μ, p and Ω such that the following estimates hold:*
(1) if $F \in L^p$, then

$$\|U\|_{W^{2,p}} \leq C\|F\|_{L^p}, \tag{2.2}$$

(2) if $F = divf$ with $f = (f_{i,j})_{2\times 2}, f_{ij} \in L^p$, then

$$\|U\|_{W^{1,p}} \leq C\|f\|_{L^p}. \tag{2.3}$$

As in [27, 28, 29], we also need an endpoint estimate for the strongly elliptic operator $\mu\Delta+(\lambda+\mu)\nabla\text{div}$ for the case $p = \infty$. This will be done with the aid of the John-Nirenberg space of bounded mean oscillation whose norm is defined by

$$\|f\|_{BMO(\Omega)} := \|f\|_{L^2(\Omega)} + [f]_{BMO(\Omega)},$$

with

$$[f]_{BMO(\Omega)} := \sup_{x\in\Omega,\ r\in(0,d)} \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} |f(y) - f_{\Omega_r(x)}|dy,$$

$$f_{\Omega_r(x)} = \frac{1}{|\Omega_r(x)|} \int_{\Omega_r(x)} f(y)dy,$$

where $\Omega_r(x) = B_r(x) \cap \Omega$, $B_r(x)$ is a ball with center x and radius r, d is the diameter of Ω and $|\Omega_r(x)|$ denotes the Lebesgue measure of $\Omega_r(x)$. We have

Lemma 2.2 ([1])**.** *If $F = divf$ with $f = (f_{ij})_{2\times 2}$, $f_{ij} \in L^\infty \cap L^2$, then $\nabla U \in BMO(\Omega)$ and there exists a constant C depending only on λ, μ such that*

$$\|\nabla U\|_{BMO(\Omega)} \leq C(\|f\|_{L^\infty} + \|f\|_{L^2}). \tag{2.4}$$

Next, we state a variant of the Brezis-Waigner inequality [4] which, together with Lemma 2.2, will be used to give the gradient estimate of ρ.

Lemma 2.3 ([27])**.** *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$ and $f \in W^{1,q}$ with $q \in (2, \infty)$, there exists a constant C depending on q such that*

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{BMO} \ln(e + \|f\|_{W^{1,q}})). \tag{2.5}$$

Remark 2.4. Lemma 2.2 and 2.3 are the two dimensional version of the corresponding lemmas in [28]. To our knowledge, when the domain Ω is bounded, the estimate (2.4) can be found in [1] for a more general setting, and for the case $\Omega = \mathbb{R}^3$, Sun, Wang and Zhang gave a proof in [28]. The Brezis-Waigner type inequality(2.5) was first established in [23] on $\mathbb{R}^3$ and the case Ω be a bounded domain in $\mathbb{R}^N(N = 2, 3)$ can be found in [27, 28].

3 Regularity of the velocity and temperature

Let $0 < T < T^*$ be arbitrary but fixed. In what follows, we assume that (ρ, u, θ) is a strong solution of (1.1)-(1.5) on $[0, T] \times \Omega$ with the regularity stated in Proposition 1.1. Suppose that $T^* < \infty$. We will prove Theorem 1.2 by a contradiction argument. To this end, we suppose that for any $T < T^*$,

$$\sup_{0\le t\le T}(\|\rho\|_{L^\infty} + \|\theta\|_{L^\infty}) \le M, \tag{3.1}$$

where M is independent of T. We will deduce a contradiction to the maximality of T^*. Hereinafter, we denote by C a general positive constant which may depend on the initial data, the domain Ω, M in (3.1) and the maximal existence time T^*.

Lemma 3.1. *Under assumption (3.1), there holds for any $T < T^*$*

$$\sup_{0\le t\le T}\int(\rho|u|^2 + \rho|\theta|^2) + \int_0^T\int(|\nabla u|^2 + |\nabla\theta|^2) \le C. \tag{3.2}$$

The proof is the same as Lemma 2 in [29], we omit it here.

Lemma 3.2. *Under assumption (3.1), there holds for any $t < T^*$*

$$\int|\nabla u|^2 + \int_0^t\int\rho|\dot{u}|^2 \le C + C\int_0^t\int\sqrt{\rho}|\dot{\theta}||\nabla u| + C\int_0^t\int|\nabla u|^4. \tag{3.3}$$

Proof. First of all, the momentum equation can be rewritten as

$$\rho\dot{u} + \nabla P = \mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u. \tag{3.4}$$

Taking the L^2 inner product of the above equation with $\dot{u}$, and integrating by parts, we get

$$\int\rho|\dot{u}|^2 = \int P\mathrm{div}\dot{u} + \int[\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u]\dot{u}. \tag{3.5}$$

Direct calculation yields

$$\begin{aligned}\int P\mathrm{div}\dot{u} &= \frac{d}{dt}\int P\mathrm{div}u - \int(P_t + \mathrm{div}(Pu))\mathrm{div}u \\ &\quad + \int P\nabla u : \nabla u^\top,\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}&\int[\mu\Delta u + (\lambda+\mu)\nabla\mathrm{div}u]\dot{u} \\ = &-\frac{1}{2}\frac{d}{dt}\int[\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div}u)^2] - \mu\int\nabla u : (\nabla u\nabla u) \\ &+\frac{\mu}{2}\int\mathrm{div}u|\nabla u|^2 - (\lambda+\mu)\int\mathrm{div}u\nabla u : \nabla u^\top + \frac{\lambda+\mu}{2}\int(\mathrm{div}u)^3.\end{aligned} \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we have

$$
\begin{aligned}
&\int \rho|\dot u|^2 + \frac{1}{2}\frac{d}{dt}\int [\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2] \\
= \ & \frac{d}{dt}\int P\mathrm{div} u - \int (P_t + \mathrm{div}(Pu))\mathrm{div} u + \int P\nabla u : \nabla u^\top \\
&-\mu\int \nabla u : (\nabla u \nabla u) + \frac{\mu}{2}\int \mathrm{div} u|\nabla u|^2 \\
&-(\lambda+\mu)\int \mathrm{div} u\nabla u : \nabla u^\top + \frac{\lambda+\mu}{2}\int (\mathrm{div} u)^3. \qquad (3.8)
\end{aligned}
$$

