SIMPLE G-GRADED ALGEBRAS AND THEIR POLYNOMIAL IDENTITIES

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July 26, 2011

Abstract. Let G be any group and F an algebraically closed field of characteristic zero. We show that any two G-graded finite dimensional G-simple algebras over F are G-graded isomorphic if and only if the satisfy the same G-graded polynomial identities. This result was proved by Koshlukov and Zaicev in case G is abelian.

Introduction

The purpose of this article is to prove that finite dimensional (associative) simple G-graded algebras over an algebraically closed field F of characteristic zero are determined up to G-graded isomorphism by their G-graded identities. Here G is any group. In case G is abelian, the result was established by Koshlukov and Zaicev [8]. Analogous results were obtained for Lie algebras by Kushkulei and Razmyslov [7] and for Jordan algebras by Drensky and Racine [6].

The structure theory of finite dimensional G-graded algebras and in particular of simple G-graded algebras plays a crucial role in the proof of the representability theorem for G-graded PI algebras and in the solution of the Specht problem (i.e. the T-ideal of G-graded identities is finitely based) for such algebras (see Aljadeff and Kanel-Belov [2]).

Recall that the representability theorem for G-graded algebras says in particular, that if W is an affine G-graded algebra which is PI as an ordinary algebra, then there exists a finite dimensional algebra A which satisfies precisely the same G-graded identities as W.

A fundamental part of the proof of the Representability Theorem is the construction of special finite dimensional G-graded algebras which are called basic. It turns out that if B is a basic algebra, then B admits G-graded polynomials which are called Kemer. These are multilinear polynomials, nonidentities, which are extremal in the sense that for every g in G, there is an alternating set of homogeneous elements of degree g of cardinality which is equal to the dimension of the g-homogeneous component of B. Clearly no nonidentity polynomial of B can have larger alternating sets.

The key point in the proof of representability is that the finite dimensional algebra A which satisfies the same G-graded identities as W can be expressed as the direct sum of basic algebras and hence the T-ideal of G-graded identities of A is the intersection of the corresponding ideals of identities of the basic algebras which appear in the decomposition. However, the basic algebras that appear in

Key words and phrases. graded algebra, polynomial identity.

the decomposition of A are not known to be unique. Furthermore, even in the nongraded case it is not true that basic algebras themselves are determined by their G-graded identities. This is because the simple components of the semisimple part of the algebra "interact" via the radical. But if the basic algebra is G-semisimple it must in fact be G-simple and the main result of the paper says that in that case the answer is positive.

To state the result precisely we recall some basic definitions. Let k be an arbitrary field and let G be a group. A k-algebra A is said to be G-graded if for each $g \in G$ there is a k-subspace A_g of A (possibly zero) such that for all $g, h \in G$, we have $A_g A_h \subseteq A_{gh}$. Such a G-graded algebra is said to be a simple G-graded algebra (or a G-simple algebra) if there are no nontrivial homogeneous ideas, or equivalently if the ideal generated by each nonzero homogeneous element is the whole algebra.

A G-graded polynomial is a polynomial in the free algebra $k\langle X_G\rangle$ where X_G is the union of sets X_g, $g \in G$ and $X_g = \{x_{1,g}, x_{2,g}, ...\}$. In other words, X_G consists of countably many variables of degree g for every $g \in G$. We say that a polynomial $p(x_{1,g_{i_1}}, ..., x_{n,g_{i_n}})$ in $k\langle X_G\rangle$ is a G-graded identity of a G-graded algebra A if p vanishes upon any graded evaluation on A. The set of G-graded identities of A is an ideal of $k\langle X_G\rangle$ which we denote by $Id_G(A)$. Moreover it is a T-ideal, that is, it is closed under G-graded endomorphisms of $k\langle X_G\rangle$.

It is known that if k has characteristic zero, the T-ideal of identities is generated as a T-ideal by multilinear polynomials, that is graded polynomials whose monomials are permutation of each other (up to a scalar from the field). Moreover we may assume in addition that all of the monomials have the same homogeneous degree. We can now state the main result of the paper.

Theorem 0.0.1. Let A and B two finite dimensional simple G-graded algebras over F where F is an algebraically closed field of characteristic zero. Then A and B are G-graded isomorphic if and only if $Id_G(A) = Id_G(B)$.

A key ingredient in the proof is the result of Bahturin, Sehgal and Zaicev ([5], Theorem 1.1) that determines the structure of a simple G-graded algebra as a combination of a fine graded algebra and an elementary graded algebra. In section 1 we state this result and use it to define the notion of a presentation of the given G-simple algebra.

Another motivation for studying G-graded polynomial identities of finite dimensional G-simple algebras is the possible existence of a "versal" object. It is well known that if A is the algebra of $n \times n$-matrices, the corresponding algebra of generic elements has a suitable central localization into an Azumaya algebra which is versal with respect to all k-forms (in the sense of Galois descent) of A where k-is any field of zero characteristic. Furthermore, extending the center to the field of fractions, one obtains a division algebra, the so called generic division algebra, which is a form of A. The algebra of generic elements can be constructed in a different way. It is well known that it is isomorphic to the the relatively free algebra of A, namely, the free algebra on a countable set of variables modulo the T-ideal of identities.

The same connection exists also in the G-graded case. Given a G-graded finite dimensional algebra one can construct the corresponding G-graded relatively free algebra, and it is of interest to know whether there exists a versal object in this case as well. It turns out that this is so for some specific cases as in [1] and [3] and [4].

Clearly, if two nonisomorphic finite dimensional G-simple algebras A and B, had the same T-ideal of identities , it would discard the possibility to have a versal object neither for A nor B. So in view of Theorem 0.0.1 it is natural to ask if for an arbitrary finite dimensional G-simple algebra there exists a corresponding versal object.

1. Preliminaries

We start by recalling some terminology. Let G be any group and A a finite dimensional simple G-graded algebra. As mentioned in the introduction, our proof is based on a result of Bahturin, Sehgal and Zaicev[5] in which they present any finite dimensional G-graded simple algebra by means of two type of G-gradings, fine and elementary. Before stating their theorem let us give two examples, one of each kind.

