A pure Dirac's canonical analysis for four-dimensional BF theories

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We perform Dirac's canonical analysis for a four-dimensional BF and for a generalized four-dimensional BF theory depending on a connection valued in the Lie algebra of $SO(3,1)$. This analysis is developed by considering the corresponding complete set of variables that define these theories as dynamical, and we find out the relevant symmetries, the constraints, the extended Hamiltonian, the extended action, gauge transformations and the counting of physical degrees of freedom. The results obtained are compared with other approaches found in the literature.

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I. INTRODUCTION

Presently, the study of topological field theories is a topic of great interest in physics. The importance to study these theories arises because they have a close relationship with general relativity. These theories are characterized by the absence of local physical degrees of freedom and by the background independence [1]. Relevant examples with close symmetries with general relativity are the so called BF theories, which are background independent and diffeomorphisms covariant, and were introduced as generalizations of three dimensional Chern-Simons action or as a zero coupling limit of Yang-Mills theories [2, 3]. We can find several examples where the BF theories come to be physically relevant in alternative formulations of gravity, such as Plebański or Macdowell-Mansouri formulations; the former consists in to obtain General Relativity by imposing extra constraints on a BF theory with the gauge group $SO(3,1)$ or $SO(4)$ [4]. The later consists in breaking down the symmetry of a BF theory from $SO(5)$ to $SO(4)$, to obtain the Palatini action plus the sum of the second Chern and Euler topological invariants [5], and since these topological classes have trivial local variations that do not contribute classically to the dynamics, one obtains

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essentially general relativity [6].

Other interesting case, where BF theories have a close relation with physical theories is found in Martellini's model [7]. This model consists in expressing Yang-Mills theory as a BF-like theory, and the BF first-order formulation is equivalent (on shell) to the usual (second-order) formulation. In fact, both formulations of the theory possess the same perturbative quantum properties; specifically the Feynman rules, the structure of one loop divergent diagrams and renormalization have been studied, founding an equivalence of the uv-behavior for both approaches [7]. Furthermore, other kind of topological BF theories are reported in [8], where by using a generalized differential calculus a geometrical relation between a generalized Chern-Simons functional and a generalized BF theory is obtained; this version corresponds to a pure BF term, plus the second Chern class and a cosmological-like term quadratic in the field B. The case of gravity viewed as a generalized topological field theory and its close relation with the generalized BF theory is also discussed [8].

In this manner, with these motivations we will perform the Hamiltonian analysis for a four-dimensional BF theory and for the generalized BF theory introduced in [8]. It is important to remark that the standard way to develop the Hamiltonian study for a BF theory is considering as dynamical variables only those ones that occur in the Lagrangian density with temporal derivative [9, 10] (called smaller phase space context). However this approach is only convenient to perform provided that the theory under study presents certain simplicity; but the price to pay for developing the standard approach is that we can not to know the full structure of the constraints and their algebra, the equations of motion and the gauge transformations. Nevertheless, the approach developed in this work will be quite different to the standard one; this means that in agreement with the background independence structure that presents the theory under study, we will develop the Hamiltonian framework by considering all the fields occurring in the theories as dynamical ones; this fact will allow us to find the complete structure of the constraints, the equations of motion, gauge transformations, the extended action as well as the extended Hamiltonian. We able to realize that developing the Hamiltonian approach on a smaller phase space context, the structure obtained for the constraints is not right. In fact, we observe in [12] that the Hamiltonian constraint for Palatini theory does not has the required structure to form a closed algebra with all constraints; this problem emerges because of by working on a smaller phase space context we lose control on the constraints, and to obtain the correct structure sometimes they need to be fixed by hand as it was done in [10] for Plebański theory. Nevertheless there are analysis on a smaller phase space performed without complications, as for instance in Maxwell and Yang-Mills theories [11]. For these reasons in this paper we develop a pure Dirac method applied to models with a close relationship to Palatini and Plebański theory just as the four-dimensional BF theories, and we will see that is not necessary to fix by hand the constraints because the method itself provides us the required structure. Thus in this paper we establish the bases for forthcoming works where will be applied the same approach to Plebański theory. We will discuss all these details along the paper and we have added an appendix to clarify these ideas.