Notice that

$$
P_t + \mathrm{div}(Pu) = R\rho\dot\theta, \qquad (3.9)
$$

in view of the state equation $P(\rho,\theta) = R\rho\theta$ and $(1.1)_1$. Substituting (3.9) into (3.8), and integrating the resulting equation over $[0,t]$, we get

$$
\begin{aligned}
&\int_0^t\int \rho|\dot u|^2 + \frac{1}{2}\int [\mu|\nabla u|^2 + (\lambda+\mu)(\mathrm{div} u)^2] \\
\le \ & C + \int P\mathrm{div} u + C\int_0^t\int \sqrt{\rho}|\dot\theta||\nabla u| + C\int_0^t\int |\nabla u|^2 + C\int_0^t\int |\nabla u|^3 \\
\le \ & C_\epsilon + \epsilon\int |\nabla u|^2 + C\int_0^t\int \sqrt{\rho}|\dot\theta||\nabla u| + C\int_0^t\int |\nabla u|^4, \qquad (3.10)
\end{aligned}
$$

where we have used Cauchy's inequality, (3.1) and (3.2). After the second term on the right hand side of (3.10) absorbed by the left hand side for a fixed ϵ small enough, we complete the proof of Lemma 3.2. □

Lemma 3.3. *Under assumption (3.1), there holds for any $t < T^*$*

$$
\int \rho|\dot u|^2 + \int_0^t\int |\nabla \dot u|^2 \le C + C\int_0^t\int \rho|\dot\theta|^2 + C\int_0^t\int |\nabla u|^4. \qquad (3.11)
$$

Proof. Here the calculations are motivated by Hoff [12]. Taking the material derivative to (3.4), one deduces that

$$
\begin{aligned}
&\rho\ddot u + \nabla P_t + \mathrm{div}(\nabla P\otimes u) \\
&= \mu[\Delta u_t + \mathrm{div}(\Delta u\otimes u)] + (\lambda+\mu)[\nabla\mathrm{div} u_t + \mathrm{div}((\nabla\mathrm{div} u)\otimes u)], \qquad (3.12)
\end{aligned}
$$

where $\mathrm{div}(f\otimes u) := \sum \partial_j(fu^j)$ as in [28, 29]. Taking the L^2 inner product of the above equation with $\dot u$, and using $(1.1)_1$, we arrive at

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int \rho|\dot u|^2 = \int P_t\mathrm{div}\dot u - \int \mathrm{div}(\nabla P\otimes u)\dot u \\
&+\int \mu[\Delta u_t + \mathrm{div}(\Delta u\otimes u)]\dot u + (\lambda+\mu)\int [\nabla\mathrm{div} u_t + \mathrm{div}((\nabla\mathrm{div} u)\otimes u)]\dot u, \ (3.13)
\end{aligned}
$$

A short computation shows that

$$\int P_t \mathrm{div}\dot{u} - \int \mathrm{div}(\nabla P \otimes u)\dot{u} = \int [P_t + \mathrm{div}(Pu)]\mathrm{div}\dot{u} - \int P\nabla u : \nabla \dot{u}^\top, \tag{3.14}$$

$$\begin{aligned} &\int \mu[\Delta u_t + \mathrm{div}(\Delta u \otimes u)]\dot{u} \\ = \quad & -\mu \int |\nabla \dot{u}|^2 + \mu \int [(\nabla u \nabla u) : \nabla \dot{u} + \nabla u : (\nabla u \nabla \dot{u}) - \mathrm{div}u \nabla u : \nabla \dot{u}], \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &(\lambda+\mu)\int [\nabla \mathrm{div} u_t + \mathrm{div}((\nabla \mathrm{div} u)\otimes u)]\dot{u} \\ = \quad & -(\lambda+\mu)\int |\mathrm{div}\dot{u}|^2 + (\lambda+\mu)\int \mathrm{div}u \nabla u : \nabla \dot{u}^\top \\ & +(\lambda+\mu)\int [\nabla u : \nabla u^\top \mathrm{div}\dot{u} - (\mathrm{div}u)^2 \mathrm{div}\dot{u}], \end{aligned} \tag{3.16}$$

Combining (3.13)-(3.16), and using (3.9) and (3.1) once more, we are led to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int \rho|\dot{u}|^2 + \mu \int |\nabla \dot{u}|^2 + (\lambda+\mu)\int |\mathrm{div}\dot{u}|^2 \\ = \quad & R\int \rho\dot{\theta}\mathrm{div}\dot{u} - \int P\nabla u : \nabla \dot{u}^\top \\ & +\mu \int [(\nabla u \nabla u) : \nabla \dot{u} + \nabla u : (\nabla u \nabla \dot{u}) - \mathrm{div}u \nabla u : \nabla \dot{u}] \\ & +(\lambda+\mu)\int [\mathrm{div}u \nabla u : \nabla \dot{u}^\top + \nabla u : \nabla u^\top \mathrm{div}\dot{u} - (\mathrm{div}u)^2 \mathrm{div}\dot{u}] \\ \leq \quad & \epsilon \int |\nabla \dot{u}|^2 + C_\epsilon \int \sqrt{\rho}\dot{\theta}^2 + C_\epsilon \int |\nabla u|^2 + C_\epsilon \int |\nabla u|^4. \end{aligned} \tag{3.17}$$