Given a finite subgroup H of G we can consider the group algebra FH with the natural H-grading. Clearly as such, it is H-simple. Moreover we can view the algebra FH as a G-graded algebra where the g-homogeneous component is set to be 0 if g is not in H. More generally we may consider any twisted group algebra $F^{\alpha}H$, where α is a 2-cocycle of H with invertible values in F, again as a G-graded algebra. As in the case where the cocycle is trivial, the algebra $F^{\alpha}H$ is finite dimensional G-simple. We refer to such grading as fine grading. The second type of grading is called elementary. Let $M_r(F)$ be the algebra of $r \times r$ matrices over the field F. Fix an r-tuple $(p_1, ..., p_r) \in G^{(r)}$, and let the elementary matrix $e_{i,j}$, $1 \leq i, j \leq n$ be of homogeneous degree $p_i^{-1}p_j$. Note that the product of the elementary matrices is compatible with their homogeneous degrees and so we obtain a G-grading on $M_r(F)$. Furthermore, since $M_r(F)$ is a simple algebra it is also G-simple.

The result of Bahturin, Sehgal and Zaicev[5] says that any finite dimensional G-simple algebra is isomorphic to a G-graded algebra which is the tensor product of two G-simple algebras, one with fine grading and the other with an elementary grading. Here is the precise result.

Theorem 1.1. [5]*Let A be a finite dimensional G-simple algebra over an algebraically closed field F of characteristic zero. Then there exists a finite subgroup H of G, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of H on F is trivial, an integer r and a r-tuple $(p_1, p_2, \ldots, p_r) \in G^{(r)}$ such that A is G-graded isomorphic to $C = F^{\alpha}H \otimes M_r(F)$ where $C_g = span_F\{u_h \otimes e_{i,j} : g = p_i^{-1}hp_j\}$. Here $u_h \in F^{\alpha}H$ is a representative of $h \in H$ and $e_{i,j} \in M_r(F)$ is the (i, j) elementary matrix.*

In particular the idempotents $1 \otimes e_{i,j}$ as well as the identity element of A are homogeneous of degree $e \in G$.

Definition 1.2. Given a finite dimensional G-simple algebra A, let H, $\alpha \in Z^2(H, F^*)$ and $(p_1, \ldots, p_r) \in G^{(r)}$ be as in the theorem above. We denote the triple $(H, \alpha, (p_1, \ldots, p_r))$ by P_A and refer to it as a <u>presentation</u> of the G-graded algebra A.

Clearly, a presentation determines the G-graded structure of A up to a G-graded isomorphism. On the other hand, a G-graded algebra may admit more than one presentation and so we need to introduce a suitable equivalence relation on presentations.

We start by establishing some conditions on presentations which yield G-graded isomorphic algebras.

Lemma 1.3. *Let A be a finite dimensional G-simple algebra with presentation $P_A = (H, \alpha, (p_1, \ldots, p_r))$. The following "moves" (and their composites) on the presentation determine G-graded algebras G-graded isomorphic to A.*

(1) *Permuting the r-th tuple, that is $A' \cong F^{\alpha_A}H_A \otimes_F M_r(F)$ and the elementary grading is given by $(p_{\pi(1)}, ..., p_{\pi(r)})$ where $\pi \in Sym(r)$.*
(2) *Replacing any entry p_i of $(p_1, ..., p_r)$ by any element $h_0 p_i \in Hp_i$ (changing right H-coset representatives).*
(3) *For an arbitrary $g \in G$,*
   (a) *replacing H with the conjugate $H^g = gHg^{-1}$,*
   (b) *replacing the cocycle α by $g(\alpha)$ where*
$$g(\alpha)(gh_1g^{-1}, gh_1g^{-1}) = \alpha(h_1, h_2)$$
   and
   (c) *shifting the tuple $(p_1, ..., p_r)$ by g, that is, replacing the tuple $(p_1, ..., p_r)$ by $(gp_1, ..., gp_r)$.*

Proof. We describe the isomorphism maps.

(1)
$$u_h \otimes e_{k,l} \longmapsto u_h \otimes e_{\pi(k),\pi(l)}$$
(2)
$$u_h \otimes e_{k,l} \longmapsto u_h \otimes e_{k,l}$$
if $k \neq i$ and $l \neq i$.
$$u_h \otimes e_{i,l} \longmapsto u_{h_0} u_h \otimes e_{i,l}$$
if $l \neq i$.
$$u_h \otimes e_{k,i} \longmapsto u_h u_{h_0}^{-1} \otimes e_{k,i}$$
if $k \neq i$.
$$u_h \otimes e_{i,i} \longmapsto u_{h_0} u_h u_{h_0}^{-1} \otimes e_{k,i}$$
(3)
$$u_h \otimes e_{k,l} \longmapsto u_g u_h u_g^{-1} \otimes e_{k,l}$$

We leave the reader the task of showing that these maps are indeed isomorphisms. □

We will call these isomorphisms basic moves of type (1), (2), or (3). We will call presentations P_A of the G-simple algebra A and P_B of the G-simple algebra B equivalent if one is obtained from the other by a (finite) sequence of basic moves. This is cleary an equivlence relation on presentations. It follows from the lemma that algebras with equivalent presentations are G-graded isomorphic.

Let A be G-simple with presentation P_A. Our proof requires, in terms of the given presentation P_A, a rather precise understanding of the structure of the subalgebra $A_N = \sum_{g \in N} A_g$ (of A) where N is an arbitrary subgroup of G. To this end we introduce an equivalence relation on the elements of the r-tuple $(p_1, ..., p_r)$: We will say $i, j \in \{1, ..., r\}$ are *N-related* in P_A if there exists $h \in H_A$ such that $p_i^{-1} h p_j \in N$. It is easy to see that this is indeed an equivalence relation. We may assume (after permuting the elements of the tuple $(p_1, ..., p_r)$ if needed) that the tuple is decomposed into subtuples whose elements are the corresponding equivalence classes. We denote the classes by $(p_{i_1}, p_{i_1+1}, ..., p_{i_1+k_1-1})$, $(p_{i_2}, p_{i_2+1}, ..., p_{i_2+k_2-1})$,..., $(p_{i_d}, p_{i_d+1}, ..., p_{i_d+k_d-1})$.

In order to get a better understanding of the N-elements in the presentation P_A, we focus our attention on one equivalence class, say $(p_{i_1}, p_{i_1+1}, ..., p_{i_1+k_1-1})$, and so, for convenience we change the notation by letting $k = k_1$ and setting $(g_1, ...g_k) = (p_{i_1}, p_{i_1+1}, ..., p_{i_1+k_1-1})$. We let B_{N_1} denote the F-space spanned by the elements $u_h \otimes e_{i,j}$ where $i, j \in \{1, ..., k\}$.

For $i = 1, ..., k$ we consider the following subgroup of N,

$$\Omega_{g_i} = g_i^{-1} H g_i \cap N$$

and let d_i be its order.