II. A PURE DIRAC'S ANALYSIS FOR A FOUR-DIMENSIONAL BF THEORY

In this section, we will develop an extension of the results reported in [9]. In the following lines, we shall study the Hamiltonian dynamics for a four-dimensional BF theory by using a pure Dirac's method. With the terminology "a pure Dirac's method" we mean that in concordance with the background independence of the theory, we will consider in the Hamiltonian framework that all the fields that define our theory are dynamical ones.

So, let us start with a four-dimensional BF theory which is described by the following action [9, 10]

$$S\left[A,\mathbf{B}\right]=\int_{M}\mathbf{B}^{IJ}\wedge\mathbf{F}_{IJ}(A), \tag{1}$$

where $F^{IJ}=dA_{IJ}+A_{I}{}^{K}\wedge A_{KJ}$ is the curvature of the Lorentz connection 1-form $A^{IJ}=A_{\alpha}{}^{IJ}dx^{\alpha}$, and $B^{IJ}=\frac{1}{2}B^{IJ}{}_{\alpha\beta}dx^{\alpha}\wedge dx^{\beta}$ is a set of six $SO(3,1)$ valued 2-forms. Here, $\mu,\nu=0,1,..,3$ are spacetime indices, x^{μ} are the coordinates that label the points for the four-dimensional Minkowski manifold M and $I,J=0,1..,3$ are internal indices that can be raised and lowered by the internal Lorentzian metric $\eta_{IJ}=(-1,1,1,1)$.

The equations of motion that arises from the variation of the action are given by

$$\begin{aligned} F &= 0, \\ DB &= 0, \end{aligned} \tag{2}$$

in this sense, both B and A are considered as dynamical variables and we will take account this fact for all our developments along the paper.

To perform the Hamiltonian framework, we will suppose that the manifold M has topology $\Sigma\times R$, where Σ corresponds to Cauchy's surfaces and R represents an evolution parameter. By performing the 3+1 decomposition, we can write the action as

$$S\left[A,\mathbf{B}\right]=\frac{1}{2}\int_{R}\int_{\Sigma}dtd^{3}x\left\{B^{IJ}{}_{0a}F_{IJbc}\epsilon^{0abc}+\epsilon^{0abc}B_{IJab}\left(\dot{A}^{IJ}{}_{c}-\partial_{c}A^{IJ}{}_{0}-A^{IL}{}_{0}A_{L}{}^{J}{}_{c}+A^{IL}{}_{c}A_{L}{}^{J}{}_{0}\right)\right\}, \tag{3}$$

where can be identified the following Lagrangian density

$$\mathcal{L}=\frac{1}{2}B^{IJ}{}_{0a}F_{IJbc}\epsilon^{0abc}+\frac{1}{2}\epsilon^{0abc}B_{IJab}\left(\dot{A}^{IJ}{}_{c}-\partial_{c}A^{IJ}{}_{0}-A^{IL}{}_{0}A_{L}{}^{J}{}_{c}+A^{IL}{}_{c}A_{L}{}^{J}{}_{0}\right). \tag{4}$$

As was commented earlier, the cornerstone of this work is to carry out the Hamiltonian analysis by considering as dynamical all the set of $A_{\alpha}{}^{IJ}$ $\rightarrow$24 and $B^{IJ}{}_{\alpha\beta}$ $\rightarrow$36 variables that define the action. We observe at this point, that our procedure is quite different to [9, 10] because in that work the Hamiltonian analysis was performed considering as dynamical variables only those with time derivative occurring explicitly in the Lagrangian (in this particular scenario only the $A^{IJ}{}_{c}$ $\rightarrow$18 are considered dynamical). It is important to observe also that strictly speaking, $A^{IJ}{}_{\alpha}$ and $B^{IJ}{}_{\alpha\beta}$ are our set of dynamical variables and the correct form to carry out the Hamiltonian analysis is taking to account that set. However, because of the action is not quadratic in the field B^{IJ}_{0i} or A^I_0, the Hamiltonian study has been performed on a smaller phase space context neglecting the variables A^{IJ}_0 and $B^{IJ}{}_{0i}$ as dynamical and being identified as Lagrange multiplier [9, 10]. Nevertheless, it is not ever easy perform the Hamiltonian analysis on a smaller phase space; example of this fact is present in Plebański's formulation [10], where Dirac's analysis needs a treatment with more details, introducing new variables and fixing the structure of the first class constraints by hand. In this manner, we attempt with the present paper to extend the standard approach developed for BF theories, by performing a full canonical analysis that will be useful in order to carry out the analysis for Plebanski's formulation or for models found in BF gravity [21] without fixing by hand the constraints as reported in [10].