Choosing a ϵ sufficiently small, the first term on the right hand side of (3.17) can be absorbed by the the left hand side, and then integrating the resulting equation over $[0,t]$, using (3.2), we complete the proof of Lemma 3.3. □

Lemma 3.4 (Crucial estimates). *Under assumption (3.1), there holds for any $T < T^*$*

$$\sup_{0\leq t\leq T}\int (|\nabla u|^2 + \rho|\dot{u}|^2 + |\nabla\theta|^2) + \int_0^T \int |\nabla \dot{u}|^2 + \int_0^T \int \rho\dot{\theta}^2 \leq C. \tag{3.18}$$

Proof. According to $(1.1)_1$, the thermal energy equation $(1.1)_3$ can be rewritten as

$$c_\nu \rho\dot{\theta} - \kappa\Delta\theta + R\rho\theta \mathrm{div}u = \frac{\mu}{2}|\nabla u + \nabla u^\top|^2 + \lambda(\mathrm{div}u)^2. \tag{3.19}$$

Multiplying the above equation by θ_t, and integrating the resulting equation over Ω, we obtain

$$c_\nu \int \rho\dot{\theta}^2 + \frac{\kappa}{2}\frac{d}{dt}\int |\nabla\theta|^2$$

$$
\begin{aligned}
&= c_\nu \int \rho\dot{\theta}u\cdot\nabla\theta - R\int \rho\theta \mathrm{div}u\dot{\theta} + R\int \rho\theta\mathrm{div}u u\cdot\nabla\theta \\
&\quad + \frac{d}{dt}\int [\frac{\mu}{2}|\nabla u+\nabla u^\top|^2 + \lambda(\mathrm{div}u)^2]\theta \\
&\quad - \int [\mu(\nabla u+\nabla u^\top):(\nabla u_t+\nabla u_t^\top) + 2\lambda \mathrm{div}u\mathrm{div}u_t]\theta. \qquad (3.20)
\end{aligned}
$$

Note that

$$
\begin{aligned}
& -2\lambda\int \mathrm{div}u\mathrm{div}u_t\theta \\
= & -2\lambda\int \mathrm{div}u\mathrm{div}\dot{u}\theta + 2\lambda\int \mathrm{div}u\mathrm{div}(u\cdot\nabla u)\theta \\
= & -2\lambda\int \mathrm{div}u\mathrm{div}\dot{u}\theta + 2\lambda\int \mathrm{div}u\partial_i u^k\partial_k u^i\theta + 2\lambda\int \mathrm{div}u u^k\partial_k \mathrm{div}u\theta \\
= & -2\lambda\int \mathrm{div}u\mathrm{div}\dot{u}\theta + 2\lambda\int \mathrm{div}u\partial_i u^k\partial_k u^i\theta + \lambda\int \theta u\cdot\nabla(\mathrm{div}u)^2 \\
= & -2\lambda\int \mathrm{div}u\mathrm{div}\dot{u}\theta + 2\lambda\int \mathrm{div}u\nabla u:\nabla u^\top\theta - \lambda\int\theta(\mathrm{div}u)^3 - \lambda\int u\cdot\nabla\theta(\mathrm{div}u)^2, \qquad (3.21)
\end{aligned}
$$

and

$$
\begin{aligned}
-\mu\int(\nabla u+\nabla u^\top):(\nabla u_t+\nabla u_t^\top)\theta &= -2\mu\int\nabla u:\nabla u_t\theta - 2\mu\int\nabla u:\nabla u_t^\top\theta \\
&= I_1 + I_2, \qquad (3.22)
\end{aligned}
$$

$$
\begin{aligned}
I_1 &= -2\mu\int\nabla u:\nabla\dot{u}\theta + 2\mu\int\partial_i u^j\partial_i(u^k\partial_k u^j)\theta \\
&= -2\mu\int\nabla u:\nabla\dot{u}\theta + 2\mu\int\partial_i u^j\partial_i u^k\partial_k u^j\theta + 2\mu\int\partial_i u^j u^k\partial_k\partial_i u^j\theta \\
&= -2\mu\int\nabla u:\nabla\dot{u}\theta + 2\mu\int\nabla u:(\nabla u\nabla u)\theta + \mu\int\theta u\cdot\nabla|\nabla u|^2 \\
&= -2\mu\int\nabla u:\nabla\dot{u}\theta + 2\mu\int\nabla u:(\nabla u\nabla u)\theta - \mu\int\theta\mathrm{div}u|\nabla u|^2 - \mu\int u\cdot\nabla\theta|\nabla u|^2,
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= -2\mu\int\nabla u:\nabla\dot{u}^\top\theta + 2\mu\int\partial_i u^j\partial_j(u^k\partial_k u^i)\theta \\
&= -2\mu\int\nabla u:\nabla\dot{u}^\top\theta + 2\mu\int\partial_i u^j\partial_j u^k\partial_k u^i\theta \\
&\quad +\mu\int\partial_i u^j u^k\partial_k\partial_j u^i\theta + \mu\int\partial_j u^i u^k\partial_k\partial_i u^j\theta \\
&= -2\mu\int\nabla u:\nabla\dot{u}^\top\theta + 2\mu\int\nabla u:(\nabla u\nabla u)^\top\theta + \mu\int\theta u^k\partial_k(\partial_i u^j\partial_j u^i)
\end{aligned}
$$

$$
\begin{aligned}
&= -2\mu\int \nabla u : \nabla\dot{u}^\top\theta + 2\mu\int \nabla u : (\nabla u\nabla u)^\top\theta \\
&\quad -\mu\int \theta\mathrm{div}u\nabla u : \nabla u^\top - \mu\int u\cdot\nabla\theta\nabla u : \nabla u^\top. \qquad (3.23)
\end{aligned}
$$