Proposition 1.4. *With the notation as above, the following hold.*

(1) *For $1 \leq i, j \leq k$ the subgroups Ω_{g_i} and Ω_{g_j} are conjugate to each other by an element of N. In particular $d_i = d_j$.*
(2) *For $i, j \in \{1, ..., k\}$ the set*
$$g_i^{-1} H g_j \cap N$$
is a left Ω_{g_i}-coset and a right Ω_{g_j}-coset . In particular the order of
$$g_i^{-1} H g_j \cap N$$
is $d_i (= d_j)$.
(3) *The subalgebra $B_{N,1}$ is G-simple with presentation*
$$P_{B_{N_1}} = (N \cap g_1^{-1} H g_1, g_1(\alpha), (n_1, ..., n_k))$$
for some elements $n_1, ..., n_k$, where $n_j \in N \cap g_1^{-1} H g_j$.

Proof. This is straightforward. We will prove only the first statement. By the equivalence condition, there are elements $h \in H$ and $n \in N$ such that $g_i^{-1} h g_j = n$. Hence

$$\begin{gathered}\Omega_{g_i} = g_i^{-1} H g_i \cap N = \\ n g_j^{-1} h^{-1} H h g_j n^{-1} \cap N = \\ n(g_j^{-1} H g_j \cap N) n^{-1} = \\ n(\Omega_{g_i}) n^{-1}\end{gathered}$$

as desired.

□

Remark 1.5. Based on the presentation of the N-simple algebra above, we see that the appearance of an N-simple component constitutes of a diagonal block of the $r \times r$-matrix algebra. We will refer to the number d_i as the number of *pages* in that component. So each N-simple component sits on the diagonal with a certain matrix size and a certain number of pages.

2. Proofs

Our aim is to show that algebras A and B (finite dimensional and G-simple) with nonequivalent presentations P_A and P_B have different T-ideals of G-graded identities and hence are G-graded nonisomorphic. This will imply

(1) G-graded (finite dimensional) G-simple algebras A and B are G-graded isomorphic if and only if any two presentations P_A and P_B are equivalent.
(2) G-graded (finite dimensional) G-simple algebras are characterized (up to G-graded isomorphism) by their T-ideal of G-graded identities.

Generally speaking, we proceed step by step where in each step we show that the presentations P_A and P_B must coincide on certain "invariants/parameters" up to applications of basic moves.

Let us start by exhibiting a list of such invariants of a presentation

$$P_A = (H_A, \alpha, (p_1, \dots, p_r))$$

of an algebra A.

(1) The multiplicities of right H-coset representatives in the r-tuple $(p_1, ..., p_r)$.
(2) The order of H.
(3) The group H up to conjugation.
(4) The group H.

Based on (4), for the rest of the invariants we will assume the subgroup H is determined. The next sequence of invariants are determined by the r-th tuple $T = (p_1, ..., p_r)$. We decompose the r-tuple $(p_1, ..., p_r)$ into subtuples where each subtuple (say of cardinality ν) constitute of all elements in $(p_1, ..., p_r)$ of the form $a_1g, ..., a_\nu g$ where g is a representative of a left coset of $N(H)$ in G and the a_i's are coset representatives of H in $N(H)$. Here, $N(H)$ denotes the normalizer of H in G.

Let us denote the full tuple by T and the subtuples by

$$T_1e, T_2g_2, ..., T_kg_k.$$

Each T_i is decomposed into representatives $t_{i,j}$ of H in $N(H)$ with multiplicity $d_{i,j}$.

(5) The vector of multiplicities of representatives in each T_i (in decreasing order).
(6) The elements of T_i up to left multiplication by a (unique) element $b_i \in N(H)$.
(7) The elements of T up to left multiplication by an element of $N(H)$. Note that by the basic moves this determines the presentation up to the two cocycle on H.

For the rest of the invariants we will assume the subgroup H and the tuple T are determined.

(8) The 2-cocycle on H is determined up to conjugation by an element of $N(H)$.

For each element $v_{i,j} \in T_i$ we consider the cocycle on H obtained by conjugation of α by $t_{i,j}$ (Note that conjugating with $t_{i,j}g_i$ gives a cocycle on H^{g_i}). Then each T_i determines a set of cocycles (on H).

(9) The set of cocycles (with multiplicities!) as determined by the elements of T_i.

We conclude by

(10) The presentation P_A of A.

Let A and B be G-graded algebras, finite dimensional G-simple with presentations

$$P_A = (H_A, \alpha, (p_1, \dots, p_r))$$

and

$$P_B = (H_B, \beta, (q_1, \dots, q_s)).$$

Suppose A and B satisfy the same G-graded identities. Our task will then be to add (in each step) an invariant from the list above on which the presentations P_A and P_B coincide (up to basic moves). The idea is to establish a suitable connection between the invariants described above and the structure of some extremal G-graded nonidentities of A. But more than that. The polynomials we construct will establish a strong connection between the above invariants and the structure of *any* nonzero evaluation of them (with a suitable basis).

Remark 2.1. Given a presentation P_A of an algebra A, it is well known that in order to test whether a G-graded multilinear polynomial is an identity of A it is sufficient to consider evaluations on any G-graded basis of A and so, for now on, we always choose the basis consisting of all elements of the form $u_h \otimes e_{i,j}$. This will play a key role in the proof since the *connection* we make via nonzero evaluations between the structure of A and structure of the polynomials will be based precisely on that particular G-graded basis of A. In particular, all subspaces we consider will be linear spans of subsets of that basis.

We want to be more precise about what we mean by "polynomials that establish a strong connection between their nonzero evaluations and the G-graded structure of A". Let $V = \oplus_g V_g \subseteq \oplus_g A_g$ be a G-graded subspace of A. Let $d_g = \dim_F(A_g)$ and $\delta_g = \dim_F(V_g)$, $g \in G$. We say that a multilinear G-graded polynomial p *determines* the G-graded subspace V of A if the following hold:

(1) $p = p(Z_G)$ is obtained from a single multilinear monomial Z_G by homogeneous multialternation. This means that we choose disjoint sets of homogeneous variables in Z_G(each set constitute of elements of the *same* homogeneous degree in G) and we alternate the elements of each set successively.
(2) For every $g \in G$ with $V_g \neq 0$, we have a subset T_g of g-variables in Z_G of cardinality d_g, and a subset S_g of T_g of cardinality δ_g such that the set T_g is alternating on $p(Z_G)$.
(3) $p(Z_G)$ is a G-graded nonidentity of A.
(4) If ϕ is *any* nonzero evaluation of $p(Z_G)$ on A (with elements of the form $u_h \otimes e_{i,j}$!), then all monomials but one vanish and for the unique monomial of $p(Z_G)$ which does not vanish, say Z_G, the elements of the set S_g assume precisely all basis elements of V_g.