Hence, by identifying our set of dynamical variables, a pure Dirac's method calls for the definition of the momenta $\left(\Pi^{\alpha}{}_{IJ}, \Pi^{\alpha\beta}{}_{IJ}\right)$ canonically conjugate to $\left(A^{IJ}{}_{\alpha}, B_{\alpha\beta}{}^{IJ}\right)$,

$$\Pi^{\alpha}{}_{IJ} = \frac{\delta L}{\delta \dot{A}^{IJ}{}_{\alpha}}, \quad \Pi^{\alpha\beta}{}_{IJ} = \frac{\delta L}{\delta \dot{B}_{\alpha\beta}{}^{IJ}}. \tag{5}$$

The matrix elements of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu(A_\alpha{}^{IJ}))\partial(\partial_\mu(A_\beta{}^{IJ}))}, \qquad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu(A_\alpha{}^{IJ}))\partial(\partial_\mu(B_{\rho\nu}{}^{IJ}))}, \qquad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu(B_{\rho\nu}{}^{IJ}))\partial(\partial_\mu(B_{\gamma\sigma}{}^{IJ}))}, \tag{6}$$

are identically zero, the rank of the Hessian is zero, thus, we expect 60 primary constraints. From the definition of the momenta (5), we identify the following 60 primary constraints

$$\begin{aligned}
\phi^{0}{}_{IJ} &: \Pi^{0}{}_{IJ} \approx 0, \\
\phi^{a}{}_{IJ} &: \Pi^{a}{}_{IJ} - \frac{1}{2}\eta^{abc} B_{bcIJ} \approx 0, \\
\phi^{0a}{}_{IJ} &: \Pi^{0a}{}_{IJ} \approx 0, \\
\phi^{ab}{}_{IJ} &: \Pi^{ab}{}_{IJ} \approx 0,
\end{aligned} \tag{7}$$

where we have defined $\epsilon^{0abc} \equiv \eta^{abc}$.

By neglecting terms on the frontier, the canonical Hamiltonian for BF theory is given by

$$H_c = -\int dx^3 \left[A^{IJ}{}_0 D_a \Pi^a{}_{IJ} + \frac{1}{2}\eta^{abc} B_{0a}{}^{IJ} F_{IJbc} \right]. \tag{8}$$

In this manner, adding the primary constraints (7) to the canonical Hamiltonian, the primary Hamiltonian is given by

$$H_P = H_c + \int dx^3 \left[\lambda^{IJ}{}_0 \phi_{IJ}{}^0 + \lambda^{IJ}{}_a \phi_{IJ}{}^a + \lambda_{0a}{}^{IJ} \phi^{0a}{}_{IJ} + \lambda_{ab}{}^{IJ} \phi^{ab}{}_{IJ} \right], \tag{9}$$

where $\lambda^{IJ}{}_0$, $\lambda^{IJ}{}_a, \lambda_{0a}{}^{IJ}$ and $\lambda_{ab}{}^{IJ}$ are Lagrange multipliers enforcing the constraints.