Substituting (3.21)-(3.23) into (3.20), using Cauchy's inequality and (3.2), we get

$$
\begin{aligned}
& c_\nu\int \rho\dot{\theta}^2 + \frac{\kappa}{2}\frac{d}{dt}\int |\nabla\theta|^2 \\
\leq\ & \frac{c_\nu}{2}\int \rho\dot{\theta}^2 + C_\epsilon\int |\nabla u|^2 + C\int |u|^2|\nabla\theta|^2 + \epsilon\int |\nabla\dot{u}|^2 \\
& +C\int |\nabla u|^4 + \frac{d}{dt}\int [\frac{\mu}{2}|\nabla u + \nabla u^\top|^2 + \lambda(\mathrm{div}u)^2]\theta. \qquad (3.24)
\end{aligned}
$$

Consequently, integrating (3.24) over $[0,t]$ and using (3.2) again, we are led to

$$
\begin{aligned}
\int_0^t\int \rho\dot{\theta}^2 + \int |\nabla\theta|^2 \ &\leq\ C_\epsilon + C\int_0^t\int |u|^2|\nabla\theta|^2 + \epsilon\int_0^t\int |\nabla\dot{u}|^2 \\
&\quad +C\int_0^t\int |\nabla u|^4 + C\int |\nabla u|^2.
\end{aligned}
$$

Substituting (3.3) and (3.11) into the above inequality, we have

$$
\begin{aligned}
& \int_0^t\int \rho\dot{\theta}^2 + \int |\nabla\theta|^2 \\
\leq\ & C_\epsilon + C\int_0^t\int |u|^2|\nabla\theta|^2 + \epsilon C\int_0^t\int \rho|\dot{\theta}|^2 \\
& +C\int_0^t\int \sqrt{\rho}|\dot{\theta}||\nabla u| + C\int_0^t\int |\nabla u|^4 \\
\leq\ & C_\epsilon + C\int_0^t\int |u|^2|\nabla\theta|^2 + \epsilon C\int_0^t\int \rho|\dot{\theta}|^2 + C\int_0^t\int |\nabla u|^4,
\end{aligned}
$$

where we have used Cauchy's inequality and (3.2). Take a ϵ sufficiently small, then there holds

$$
\int_0^t\int \rho\dot{\theta}^2 + \int |\nabla\theta|^2 \ \leq\ C + C\int_0^t\int |u|^2|\nabla\theta|^2 + C\int_0^t\int |\nabla u|^4. \qquad (3.25)
$$

It follows from Lemma 2.1, Gagliardo-Nirenberg' inequality and (3.1) that

$$
\begin{aligned}
\int_0^t\int |\nabla u|^4 \ &\leq\ \int_0^t\int |\nabla v|^4 + \int_0^t\int |\nabla w|^4 \\
&\leq\ C\int_0^t\int |P|^4 + C\int_0^t \|\nabla w\|_{L^2}^2\|\nabla w\|_{H^1}^2 \\
&\leq\ C + C\int_0^t (\|\nabla u\|_{L^2}^2 + \|\nabla v\|_{L^2}^2)\|\sqrt{\rho}\dot{u}\|_{L^2}^2
\end{aligned}
$$

$$
\begin{aligned}
&\le C + C\int_0^t (\|P\|_{L^2}^2 + \|\nabla u\|_{L^2}^2)\|\sqrt{\rho}\dot{u}\|_{L^2}^2 \\
&\le C + C\int_0^t (1 + \|\nabla u\|_{L^2}^2)\|\sqrt{\rho}\dot{u}\|_{L^2}^2, \qquad (3.26)
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\|\nabla u\|_{L^4}^2 &\le C\|P\|_{L^4}^2 + C(\|P\|_{L^2} + \|\nabla u\|_{L^2})\|\sqrt{\rho}\dot{u}\|_{L^2} \\
&\le C + C(1 + \|\nabla u\|_{L^2})\|\sqrt{\rho}\dot{u}\|_{L^2}. \qquad (3.27)
\end{aligned}
$$

Moreover,

$$
\int_0^t \int |u|^2|\nabla\theta|^2 \le \int_0^t \|u\|_{L^\infty}^2\|\nabla\theta\|_{L^2}^2 \le C\int_0^t \|u\|_{L^\infty}^4 + C\int_0^t \|\nabla\theta\|_{L^2}^4. \qquad (3.28)
$$

By Sobolev imbedding $W^{1,4} \hookrightarrow L^\infty$,

$$
\|u\|_{L^\infty}^4 \le C\|u\|_{L^4}^4 + C\|\nabla u\|_{L^4}^4 \le C\|\nabla u\|_{L^4}^4, \qquad (3.29)
$$

where we have used Poincaré's inequality. Then (3.28) and (3.29) imply that

$$
\int_0^t \int |u|^2|\nabla\theta|^2 \le C\int_0^t (\|\nabla u\|_{L^4}^4 + \|\nabla\theta\|_{L^2}^4). \qquad (3.30)
$$

Substituting (3.26) and (3.30) into (3.25), we obtain

$$
\begin{aligned}
&\int_0^t \int \rho\dot{\theta}^2 + \int |\nabla\theta|^2 \\
&\le C + C\int_0^t \|\nabla\theta\|_{L^2}^4 + C\int_0^t (1 + \|\nabla u\|_{L^2}^2)\|\sqrt{\rho}\dot{u}\|_{L^2}^2. \qquad (3.31)
\end{aligned}
$$