Roughly speaking we construct alternating polynomials which are not only nonidentities of A, but also have the property that by means of *any* nonvanishing evaluation we are able to "allocate" the elements of V_g, $g \in G$. The upshot of this is that since A and B satisfy the same G-graded identities, we will be able to allocate elements of B in terms of the presentation P_B.

In what follows we will show how to construct such polynomials for certain G-graded subspaces V of A which correspond to the invariants mentioned above.

Before we do it let us generalize (twice) the allocation property just described. Assume the algebra A contains the direct sum of k subspaces $V_1, \ldots, V_k$ which are G-graded isomorphic. We say that a multilinear polynomial *p determines* the G-graded spaces $V_1, \ldots, V_k$ if the following hold:

(1) $p = p(Z_G)$ is obtained from a single multilinear monomial Z_G by homogeneous multialternation as above.

(2) For every $g \in G$ with $V_g \neq 0$, we have a subsets T_g of g-variables in Z_G of cardinality d_g, and subsets $S_{1,g}, \ldots, S_{k,g}$ of T_g, each of cardinality δ_g, such that the set T_g is alternating on $p(Z_G)$.
(3) $p(Z_G)$ is a G-graded nonidentity of A.
(4) If ϕ is any nonzero evaluation of $p(Z_G)$ on A, then all monomials but one vanish and for the unique monomial of $p(Z_G)$ which does not vanish, say Z_G, the elements of the sets $S_{1,g}, \ldots, S_{k,g}$ assume precisely all basis elements of $V_1, \ldots, V_k$ such that up to a permutation of the spaces V_i, the set S_i assumes the basis elements of V_i.

Let us generalize once more the allocation property. Assume that the algebra A contains the direct sum of r subspaces $V_1, \ldots, V_r$ and each one is the direct sum of G-graded isomorphic subspaces $V_{i,t}$, $t = 1, ..., i_d$. We say that a multilinear polynomial p *determines* the G-graded spaces $V_{i,j}$ if the following hold:

(1) p is obtained from a single multilinear monomial Z_G by homogeneous multialternation.
(2) For every $g \in G$ with $V_{i,j,g} \neq 0$ (some (i,j)), we have a subsets T_g of g-variables in Z_G, of cardinality d_g and subsets $S_{i,j,g}$ of T_g each of appropriate cardinality such that the set T_g is alternating on $p(Z_G)$.
(3) $p(Z_G)$ is a G-graded nonidentity of A.
(4) If ϕ is any nonzero evaluation of $p(Z_G)$ on A, then all monomials but one vanish and for the unique monomial of $p(Z_G)$ which does not vanish, say Z_G, the elements of the sets $S_{i,j,g}$ assume precisely all basis elements of $V_{i,j,g}$ up to a permutation on the second index.

In order to construct the polynomials (roughly speaking) we proceed as follows. We identify in the algebra A (say) the spaces $(V_{i,j,g})$ as well as the full g-component of A for any g which appears as a homogeneous degree in the $V_{i,j,g}$'s (no damage if we add a homogeneous degree g for which $V_g = 0$). We write a *nonzero* monomial with the basis elements $u_h \otimes e_{i,j}$ where we pay special attention to the spaces $V_{i,j,g}$'s. We border the elements of the sets which are about to alternate with idempotents in the e-component (of the form $1 \otimes e_{i,i}$). Next we consider the homogeneous degrees of the basis elements and we construct a (long!) multilinear monomial, denoted by Z_G, with homogeneous variables whose homogeneous degrees are as prescribed by the just constructed homogeneous monomial in A. Finally we alternate the homogeneous sets (of cardinality equal to the full dimension of the g-homogeneous component in A).

We start with the following case. Consider the e-component of A. By Proposition 1.4 and Remark 1.5, it is isomorphic to the direct sum of simple algebras which can be realized in blocks along the diagonal. By permuting the tuple elements we can order them in such a way the e-blocks are in decreasing order. Construct a monomial Z_G with segments which pass through each one of the e-blocks, bridged by an element (necessarily) outside the e-component. We border the elements of the e-blocks by idempotents. The prescribed sets V here are determined as follows. For the maximal size (say d_1) of e-blocks, we have r_1 blocks, for the second size (d_2) we have r_2 blocks and so on. So we have r_1 e-spaces of the largest dimension $(d_1)^2$, and so on. We produce the alternating polynomial as above.

Proposition 2.2. *The polynomial above allocates the e-blocks, where the e-blocks of the same dimension are determined up to permutation.*

Proof. First note that the polynomial p is a nonidentity of A. To see this let us show that the evaluation (which determined the monomial Z_G) is indeed a nonzero evaluation. Clearly the monomial Z_G does not vanish by construction. On the other hand in any nontrivial alternation, elements of the e-blocks will meet the wrong idempotent borderings and so we get 0.

Next let us show that for any nonzero evaluation of the polynomial (with basis elements $u_h \otimes e_{i,j}$) we have that all monomials but one vanish and for the one that does not vanish, the evaluation allocates the e-blocks as prescribed. By the alternation property we are forced to evaluate the full e-alternating set by a full basis of the e-component (for otherwise we get zero) so taking a basis of e-elements of the form $u_h \otimes e_{i,j}$ we are forced to use all of them and each exactly once.

Now, lets analyze the evaluation of any monomial whose value is nonzero. Elements of the e-component that substitute variables in the same segment must belong to the same block for otherwise we obtain zero. So elements of segments must be evaluated only by elements of the same e-block. Consider a segment of largest size. Since it must be evaluated by elements of one single block, it must exhaust the block (since we don't have blocks of larger size). We continue in this way until we exhaust all e-elements. This demonstrates the allocation property. □

Having constructed the polynomial p, we would like to see what can be deduced from the fact that p is also a nonidentity of the G-graded algebra B. Without loss of generality let us assume the the configuration of the multiplicities (i.e. the sizes of the e-blocks) for A is larger than for B (with the lexicographic order). It follows that in the largest e-segment we must put a full e-block and so we must have an e-block of the corresponding size. We continue in this way and we obtain that the multiplicities in B must be the same as in A.