The non-vanishing fundamental Poisson brackets for the theory under study are given by

$$\begin{aligned}
\{A^{IJ}{}_{\mu}(x^0,x), \Pi^{\nu}{}_{KL}(x^0,y)\} &= \delta^{\nu}{}_{\mu} \frac{1}{2}\left(\delta^I{}_K \delta^J{}_L - \delta^I{}_L \delta^J{}_K\right) \delta^3(x-y), \\
\{B_{\alpha\beta}{}^{IJ}(x^0,x), \Pi^{\mu\nu}{}_{KL}(x^0,y)\} &= \frac{1}{4}\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right)\left(\delta^I{}_K \delta^J{}_L - \delta^I{}_L \delta^J{}_K\right) \delta^3(x-y).
\end{aligned} \tag{10}$$

Now, we need to identify whether the theory presents secondary constraints. For this aim, we compute the 60×60 matrix whose entries are the Poisson brackets among the primary constraints (7), the nonzero brackets are given by

$$\{\phi^a{}_{IJ}(x), \phi^{bc}{}_{KL}(y)\} \;=\; -\frac{1}{4}\eta^{abc}(\eta_{IK}\eta_{JL} - \eta_{IL}\eta_{JK})\delta^3(x,y), \tag{11}$$

this matrix has rank= 36, and 24 linearly independent null-vectors. This result suggests that consistency conditions imply 24 secondary constraints. From the temporal evolution of the constraints (7) and the contraction with the 24 null vectors, it follows that the following 24 secondary constraints arise

$$\begin{aligned}
\dot{\phi}^0{}_{IJ} &= \{\phi^0{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi_{IJ} := D_a\Pi^a{}_{IJ} \approx 0. \\
\dot{\phi}^{0a}{}_{IJ} &= \{\phi^{0a}{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi^{0a}{}_{IJ} := \frac{1}{2}\eta^{abc}F_{bcIJ} \approx 0,
\end{aligned} \tag{12}$$

and the rank allows us fix the next values for the Lagrange multipliers

$$\begin{aligned}
\dot{\phi}^a{}_{IJ} = \{\phi^a{}_{IJ}(x), H_P\} \approx 0 \quad &\Rightarrow \quad [\Pi^a{}_{JL}\eta_{KI} - \Pi^a{}_{IL}\eta_{KJ}]\, A_0{}^{KL} + \eta^{abc}D_bB_{0cIJ} \\
&- \;\; \frac{1}{2}\eta^{abc}\lambda_{bcIJ} \approx 0, \\
\dot{\phi}^{ab}{}_{IJ} = \{\phi^{ab}{}_{IJ}(x), H_P\} \approx 0 \quad &\Rightarrow \quad \eta^{abc}\lambda_{cIJ} \approx 0.
\end{aligned} \tag{13}$$

For this theory there are not third constraints. By following with the method, we need to separate from the primary and secondary constraints which ones correspond to first and second class. In order to achive this aim, we need to calculate the Poisson brackets among primary and secondary constraints, which the nonzero brackets are given by

$$\begin{aligned}
\{\phi^a{}_{IJ}(x), \phi^{bc}{}_{KL}(y)\} &= -\frac{1}{4}\eta^{abc}(\eta_{IK}\eta_{JL} - \eta_{IL}\eta_{JK})\delta^3(x,y), \\
\{\phi^a{}_{IJ}(x), \Psi_{KL}(y)\} &= \frac{1}{2}\left(\Pi^a{}_{JL}\eta_{IK} - \Pi^a{}_{IL}\eta_{KJ} + \Pi^a{}_{KJ}\eta_{IL} - \Pi^a{}_{KI}\eta_{LJ}\right)\delta^3(x-y), \\
\{\phi^a{}_{IJ}(x), \Psi^b{}_{KL}(y)\} &= \frac{1}{2}\eta^{abc}\Big[(\eta_{IK}\eta_{JL} - \eta_{KJ}\eta_{IL})\partial_c + (A_{JLc}\eta_{IK} - A_{ILc}\eta_{KJ}) \\
&- \;(A_{KI}\eta_{LJ} - A_{KJc}\eta_{LI})\Big]\delta^3(x-y), \\
\{\Psi_{IJ}(x), \Psi_{KL}(y)\} &= -\frac{1}{2}(\Psi_{LJ}\eta_{IK} - \Psi_{KJ}\eta_{IL} + \Psi_{IL}\eta_{KJ} - \Psi_{IK}\eta_{LJ})\delta^3(x-y) \approx 0, \\
\{\Psi_{IJ}(x), \Psi^a{}_{KL}(y)\} &= \frac{1}{2}\left(\Psi^a{}_{LJ}\eta_{IK} - \Psi^a{}_{KJ}\eta_{IL} + \Psi^a{}_{IL}\eta_{KJ} - \Psi^a{}_{IK}\eta_{LJ}\right)\delta^3(x-y). \approx 0,
\end{aligned} \tag{14}$$