Now we are in a position to close all the estimates above. Indeed, $4C\times$(3.31)+(3.3)+(3.11) implies

$$
\begin{aligned}
&\int (|\nabla u|^2 + \rho|\dot{u}|^2 + |\nabla\theta|^2) + \int_0^t \int |\nabla\dot{u}|^2 + \int_0^t \int \rho\dot{\theta}^2 \\
&\le C + C\int_0^t \|\nabla\theta\|_{L^2}^4 + C\int_0^t (1 + \|\nabla u\|_{L^2}^2)\|\sqrt{\rho}\dot{u}\|_{L^2}^2 \\
&\le C + C\int_0^t (1 + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)(\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2). \qquad (3.32)
\end{aligned}
$$

By virtue of Gronwall's inequality and (3.2), we complete the proof of Lemma 3.4. □

Corollary 3.5. *Under assumption (3.1), there holds for any $T < T^*$*

$$
\sup_{0\le t\le T} (\|u\|_{L^\infty} + \|\nabla u\|_{L^4}) \le C, \qquad (3.33)
$$

$$
\int_0^T \int |\nabla^2\theta|^2 \le C. \qquad (3.34)
$$

Proof. Clearly, (3.33) is a direct consequence of (3.18), (3.27) and (3.29). In view of (3.18), (3.33) and (3.1), applying the standard elliptic regularity theory to (3.19), then (3.34) follows immediately. □

Lemma 3.6. *Under assumption (3.1), there holds for any $T < T^*$*

$$\sup_{0\le t\le T}\int\rho\dot\theta^2+\int_0^T\int|\nabla\dot\theta|^2\le C. \tag{3.35}$$

$$\sup_{0\le t\le T}\|\theta\|_{H^2}\le C. \tag{3.36}$$

Proof. Taking the material derivative to (3.19), we find

$$\begin{aligned} & c_\nu\rho\ddot\theta-\kappa(\Delta\theta)^{\cdot}=(c_\nu-R)\rho\dot\theta\mathrm{div}u+R\rho\theta[(\mathrm{div}u)^2+\nabla u:\nabla u^\top]\\ - \ & R\rho\theta\mathrm{div}\dot u+[\mu(\nabla u+\nabla u^\top):(\nabla\dot u+\nabla\dot u^T)+2\lambda\mathrm{div}u\mathrm{div}\dot u]\\ - \ & [\mu(\nabla u+\nabla u^\top):(\nabla u\nabla u+\nabla u^\top\nabla u^\top)+2\lambda\mathrm{div}u\nabla u:\nabla u^\top], \end{aligned}\tag{3.37}$$

where we have used the facts

$$\dot\rho=-\rho\mathrm{div}u,\quad (\mathrm{div}u)^{\cdot}=\mathrm{div}\dot u-\nabla u:\nabla u^\top$$

and

$$(\nabla u)^{\cdot}=\nabla\dot u-\nabla u\nabla u,\quad (\nabla u^\top)^{\cdot}=\nabla\dot u^\top-\nabla u^\top\nabla u^\top.$$

Taking the L^2 inner product of the (3.37) with $\dot\theta$, and using $(1.1)_1$, we arrive at

$$\begin{aligned} & \frac{c_\nu}{2}\frac{d}{dt}\int\rho\dot\theta^2-\kappa\int(\Delta\theta)^{\cdot}\dot\theta\\ \le \ & (c_\nu-R)\int\rho\dot\theta^2\mathrm{div}u+R\int\rho\dot\theta\theta[(\mathrm{div}u)^2+\nabla u:\nabla u^\top]\\ & -R\int\rho\dot\theta\theta\mathrm{div}\dot u+C\int|\nabla u||\nabla\dot u||\dot\theta|+C\int|\nabla u|^3|\dot\theta|. \end{aligned}\tag{3.38}$$

Note that

$$\begin{aligned} & -\kappa\int(\Delta\theta)^{\cdot}\dot\theta=-\kappa\int(\Delta\theta_t+u\cdot\nabla\Delta\theta)\dot\theta\\ = \ & \kappa\int\nabla\theta_t\nabla\dot\theta+\kappa\int\partial_iu^k\partial_{ki}\theta\dot\theta+\kappa\int u^k\partial_{ki}\theta\partial_i\dot\theta\\ = \ & \kappa\int|\nabla\dot\theta|^2-\kappa\int\nabla(u\cdot\nabla\theta)\cdot\nabla\dot\theta+\kappa\int\partial_iu^k\partial_{ki}\theta\dot\theta+\kappa\int u^k\partial_{ki}\theta\partial_i\dot\theta\\ = \ & \kappa\int|\nabla\dot\theta|^2-\kappa\int\nabla\dot\theta\nabla u\nabla\theta+\kappa\int\nabla u:\nabla^2\theta\dot\theta. \end{aligned}\tag{3.39}$$

We infer from (3.38), (3.39) and (3.1) that

$$\frac{c_\nu}{2}\frac{d}{dt}\int\rho\dot\theta^2+\kappa\int|\nabla\dot\theta|^2$$

$$
\begin{aligned}
&\le C\int \rho\dot\theta^2|\nabla u| + C\int \rho|\dot\theta||\nabla u|^2 + C\int \rho|\dot\theta||\nabla\dot u| + C\int |\nabla u||\nabla\dot u||\dot\theta| \\
&\quad + C\int |\nabla u|^3|\dot\theta| + \kappa\int |\nabla\dot\theta||\nabla u||\nabla\theta| + \kappa\int |\nabla u||\nabla^2\theta||\dot\theta| \\
&\le \frac{\kappa}{2}\int |\nabla\dot\theta|^2 + \epsilon\int |\nabla u|^2|\dot\theta|^2 + C_\epsilon\int \rho\dot\theta^2 + C_\epsilon\int |\nabla u|^4 \\
&\quad + C_\epsilon\int |\nabla\dot u|^2 + C_\epsilon\int |\nabla^2\theta|^2 + C\int |\nabla u|^2|\nabla\theta|^2,
\end{aligned}
\tag{3.40}
$$