Corollary 2.3. *The matrix size of the presentations P_A and P_B coincide. Consequently, the subgroups H_A and H_B have the same order.*

Proof. Clearly, the size of the matrix part in P_A (resp. P_B) is the sum of the sizes of the e-blocks of P_A (resp. P_B) and so they must be equal. For the second part we know that the dimensions if the homogeneous components of A and B are equal: Indeed, for the G-graded algebra A, there exists a nonidentity polynomial which is multilinear and alternates on sets of variables with homogeneous degree g of cardinality which is equal to the dimension of A_g (see [2]: The authors there considered the case where G is finite. The same proof holds in case the group G is arbitrary). Clearly, any polynomial with larger alternating sets of g-variables is necessary an identity of A. It follows that since A and B have the same identities, the dimensions of A_g and B_g must coincide. In particular A and B have the same dimension. We can now conclude that the subgroups H_A and H_B have the same order. □

The next step is to show that the subgroups H_A and H_B are conjugate in G. By shifting the tuple of the elementary grading (basic move (3)) we may assume that e is an element in the tuple $(p_1, ..., p_r)$ and that e is the representative with highest multiplicity.

Let $H = H_A$. In view of the Proposition 1.4 and Remark 1.5, the subalgebra A_H decomposes into a direct sum of H-simple subalgebras where each H-simple is a diagonal block with a certain number of pages. Note that at least one of the blocks (for example any block coming from the representative e) is "full" in the sense that it has the maximal possible number of pages (namely, $|H|$). In fact we get this maximal number precisely when the coset representative is in the normalizer $N(H)$ of H. We produce a product of basis elements $u_h \otimes e_{i,j}$ where one of the segments corresponds to an H-simple algebra with a full number of pages. We can produce such a product where we visit all basis elements of the H-simple algebra and such that all bridges between basis elements are H-elements. Specifically, we know we can form a nonzero product of all d^2 elementary matrices of a $d \times d$-matrix algebra (any d) which starts with $e_{1,1}$ and ends with $e_{i,1}$, some i. In particular the product is $e_{1,1}$. We produce a copy of such a product, one for each element h of H (that is we replace the d^2 elementary matrices $e_{i,j}$ by $u_h \otimes e_{i,j}$, $h \in H$). Putting these together we obtain a product of the form $\lambda u_{h_0} \otimes e_{1,1}$. Finally, multiplying with a suitable h_0-element we may assume the value of the product is $1 \otimes e_{1,1}$. This can be done for each block with a full number of pages. In addition we introduce borders between the basis elements which are idempotents of the form $1 \otimes e_{i,i}$.

Now for the H-simple components which appear in blocks where the H-coset representatives are outside $N(H)$, we know that the corresponding H-simple algebra will *not* have a full number of pages. It follows that if k is the position of a representative in that block, then there exists $h \in H$ such that the homogeneous degree of $u_h \otimes e_{k,k}$ in not in H. The point of this analysis is that we may use these elements as bridges of the basis elements in that H-block. Following these rules we construct a product of basis elements that passes through all H-elements of A.

Now we produce the monomial which corresponds to that product, namely a multilinear monomial Z_G where each variable (with the corresponding homogeneous degree) replaces basis elements. In terms of the discussion above, the designated segments S_h are those which correspond to blocks with full number of pages and the T_h are the variables which correspond to any block. The different bridges, either with homogeneous degrees in H or outside H, are variables in a set which we denote by Y_G. Next we alternate the homogeneous elements of T_h.

We claim that the polynomial $p(Z_G)$ is a G-graded nonidentity of A. Indeed, replacing the monomial Z_G with the original basis elements we obtain a nonzero product. Let us show now that for any nontrivial alternation, an h-element of T_h (some $h \in H$) will be bordered by elements which annihilate it. To see this note that two basis elements with the same (i,j) position cannot have the same homogeneous degree. This shows that elements with equal homogeneous degrees are bordered by basis elements of the form $u_{h_1} \otimes e_{i,i}, u_{h_2} \otimes e_{j,j}$ and $u_{h_3} \otimes e_{i',i'}, u_{h_4} \otimes e_{j',j'}$ where the pairs (i,j) and $(i^{'},j^{'})$ are different. It follows easily that any nontrivial alternation yields a zero value. This proves the claim.

Let us show now that any nonzero evaluation of the polynomial $p(Z_G)$ on A (with basis elements $u_h \otimes e_{i,j}$) has the property that only one monomial does not vanish and the designated variables from S_h get values from the blocks with the full number of pages. Furthermore, the evaluations are such that H-simple components (with full number of pages) substitute entire designated segments. Without loss of generality we may assume that the monomial Z_G is nonzero (upon the evaluation). Because of the alternation on elements of T_h we must use precisely a basis of the h-component. Next, we must replace designated segments by block elements with full number of pages for otherwise they must admit bridges outside H and this is not possible.

Now suppose H_B is not conjugate to $H = H_A$. In that case the nonidentity polynomial corresponding to H in B (produced as in the preceding paragraph) will have all its segments with bridges outside H. We claim that this polynomial is an identity for A. All elements of H are in segments bridged by homogeneous elements of degree not in H. But the elements of any block in A with a full number of pages cannot be bridged by elements outside H. Since it is the largest (we can make it the largest from both algebras) we get a contradiction. Note that we needed to know that the order of the groups were the same since otherwise, the group H_B could be a subgroup of H_A and in both cases we could get the full number of pages.

So we are now in the situation where the groups H are the same (applying a basic move if necessary) and the two r-tuples for the elementary grading have the same configuration of multiplicities. As before we take the same left H-coset representative in case the elements represent that same left H-coset.

We can now decompose the r-tuples into blocks which determine the same left coset of $N(H)$ in G. We want to show that the multiplicities inside each block coming from any $N(H)$-coset representative are the same in A and B. To this end we produce alternating sets for H, for $g_2^{-1}Hg_2$, and so on. We see that each designated segment must be evaluated within the same $N(H)$-block and so the multiplicities there must coincide. We obtain that the tuple of the elementary grading for B is obtained from the tuple of A multiplied by (possibly different) elements from $N(H)$.

Remark 2.4. Note that if H is e then all we have so far is that the size of the matrix algebra is the same (for A and B).

Our next step is to show that the tuples of the elementary grading in A and in B are obtained one from the other by multiplication on the left by an element of $N(H)$. This will lead to the situation where the groups H_A and H_B are the same and the tuples are the same. Then the only parameter we'll need to deal with is the 2-cocycle on the group H.

Lets consider a (very) special case of the statement above, namely where H is e. We have the tuple for A and based on it we can construct a monomial for the e-component of the algebra. Let us recall what does it mean. We consider the e-blocks arising from the multiple representatives. We produce e-segments for each block bridged by non-e-elements. We know that the monomial is a nonidentity of A and if we put frames we know that any nontrivial permutation of the designated e-elements gives a zero product of basis elements. It follows that if we construct a monomial out of the product above and alternate the designated variables we obtain a nonidentity of A with the properties described above. By assumption, we know that the polynomial is also a nonidentity of B.