Thus, we can observe that this matrix has a rank= 36 and 48 null-vectors. In this manner, we find that our theory presents a set of 48 first class constraints and 36 second class constraints. By using the contraction of the null vectors with the constraints (7) and (12), we identify the following 48 first class constraints

$$\begin{aligned}
\gamma^0{}_{IJ} &: \Pi^0{}_{IJ}, \\
\gamma^{0a}_{IJ} &: \Pi^{0a}{}_{IJ}, \\
\gamma_{IJ} &: D_a\Pi^a{}_{IJ} - \eta_{abc}\left[\Pi^a{}_{IP}\Pi^{bc}{}_{QJ}\eta^{PQ} - \Pi^a{}_{JP}\Pi^{bc}{}_{QI}\eta^{PQ}\right], \\
\gamma^a_{IJ} &: \frac{1}{2}\eta^{abc}F_{IJbc} + 2D_c\Pi^{ca}{}_{IJ},
\end{aligned} \tag{15}$$

we identify the third constraint as the Gauss constraint for BF theory, corresponding to the generator of $SO(3,1)$ transformations. It is important to remark, that the null vectors obtained from (14) provide us the complete form of the first class constraints, and we have not fixed by hand their structure.

The rank obtained form the matrix (14) yields identifying the following 36 second class constraints

$$\begin{aligned} \chi^{a}{}_{IJ} &: \Pi^{a}{}_{IJ} - \frac{1}{2}\eta^{abc}B_{IJbc}, \\ \chi^{ab}{}_{IJ} &: \Pi^{ab}{}_{IJ}. \end{aligned} \tag{16}$$

On the other hand, we can observe that the 48 first class constraints given in (15) are not all independent. The reason for that is because of in virtue of Bianchi's identity $DF_{IJ} = 0$ one finds that

$$D_a\gamma^{a}_{IJ} - \frac{1}{2}\left[\Pi^{ab}{}_{IK}F_{abJ}{}^{K} - \Pi^{ab}{}_{KJ}F_{abI}{}^{K}\right] = 0. \tag{17}$$

Thus, from the γ^{a}_{IJ}=18 first class constraints we identify that [18-6]= 12 are independent. Therefore, we procede to calculate the physical degrees of freedom as follows; there are 120 canonical variables, [48-6]=42 independent first class constraints and 36 independent second class constraints. With this information, we conclude that four-dimensional BF theory is devoid of local degrees of freedom, hence in this sense we can say that the theory is topological; although this theory has global degrees of freedom due to the nontrivial topology of the manifold on which is defined [19].

Additionally we observe that the complete structure of the constraints (15) and (16) as well as the full structure of reducibility conditions were not reported in [9]. The reason is because in [9, 10] Dirac's canonical method was performed on a smaller phase space context, thus the complete structure of the constraints and reducibility conditions were not found. Of course, in our results, by considering the second class constraints (16) as strong equations the above results are reduced to those reported in [9], thus our results extend and complete those ones. The correct identification of the constraints is a very important step because are used to carry out the counting of the physical degrees of freedom and to identify the gauge transformations if there exist first class constraints. On the other hand, the constraints are the guideline to make the best progress for the quantization of the theory. We need to remember that the quantization scheme for gauge theories as Maxwell or Yang-Mills can not be directly applied to theories with the symmetry of covariance under diffeomorphisms (as for instance BF theories) because we lose relevant physical information [13].