From (3.33), Gagliardo-Nirenberg' inequality and Poincaré's inequalty, we have

$$
\epsilon\int |\nabla u|^2|\dot\theta|^2 \le \epsilon\|\nabla u\|_{L^4}^2\|\dot\theta\|_{L^4}^2 \le \epsilon C\|\dot\theta\|_{L^2}\|\nabla\dot\theta\|_{L^2} \le C\|\nabla\dot\theta\|_{L^2}^2, \tag{3.41}
$$

and

$$
\begin{aligned}
\int |\nabla u|^2|\nabla\theta|^2 &\le \|\nabla u\|_{L^4}^2\|\nabla\theta\|_{L^4}^2 \le C\|\nabla\theta\|_{L^2}\|\nabla\theta\|_{H^1} \\
&\le C\|\nabla\theta\|_{L^2}^2 + C\|\nabla^2\theta\|_{L^2}^2 \le C + C\|\nabla^2\theta\|_{L^2}^2,
\end{aligned}
\tag{3.42}
$$

where we have used (3.18) in the last inequality above. Substituting (3.41) and (3.42) into (3.40), choosing a ϵ small enough and using (3.33), we obtian

$$
c_\nu\frac{d}{dt}\int \rho\dot\theta^2 + \kappa\int |\nabla\dot\theta|^2 \le C + C\int |\nabla^2\theta|^2 + C\int |\nabla\dot u|^2 + C\int \rho\dot\theta^2.
$$

Integrating the above equation over $[0,t]$, and using (3.18) and (3.34) once more, we get (3.35). Finally, (3.36) follows from (3.19), (3.18), (3.33), (3.35) and (3.1) immediately. This completes the proof of Lemma 3.6. □

4 Improved regularity of the density and velocity

Generally speaking, for the continuity equation $(1.1)_1$, the gradient estimate on ρ rely on the boundedness of $\int_0^T \|\nabla u\|_{L^\infty}$. On the other hand, due to the lower regularity on ρ, we can not obtain the higher order regularity of u through the momentum equation (3.4). Nevertheless, with the help of the decomposition $u = v + w$, we can close the gradient estimate of ρ based on a logarithmic estimate for the strongly elliptic operator $\mu\Delta + (\lambda+\mu)\nabla\mathrm{div}$.

We first give some higher regularity on w below.

Lemma 4.1. *Under assumption (3.1), there holds for any $T < T^*$*

$$
\sup_{0\le t\le T}\|w\|_{H^2} + \int_0^T (\|\nabla^2 w\|_{L^q}^2 + \|\nabla w\|_{L^\infty}^2) \le C, \quad q\in(2,\infty). \tag{4.1}
$$

Proof. Using (2.2) with $U = w, F = \rho\dot u$ and $p = 2$, we have

$$
\|w\|_{H^2} \le C\|\rho\dot u\|_{L^2} \le C\|\sqrt{\rho}\dot u\|_{L^2}, \tag{4.2}
$$

where we have used (3.1). Then it follows from (3.18) that

$$\sup_{0\le t\le T}\|w\|_{H^2}\le C. \tag{4.3}$$

For $q>2$, using (2.2), (3.1) again, and by virtue of Gagliardo-Nirenberg inequality, we find

$$\begin{aligned}\|\nabla^2 w\|_{L^q} &\le C\|\rho\dot u\|_{L^q}\le C\|\dot u\|_{L^q}\\ &\le C\|\dot u\|_{L^2}^{\frac{2}{q}}\|\nabla\dot u\|_{L^2}^{\frac{q-2}{q}}\le C\|\nabla\dot u\|_{L^2},\end{aligned}$$

where we have used Poincaré's inequality. Next, we use the Sobolev imbeding $W^{1,q}\hookrightarrow L^\infty$ for $q>2$ to get

$$\|\nabla w\|_{L^\infty}\le C(\|\nabla w\|_{L^q}+\|\nabla^2 w\|_{L^q})\le C(\|\nabla w\|_{H^1}+\|\nabla^2 w\|_{L^q}), \tag{4.4}$$

then we infer from (3.18), (4.3) and (4.4) that

$$\int_0^T(\|\nabla^2 w\|_{L^q}^2+\|\nabla w\|_{L^\infty}^2)\le C. \tag{4.5}$$

This completes the proof of Lemma 4.1. □

Thanks to all the estimates obtained above, we will get the gradient estimates of the density ρ next.

Lemma 4.2. *Under assumption (3.1), there holds for any $T<T^*$*

$$\sup_{0\le t\le T}(\|\rho\|_{W^{1,q}}+\|u\|_{H^2})+\int_0^T\|\nabla u\|_{L^\infty}^2\le C. \tag{4.6}$$

Proof. The proof follows the ideas of Sun, Wang and Zhang [27, 28, 29], we sketch it here for completeness. First of all, take ∇ to the continuity equation $(1.1)_1$ to find

$$\partial_t\nabla\rho+(u\cdot\nabla)\nabla\rho+\nabla u\nabla\rho+\mathrm{div}u\nabla\rho+\rho\nabla\mathrm{div}u=0. \tag{4.7}$$