We denote by $\sigma_1, ..., \sigma_n$ the distinct coset representatives in the tuple for A and by $\tau_1, ..., \tau_n$ the distinct coset representatives in the tuple for B. Note that here, since $H = \{e\}$, distinct coset representatives just means distinct elements. Also, we remind the reader that by previous steps, the vector of multiplicities of $\sigma_1, ..., \sigma_n$ and $\tau_1, ..., \tau_n$ is the same. Let $d_1, ..., d_n$ be the vector of multiplicities (in decreasing order). A nonzero evaluation on B gives rise to a permutation π on $\{\tau_1, ..., \tau_n\}$ so that the segment for σ_i is being evaluated by the elements in the $\tau_{\pi(i)}$ block.

In particular for every pair $i, j \in \{1, 2, \ldots, n\}$ the corresponding bridging elements must have the same weight and so we obtain the relations $\sigma_i^{-1}\sigma_j = \tau_{\pi(i)}^{-1}\tau_{\pi(j)}$.

Remark 2.5. Note that not every permutation is allowed. For instance, a permutation that exchanges elements with different multiplicities would lead to a contradiction. In other words we cannot substitute an e-block of size d_i with an e-block of size $d_j \neq d_i$. It is important to note (as mentioned above) that if a segment was determined by a block of size d_i, arising from an element σ_i (say) (with multiplicity d_i) then in any nonzero evaluation on B (or on A) the segment will assume values precisely of one block arising from τ_j where necessarily $d_j = d_i$. Nevertheless, for the proof, we only need to know the existence of a permutation π as above.

So we now have that for all $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, $\sigma_j = (\sigma_i\tau_{\pi(i)}^{-1})\tau_{\pi(j)}$. Moreover, rewriting these equations, we see that for all $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$, $\sigma_j\tau_{\pi(j)}^{-1} = \sigma_i\tau_{\pi(i)}^{-1}$ and so all the multipliers $\sigma_i\tau_{\pi(i)}^{-1}$ are the same. We see then that for all $i > 1$, $\sigma_i = (\sigma_1\tau_{\pi(1)}^{-1})\tau_{\pi(i)}$. We are missing only σ_1. But clearly $\sigma_1 = (\sigma_1\tau_{\pi(1)}^{-1})\tau_{\pi(1)}$, and we are done.

The same argument applies when H is normal in G.

We can now consider the general case where the group H is not necessarily normal in G. Then we decompose the tuple into subtuples coming from different $N(H)$-representatives in G. We will refer to these subtuples as "big blocks". We know that the multiplicities in each subtuple coincide. We construct a monomial which corresponds to that configuration: We start with a monomial for H (full number of pages) coming from the representatives in $N(H)$. Then for H^{g_2} (again, for that group with full number of pages!) and so on. At the end of the monomial we append monomials which will take care of all H elements that appear *outside* the block $N(H)$, all elements of H^{g_2} that appear in blocks *outside* the $N(H)$-coset representative g_2 and so on. We know that the appearance of these elements is in blocks with fewer pages and so we bridge them (as we can) by elements outside H, H^{g_2}, etc. This forces any nonzero evaluation on B to determine a permutation π which is not arbitrary but respects the $N(H)$-representatives in G and also the multiplicities. We can then obtain equations similar to those given above in the case where $H = \{e\}$: we denote by $\sigma_i g_s$ the representatives of the cosets of H in G which are contained in $N(H)g_s$ (here $\sigma_i \in N(H)$) then we must have that the permutation π permutes representatives within the same $N(H)$-coset in G (and also with the same multiplicity). It follows that once again the bridges must have the same weights and so

$$g_s^{-1}\sigma_i^{-1}\sigma_{i+1}g_t = g_s^{-1}\tau_{\pi(i)}^{-1}\tau_{\pi(i+1)}g_t$$

This leads to equations as in the normal case and the result follows.

So by applying basic moves, we may now assume that the fine gradings of A and B are determined by the same group H and the elementary grading is determined by the same r-tuple $(p_1, ..., p_r) \in G^{(r)}$. We proceed now to show that the cocycles α and β may be assumed to be the same. Let us be more precise. Assume we can find an element b of $N(H)$ such that left multiplication of the r-tuple $(p_1, ..., p_r)$ permutes the representatives in each big block in such a way that it preserves multiplicities. That means that a representative σ_i will be moved to $\sigma_j = b\sigma_i$ (up to left multiplication with elements of H) where σ_i and σ_j belong to the same

$N(H)$-coset and have the same multiplicity. So assume we have such an element b which also conjugates α to β. Then by our basic moves, the presentations P_A and P_B are equivalent. So our task is to show that if A and B satisfy the same identities then there is such element b in $N(H)$.

We start with the case where the grading is fine, that is, A and B are twisted group algebras. Before stating the lemma recall (Aljadeff, Haile and Natapov[1]) that the T-ideal of H-graded identities of a twisted group algebra $F^{\alpha}H$ is spanned over F by binomial identities of the form

$$B(\alpha) = x_{h_1}x_{h_2}\cdots x_{h_s} - \lambda_{((h_1,\ldots,h_s),\pi)}x_{h_{\pi(1)}}x_{h_{\pi(2)}}\cdots x_{h_{\pi(s)}},$$

where $h_i \in H$, $i = 1, ..., s$, $\pi \in Sym(s)$ and λ is a suitable nonzero element (root of unity) $\in F$.

Proposition 2.6. *Given twisted group algebras $F^{\alpha}H$ and $F^{\beta}H$, then the cocycles are cohomologous if and only if the algebras satisfy the same identities.*

Proof. The idea of the proof appeared already in [1]. There, for the particular case where the group H is of central type and the twisted group algebra $F^{\alpha}H$ is the algebra of $k \times k$-matrices over F where $ord(H) = k^2$. However, the same construction holds in general. For the reader convenience, let us recall here the construction.

It is well known, by the Universal Coefficient Theorem, that the cohomology group $H^2(H, F^*)$ is naturally isomorphic to $Hom(M(H), F^*)$ where $M(H)$ denotes the Schur multiplier of H. It is also well known that $M(H)$ can be described by means of presentations of H, namely the Hopf formula. Indeed, let $\Gamma = \Gamma\langle x_{h_1}, \ldots, x_{h_s}\rangle$ be the free group on the variables x_{h_i}'s where $H = \{h_1, \ldots, h_s\}$. Consider the presentation

$$\{1\} \to R \to \Gamma \to H \to \{1\}$$

where the epimorphism is given by $x_{h_i} \longrightarrow h_i$.