Now, we will calculate the algebra of the constraints; smearing the constraints with test fields

$$
\begin{aligned}
\phi_1 &:= \gamma^0{}_{IJ}\left[\mathbf{A}\right] = \int dx^3 A^{IJ}\left[\Pi^0{}_{IJ}\right], \\
\phi_2 &:= \gamma^{0a}{}_{IJ}\left[\mathbf{B}\right] = \int dx^3 B_a{}^{IJ}\left[\Pi^{0a}{}_{IJ}\right], \\
\phi_3 &:= \gamma_{IJ}\left[\mathbf{C}\right] = \int dx^3 C_a{}^{IJ}\left[D_a\Pi^a{}_{IJ} - \eta_{abc}\left[\Pi^a{}_{IP}\Pi^{bc}{}_{QJ}\eta^{PQ} - \Pi^a{}_{JP}\Pi^{bc}{}_{QI}\eta^{PQ}\right]\right], \\
\phi_4 &:= \gamma^{0a}{}_{IJ}\left[\mathbf{D}\right] = \int dx^3 \mathbf{D}_{0a}{}^{IJ}\left[\frac{1}{2}\eta^{abc}F_{bcIJ} - 2D_b\Pi^{ab}{}_{IJ}\right], \\
\phi_5 &:= \chi^a{}_{IJ}\left[\mathbf{H}\right] = \int dx^3 H_a{}^{IJ}\left[\Pi^a{}_{IJ} - \frac{1}{2}\eta^{abc}B_{bc}{}^{IJ}\right], \\
\phi_6 &:= \chi^{ab}{}_{IJ}\left[\mathbf{G}\right] = \int dx^3 G_{ab}{}^{IJ}\left[\Pi^{ab}{}_{IJ}\right],
\end{aligned}
\tag{18}
$$

the nonzero brackets of the constraints are given by

$$
\begin{aligned}
\left\{\phi_3\left[\mathbf{C}^{IJ}\right], \phi_3\left[\mathbf{C}'^{KL}\right]\right\} &= \int dx^3\left[\mathbf{C}^{IK}\mathbf{C}'_K{}^J - \mathbf{C}^{JK}\mathbf{C}'_K{}^I\right]\gamma_{IJ} \approx 0, \\
\left\{\phi_3\left[\mathbf{C}^{IJ}\right], \phi_4\left[\mathbf{D}_{0a}{}^{KL}\right]\right\} &= \int dx^3\left[\mathbf{C}^{IK}\mathbf{D}_{0aK}{}^J - \mathbf{C}^{JK}\mathbf{D}_{0aK}{}^I\right]\gamma^{0a}{}_{IJ} \approx 0, \\
\left\{\phi_3\left[\mathbf{C}^{IJ}\right], \phi_5\left[\mathbf{H}_a{}^{KL}\right]\right\} &= \int dx^3\left[\mathbf{C}^{IK}\mathbf{H}_{aK}{}^J - \mathbf{C}^{JK}\mathbf{H}_{aK}{}^I\right]\chi^a{}_{IJ} \approx 0, \\
\left\{\phi_5\left[\mathbf{H}_a{}^{IJ}\right], \phi_6\left[\mathbf{G}'_{ab}{}^{KL}\right]\right\} &= -\frac{1}{2}\eta^{abc}\int dx^3\left[\mathbf{H}_{aKH}\mathbf{G}_{bc}{}^{KH}\right],
\end{aligned}
\tag{19}
$$

where we are able to appreciate that the constraints form a set of first and second class constraints as expected. From the constraint algebra (19), we are able to identify the Dirac brackets for the theory, by observing that the matrix whose elements are only the Poisson brackets among second class constraints is given by

$$
C_{\alpha\beta} = \begin{pmatrix} 0 & -\frac{1}{4}\eta^{abc}(\eta_{IK}\eta_{JL} - \eta_{IL}\eta_{JK})\delta^3(x-y) \\ \frac{1}{4}\eta^{abc}(\eta_{IK}\eta_{JL} - \eta_{IL}\eta_{JK})\delta^3(x-y) & 0 \end{pmatrix}. \tag{20}
$$