Multiplying (4.7) by $q|\nabla\rho|^{q-2}\nabla\rho$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\|\nabla\rho\|_{L^q} &\le C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}+C\|\nabla^2 u\|_{L^q}\\ &\le C(\|\nabla v\|_{L^\infty}+\|\nabla w\|_{L^\infty})\|\nabla\rho\|_{L^q}+C(\|\nabla^2 v\|_{L^q}+\|\nabla^2 w\|_{L^q})\\ &\le C(1+\|\nabla v\|_{L^\infty}+\|\nabla w\|_{L^\infty})\|\nabla\rho\|_{L^q}+C(\|\nabla^2 w\|_{L^q}+\|\nabla\theta\|_{L^q}),\end{aligned} \tag{4.8}$$

where we have used (2.2). To close (4.8), we have to bound $\|\nabla v\|_{L^\infty}$, and it is just this term leads us to show the endpoint estimate for the strongly elliptic operator $\mu\Delta+(\lambda+\mu)\nabla\mathrm{div}$. In fact, Lemma 2.1-2.3 imply that if $q>2$

$$\|\nabla v\|_{L^\infty}\le C(1+\|\nabla v\|_{BMO}\ln(e+\|\nabla^2 v\|_{L^q}))$$

$$
\begin{aligned}
&\le C(1+(\|P\|_{L^\infty}+\|P\|_{L^2})\ln(e+\|\nabla^2 v\|_{L^q}))\\
&\le C(1+\ln(e+\|\nabla\rho\|_{L^q})+\ln(e+\|\nabla\theta\|_{L^q}))\\
&\le C(1+\|\nabla\theta\|_{L^q}+\ln(e+\|\nabla\rho\|_{L^q})). \qquad (4.9)
\end{aligned}
$$

Substituting (4.9) into (4.8), we get

$$
\begin{aligned}
\frac{d}{dt}(e+\|\nabla\rho\|_{L^q}) \le\; & C(1+\|\nabla\theta\|_{L^q}+\|\nabla w\|_{L^\infty})\|\nabla\rho\|_{L^q}+C\ln(e+\|\nabla\rho\|_{L^q})\|\nabla\rho\|_{L^q}\\
&+C(\|\nabla^2 w\|_{L^q}+\|\nabla\theta\|_{L^q}), \qquad (4.10)
\end{aligned}
$$

then using (4.1), (3.36) and Gronwall's inequality, one deduces that

$$
\sup_{0\le t\le T}\|\nabla\rho\|_{L^q}\le C. \qquad (4.11)
$$

From (3.36), (4.1), (4.11) and Lemma 2.1, there holds for $q>2$

$$
\begin{aligned}
\int_0^T\|\nabla u\|_{L^\infty}^2 &\le C\int_0^T\|\nabla v\|_{L^\infty}^2+\int_0^T\|\nabla w\|_{L^\infty}^2\\
&\le C+C\int_0^T\|\nabla v\|_{L^q}^2+C\int_0^T\|\nabla^2 v\|_{L^q}^2\le C. \qquad (4.12)
\end{aligned}
$$

Moreover, it follows from (4.1), (3.36) and (4.11)

$$
\|\nabla^2 u\|_{L^2}\le\|\nabla^2 w\|_{L^2}+\|\nabla^2 v\|_{L^2}\le C+C\|\nabla\rho\|_{L^2}\le C. \qquad (4.13)
$$

This completes the proof of Lemma 4.2. □

5 Proof of Theorem 1.2

The combination of Lemma 3.4, Lemma 3.6 and Lemma 4.2 will enable us to extend the strong solution (ρ,u,θ) beyond the maximal existence time T^*. Indeed, by virtue of (4.6), (3.36) and time continuity stated in (1.13), we can define

$$
(\rho,u,\theta)|_{t=T^*}=\lim_{t\to T^*}(\rho,u,\theta), \qquad (5.1)
$$

and

$$
h:=\rho\dot u|_{t=T^*}=\lim_{t\to T^*}(\mu\Delta u+(\lambda+\mu)\nabla\mathrm{div}u-\nabla P) \quad \text{strongly in } L^2. \qquad (5.2)
$$

On the other hand, using Sobolev imbedding $W^{1,q}\hookrightarrow C^{0,1-\frac{2}{q}}$ for $q>2$, (4.6) and the time continuity on ρ stated in (1.13), one easily deduces that

$$
\rho|_{t=T^*}(x)=\lim_{t\to T^*}\rho(x) \quad \text{uniformly in } \Omega,
$$

and hence

$$
\sqrt{\rho}|_{t=T^*}(x)=\lim_{t\to T^*}\sqrt{\rho}(x) \quad \text{pointwise in } \Omega, \qquad (5.3)
$$

then (5.3), the assumption (3.1) that ρ is upper bounded and Lebesgue's Dominated Convergence Theorem imply that

$$\sqrt{\rho}|_{t=T^*} = \lim_{t\to T^*} \sqrt{\rho} \quad \text{strongly in } L^2. \tag{5.4}$$

Besides, we infer from (3.18) that there exists a sequence $t_k \to T^*$ as $k \to \infty$ and a function $\tilde{g}_1 \in L^2$, such that

$$\tilde{g}_1 = \lim_{k\to\infty} (\sqrt{\rho}\dot{u})(t_k) \quad \text{weakly in } L^2. \tag{5.5}$$

It follows from (5.4) and (5.5) that

$$\sqrt{\rho}|_{t=T^*}\tilde{g}_1 = \lim_{k\to\infty} (\rho\dot{u})(t_k) \quad \text{in the sence of distribution.} \tag{5.6}$$

Comparing (5.2) with (5.6), we obtain

$$h = \sqrt{\rho}|_{t=T^*}\tilde{g}_1,$$

i.e.

$$(\mu\Delta u + (\lambda+\mu)\nabla \mathrm{div} u - \nabla P)|_{t=T^*} = \sqrt{\rho}|_{t=T^*}\tilde{g}_1. \tag{5.7}$$

Similarly, there exists a function $\tilde{g}_2 \in L^2$ such that

$$(\kappa\Delta\theta + \frac{\mu}{2}|\nabla u + \nabla u^\top|^2 + \lambda(\mathrm{div} u)^2)|_{t=T^*} = \sqrt{\rho}|_{t=T^*}\tilde{g}_2. \tag{5.8}$$

Now (5.1), (5.7) and (5.8) assure that we can take $(\rho, u, \theta)|_{t=T^*}$ as the initial data and apply Proposition 1.1 to extend the local strong solution beyond T^*, which contradicts the maximality of T^*. This completes the proof of Theorem 1.2. □

Acknowledgment. The second author would like to thank Professor Zhifei Zhang and Dr. Huanyao Wen for their sincere help.