One knows that the Schur multiplier $M(H)$ is isomorphic to

$$R \cap [\Gamma, \Gamma]/[R, \Gamma].$$

Given a 2-cocycle α on H (representing $[\alpha] \in H^2(H, F^*)$) it determines an element of $Hom(M(H), F^*)$ as follows: Let $[z]$ be an element in $M(H)$ where z is a representing word in $R \cap [\Gamma, \Gamma]$. For each variable x_h consider the element u_h in the twisted group algebra $F^{\alpha}H$ representing h. Then the value of α on z is the root of unity which is the product in $F^{\alpha}H$ of the elements u_h (which correspond to the variables x_h of z). One knows that the value $[\alpha]([z])$ depends on the classes $[\alpha] \in H^2(H, F^*)$ and $[z] \in M(H)$ and not on their representatives. Note that by the isomorphism of $H^2(H, F^*))$ with $Hom(M(H), F^*)$ we have that for two noncohomologous 2-cocycles α and β there is $z \in R \cap [\Gamma, \Gamma]$ with $\alpha(z) \neq \beta(z)$. Let us show now how H-graded polynomial identities come into play.

Let

$$z = x_{h_{i_1}}^{\epsilon_1} x_{h_{i_2}}^{\epsilon_2} \cdots x_{h_{i_r}}^{\epsilon_r}$$

where $\epsilon_i = \{\pm 1\}$. Being z in R implies that $h_{i_1}^{\epsilon_1} h_{i_2}^{\epsilon_2} \cdots h_{i_r}^{\epsilon_r} = e$ whereas being in $[\Gamma, \Gamma]$ says that the sum of the exponents ϵ_i which decorate any variable x_h which appears in z, is zero.

Our task is to "produce" out of z and the value $\alpha(z) \in F^*$ an H-graded binomial identity of the twisted group algebra $F^\alpha H$. Pick any variable x_h in z and let n be the order of h (in H). Clearly the commutator $[x_h^n, y]$, $y \in \Gamma$, is in $[R, \Gamma]$ and so multiplying z (say on the left) with elements x_h^n and x_h^{-n}, and moving them (to the right) successively along z by means of the relation $[x_h^n, y]$, we obtain a representative of $[z]$ in $M(H)$ of the form

$$z_1 z_2^{-1}$$

where the variables in z_1 and z_2 appear only with positive exponents.

The binomial identity which corresponds to z and $\alpha(z)$ is given by

$$Z_1 - \alpha(z) Z_2$$

where Z_i is the monomial in the free H-graded algebra whose variables are in one to one correspondence with the variables of z_i. We leave the reader the task to show that indeed $Z_1 - \alpha(z)Z_2$ is a G-graded identity. Clearly, from the construction it follows that twisted group algebras $F^\alpha H$ and $F^\beta H$ satisfy the same G-graded identities if and only if the cocycles α and β are cohomologous. This completes the proof of the proposition. □

Remark 2.7.
1. The binomial identity obtained above, say for α, may not be multilinear. In order to obtain a mulilinear binomial identity assume x_h appears k-times in each monomial Z_i, $i = 1, 2$. Then replacing the variables by k different variables $x_{1,h}, \ldots, x_{k,h}$ in each monomial (any order!) we obtain an H-graded (binomial) identity which is on variables whose homogeneous degree is h. Repeating this process for every $h \in H$ gives a multilinear (binomial) identity.
2. It follows that any two noncohomologous cocycles can be separated by suitable binomial identities in the sense that for any ordered pair, (α, β) (where $(\alpha \neq \beta)$) there is a binomial $B(\widehat{\alpha}, \beta)$ which is an identity of β (abuse of language) and not an identity of α.
3. Assume $\beta_1, \ldots, \beta_k$ are cocycles on H which are different from α (noncohomologous to α). Then by the previous Proposition there is a binomial identity $B(\widehat{\alpha}, \beta_i)$ of β_i which is a nonidentity of α. Then if we take the product of these binomials (with different variables), we see that the product is a multilinear identity of any of the β_i's and not an identity of α. To see the last statement note that the value of any evaluation of $B(\widehat{\alpha}, \beta_i)$ on $F^\alpha H$ is an invertible element (homogeneous) and hence the product is nonzero.

We now come to an important lemma which is due to Yaakov Karasik, in which we extend the preceding lemma to algebras which have a presentation P_A where the tuple that determines the elementary grading is trivial, that is,, $(\sigma_1, ..., \sigma_k) = (e, ..., e)$.

Lemma 2.8. *Let A and B be finite dimensional G-simple algebras with presentations P_A and P_B respectively. Suppose P_A and P_B are given by $F^\alpha H \otimes M_r(F)$ and $F^\beta H \otimes M_r(F)$ respectively, both with trivial elementary grading on $M_r(F)$. If α and β are noncohomologous, then there is an identity of A (resp. B) which is a nonidentity of B (resp. A).*

Remark 2.9. In case the group G is abelian, this was proved by Koshlukov and Zaicev [8] using certain modifications of the standard polynomial. However this approach (at least in its straightforward generalization) seems to fail for nonabelian groups.

Proof. As above let $B(\widehat{\alpha},\beta)$ denote a binomial identity of $F^{\beta}H$ which is a nonidentity of $F^{\alpha}H$. Then $B(\widehat{\alpha},\beta)$ has the form

$$B(\widehat{\alpha},\beta)=z_{h_1}z_{h_2}\cdots z_{h_s}-\lambda_{(\widehat{\alpha},\beta,(h_1,\ldots,h_s),\pi)}z_{h_{\pi(1)}}z_{h_{\pi(2)}}\cdots z_{h_{\pi(s)}}.$$

Next, consider the Regev polynomial $p(X,Y)$ on $2r^2$ variables (each of the sets X and Y consists of r^2 variables). It is multilinear (of degree $2r^2$) and central on $M_r(F)$. Any evaluation of X or Y on a proper subset of the r^2 elementary matrices $e_{i,j}$ yields zero whereas in case X and Y assume the full set of elementary matrices the value is central, nonzero (and hence invertible) matrix.