References

[1] Acquistapace, P.: On BMO regularity for linear elliptic systems. Ann. Mat. Pura Appl. 161, 231-269(1992).

[2] Bresch, D., Desjardins,B.,: On the existence of global weak solutions to the Navier-Stokes equations for viscous compressible and heat conducting fluids. J. Math. Pures Appl. 87, 57-90 (2007)

[3] Beal, J.T., Kato, T., Majda, A.: Remarks on the breakdown of smooth solutions for the 3-D Euler equations. Commun. Math. Phys. 94, 61-66 (1984)

[4] Brézis, H., Wainger, S.: A note on limiting cases of Sobolev embeddings and convolution inequalities, Comm. Partial Differential Equations 5, 773-789 (1980)

[5] Cho, Y., Kim, H.: Existence results for viscous polytropic fluids with vacuum. J. Diff. Eqs. 228, 377-411 (2006)

[6] Danchin, R.: Local theory in critical spaces for compressible viscous and heatconductive gases. Comm. Partial Differ. Equ. 26, 1183-1233 (2001)

[7] Danchin, R.: Global existence in critical spaces for flows of compressible viscous and heat-conductive gases. Arch. Rational Mech. Anal. 160, 1-39 (2001)

[8] Fan, J.S., Jiang, S.: Blow-up criteria for the navier-stokes equations of compressible fluids. J.Hyper. Diff. Eqs. 5(1), 167-185 (2008)

[9] Fan, J., Jiang, S., Ou, Y. A blow-up criterion for the compressible viscous heat-conductive flows, Ann. Inst. H. Poincaré Anal. Non Linéaire 27, 337-350 (2010)

[10] Feireisl, E., Novotny, A., Petzeltov, H.: On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. 3(4), 358-392 (2001)

[11] Feireisl, E.: Dynamics of Viscous Compressible Fluids. Oxford: Oxford University Press, 2004

[12] Hoff, D.: Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data. J. Differential Equations 120, 215-254 (1995)

[13] Hoff,D.: Discontinuous solutions of the Navier-Stokes equations for multidimensional flows of heat-conducting fluids. Arch. Rational Mech. Anal. 139, 303-354 (1997)

[14] Hoff, D.: Compressible flow in a half-space with Navier boundary conditions. J. Math. Fluid Mech. 7, 315-338 (2005)

[15] Huang, X.D.: Some results on blowup of solutions to the compressible Navier-Stokes equations. PhD Thesis, Chinese University of Hong Kong, 2009

[16] Huang, X.D., Xin, Z.P.: A blow-up criterion for classical solutions to the compressible Navier-Stokes equations. Sci. China Math 53(3), 671-686 (2010)

[17] Huang, X. D., Li, J., Xin, Z. P.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier-Stokes equations, arXiv:1004.4749v2

[18] Huang, X. D., Li, J., Xin, Z. P.: Blowup criterion for viscous baratropic flows with vacuum states. Commun. Math. Phys. 301, 23-35 (2011)

[19] Jiang, S.: Global spherically symmetric solutions to the equations of a viscous polytropic ideal gas in an exterior domain. Comm. Math. Phys. 178, 339-374 (1996)

[20] Jiang, S., Ou, Y. B.: A blow-up criterion for compressible viscous heat-conductive flows. Acta Math. Sci. 30B(6):1851-1864 (2010)

[21] Jiang, S., Zhang, P.: On sperically symmetric solutions of the compressible isentropic Navier-Stokes equations. Commun. Math. Phys. 215, 559-581 (2001)

[22] Jiang, S., Zhang, P.: Axisymmetric solutions of the 3-D Navier-Stokes equations for compressible isentropic flows, J. Math. Pure Appl. 82, 949-973 (2003)

[23] Kozono, H. Taniuchi, Y.: Limiting case of the Sobolev inequality in BMO, with application to the Euler equations. Comm. Math. Phys. 214, 191-200 (2000)

[24] Lions, P.L.: Mathematical topics in fluid mechanics. Vol. 2. Compressible models. New York: Oxford University Press, 1998

[25] A. Matsumura, T. Nishida, Initial boundary value problems for the equations of motion of compressible viscous and heat conductive fluids. Comm. Math. Phys. 89, 445-464 (1983)

[26] Rozanova, O.: Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes equations. J. Differ. Equ. 245, 1762-1774 (2008)

[27] Sun, Y. Z., Zhang, Z. F.: A blow-up criterion of strong solution for the 2-D compressible Navier-Stokes equations. Sci. China Math. 54, 105-116 (2011)

[28] Sun, Y. Z., Wang, C., Zhang, Z. F.: A Beal-Kato-Majda blow-up criterion for the 3D compressible Navier-Stokes equations. J. Math. Pures Appl. 95, 36-47 (2011)

[29] Sun, Y. Z., Wang, C., Zhang, Z. F.: A Beale-Kato-Majda criterion for three dimensional compressible viscous heat-conductive flows. Arch. Rational Mech. Anal. DOI: 10.1007/s00205-011-0407-1

[30] Wen, H. Y., Zhu, C. J.,: Global classical large solutions to Navier-Stokes equations for viscous compressible and heat conducting fluids with vacuum. arXiv:1103.1421v1

[31] Xin, Z. P.: Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density. Comm. Pure Appl. Math. 51, 229-240 (1998)