Now, for each variable z_h of $B(\widehat{\alpha},\beta)$ we construct a Regev polynomial on $2r^2$ variables where we pick one variable from X (no matter which) and we determine its homogeneous degree to be h. The rest of the x's and all the y's in Y are determined as variables of homogeneous degree e. We denote the corresponding Regev polynomial by $p_h(X_{r^2},Y_{r^2})$. Now, we consider a basis of the algebra $F^{\alpha}H\otimes M_k(F)$ consisting of elements of the form $u_h\otimes e_{i,j}$. Note that there are precisely r^2 basis elements of degree e and r^2 basis elements of degree h. We see that if we evaluate the polynomial $p_h(X_{r^2},Y_{r^2})$ with elements in $\{1\otimes e_{i,j},u_h\otimes e_{i,j}\}_{i,j}$ we'll get zero as long as the elementary matrix constituent of the basis elements is *not* the full set of r^2 matrices (either for X or for Y) and $u_h\otimes\lambda\,\mathrm{Id}$ otherwise. It follows that if we replace every variable z_h in $B(\widehat{\alpha},\beta)$ by the Regev polynomial $p_h(X_{r^2},Y_{r^2})$ we obtain a polynomial

$$R(\widehat{\alpha},\beta,r)$$

which is an identity of B and a nonidentity of A. □

Before we continue, recall that a big block of $M_r(F)$ is any block which is determined by elements of the tuple $\{p_1,...,p_r\}$ which belong to the same $N(H)$-coset of G. A subblock of a big block is called "basic" if it is determined by elements of the tuple $\{p_1,...,p_r\}$ which belong to the same $N(H)$-coset of G <u>and</u> have the same multiplicity.

Consider the cocycles that appear along the diagonal blocks of the algebra A. For each big block, we consider the representatives of H in $N(H)$ (multiplied by a coset representative of $N(H)$ in G). Each one of these representatives, say $\sigma g\in N(H)g$, conjugates the cocycle α into a cocycle $\alpha^{\sigma g}$ of the group H^g. Now, for the coset representatives of each basic block, (i.e. in the same big block and the same multiplicity) we consider the different representatives of H in $N(H)$ and the corresponding cocycles obtained by conjugation of α by these representatives. We claim that the set of cocycles obtained, in A and in B are the same, where we take multiplicities into account. To see this we consider representatives of one basic block $\sigma_1g,...,\sigma_rg$, where g is a representative of $N(H)$ in G and $\sigma_1,...,\sigma_r$ are representatives of H in $N(H)$. Assume the multiplicity is d. For simplicity, let us assume $g=e$. We consider the conjugation of α by the σ's and the conjugation of β by the σ's. We want to show these two sets are the same. For any pair $\alpha^{\sigma_i},\beta^{\sigma_j}$ we consider the corresponding polynomial $R(\widehat{\alpha^{\sigma_i}},\beta^{\sigma_j},d)$. Now we produce as above a long monomial arising from the algebra A, with segments that correspond to H-blocks, then to H^g-blocks for the second g and so on. These are the special segments, and in the "back" part of the monomial we complete the homogeneous sets (again for H, H^g,...) with bridges outside the corresponding group. Now, in the special segments we insert the polynomials of the form $R(\widehat{\alpha^{\sigma_i}}, \beta^{\sigma_j}, d)$.

Take a cocycle τ which appears in a basic block of A. Assume the cocycle appears with multiplicity t. On the other hand assume it appears in B with multiplicity $t^{'}$ where $t^{'} > t$. Let us consider the cocycles γ_i which appear in the same basic block and are different from τ. Their number is $r - t$ in A and $r - t^{'}$ in B. In the monomial built for the algebra A we insert in the last $r - t$ segments a polynomial

$$R(\widehat{\gamma_i}, \tau, d)$$

(in the first t segments we don't insert any polynomial). Since our nonzero evaluation is such that the blocks with cocycle τ may "go" to the first segments, we obtain a nonidentity of A. It is therefore a nonidentity of B. But by the pigeonhole principle, one of the blocks with cocycle τ will meet a polynomial of the form $R(\widehat{\gamma_i}, \tau, d)$ and so we get zero. This shows the multiplicities must coincide. We could argue more generally. In the algebra B we can insert products of $R(\widehat{\gamma_i}, \tau, d)$ which are identities for all cocycles in the Schur multiplier of H but not for the cocycle which appear in the block. It is a nonidentity for A and therefore a nonidentity for B. Assume a certain cocycle in B has larger multiplicity than the multiplicity it appears in A. Then we get zero. What happens if there is a cocycle in B which does not appear in A? We can fix it by inserting products of polynomials for all cocycles but τ. Note that there is a finite number since the Schur multiplier is a finite group.

The argument above leads to the situation where the multiplicities of the cocycles appearing in each basic block for the algebras A and B coincide. In particular we know that the cocycles α and β are conjugate by an element of $N(H)$.

Our final step is to show that the presentations P_A and P_B are equivalent. The polynomial we construct here will be based on the same polynomial we used above. We have the algebras A and B with presentations P_A and P_B where $H_A = H_B$, same tuple (which determines the elementary grading) and the cocycles in each basic block are the same with multiplicities. We produce a polynomial for A with special segments as above. Recall there is a correspondence between segments of the polynomial and basic blocks of A. Each basic block is determined by representatives $\sigma_1 g, ..., \sigma_1 g$ where $\sigma_i \in N(H)$. It follows that the cocycles on that basic block is $\alpha^{\sigma_i g}$. Let us assume again for simplicity that $g = e$. Consider polynomials $R(\widehat{\alpha^{\sigma}}, \gamma, d)$ where γ is any cocycle of the Schur multiplier $M(H)$ *different* from $\alpha^{\sigma g}$ and d is the size of the block determined by σ. Let R_σ be the product of these polynomials. We insert in each segment such a polynomial. We note that this is a nonidentity for the algebra A and hence a nonidentity for the algebra B.

A nonzero evaluation on B determines a permutation of the H-coset representatives inside a basic block. But more than that, a nonzero evaluation on B determines a permutation among the H-representatives of a basic block which determine the same cocycle. We want to show that this determines an element in $N(H)$ such that multiplication on the left (of the tuple) leaves the basic blocks invariant and also, it conjugates the cocycle α to β. Indeed, denote the permutation on the H-coset representative by π. We obtain equations

$$\sigma_i^{-1}\sigma_{i+1} = \sigma_{\pi(i)}^{-1}\sigma_{\pi(i+1)}.$$

We know that the shifting element is

$$\sigma_i\sigma_{\pi(i)}^{-1}$$

and so we would be done if we show that

$$\alpha^{\sigma_i\sigma_{\pi(i)}^{-1}} = \beta.$$

But, by the evaluation on B we know that

$$\alpha^{\sigma} = \beta^{\sigma_{\pi(i)}}$$

and we are done.

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