In this manner, the Dirac bracket among two functionals A, B is expressed by

$$
\{A(x), B(y)\}_D = \{A(x), B(y)\}_P + \int dudv\{A(x), \zeta^\alpha(u)\}C^{-1}_{\alpha\beta}(u,v)\{\zeta^\beta(v), B(y)\}, \tag{21}
$$

where $\{A(x), B(y)\}_P$ is the usual Poisson bracket between the functionals A, B, $\zeta^\alpha(u) = (\chi_{IJ}{}^a, \chi_{IJ}{}^{ab})$, with $C^{-1}_{\alpha\beta}(u,v)$ being the inverse of (20) which has a trivial form. It is well known that Dirac's bracket (21) will be an essential ingredient to make progress in the quantization of the theory [16, 17].

Furthermore, the identification of the constraints will allow us to identify the extended action. By using the first class constraints (15), the second class constraints (16), and the Lagrange multipliers (13) we find that the extended action takes the form

$$
\begin{aligned}
S_E &\left[A_\alpha{}^{IJ}, \Pi^\alpha{}_{IJ}, B_{\mu\nu}{}^{IJ}, \Pi^{\mu\nu}{}_{IJ}, u_0{}^{IJ}, u_{0a}{}^{IJ}, u^{IJ}, u_a{}^{IJ}, v_a{}^{IJ}, v_{ab}{}^{IJ}\right] = \int\Big\{\dot{A}_\alpha{}^{IJ}\Pi^\alpha{}_{IJ} + \dot{B}_{0a}{}^{IJ}\Pi^{0a}{}_{IJ} \\
&+ \dot{B}_{ab}{}^{IJ}\Pi^{ab}{}_{IJ} - H - u_0{}^{IJ}\gamma^0{}_{IJ} - u_{0a}{}^{IJ}\gamma^{0a}{}_{IJ} - u^{IJ}\gamma_{IJ} - u_a{}^{IJ}\gamma^{0a}{}_{IJ} - v_a{}^{IJ}\chi^a{}_{IJ} - v_{ab}{}^{IJ}\chi^{ab}{}_{IJ}\Big\}dx^4,
\end{aligned}
\tag{22}
$$

where H is linear combination of first class constraints

$$H = \frac{1}{2}A_0{}^{IJ}\left[D_a\Pi^a{}_{IJ} - \eta_{abc}\left[\Pi^a{}_{IP}\Pi^{bc}{}_{QJ}\eta^{PQ} - \Pi^a{}_{JP}\Pi^{bc}{}_{QI}\eta^{PQ}\right]\right] - B_{0a}{}^{IJ}\left[\frac{1}{2}\eta^{abc}F_{bcIJ} - 2D_b\Pi^{ab}{}_{IJ}\right], \tag{23}$$

and $u_0{}^{IJ}, u_{0a}{}^{IJ}, u^{IJ}, u_a{}^{IJ}, v_a{}^{IJ}, v_{ab}{}^{IJ}$ are the Lagrange multipliers enforcing the first and second class constraints. We can observe that by considering the second class constraints as strong equations the Hamiltonian (23) is reduced to the Hamiltonian found in [9] where was performed the Hamiltonian analysis on a smaller phase space context. In this manner, we have developed in this work a best and complete description at classical level.

From the extended action we can identify the extended Hamiltonian given by

$$H_E = H - u_0{}^{IJ}\gamma^0{}_{IJ} - u_{0a}{}^{IJ}\gamma^{0a}{}_{IJ} - u^{IJ}\gamma_{IJ} - u_a{}^{IJ}\gamma^{0a}{}_{IJ}. \tag{24}$$

It is well know that the equations of motion obtained by means of the extended Hamiltonian, in general, are mathematically different from the Euler-Lagrange equations, but the difference is unphysical [11].

It is important to remark, that the theory under study has an extended Hamiltonian which is linear combination of first class constraints reflecting the general covariance of the theory, just as General Relativity, thus, it is not possible to construct the Schrodinger equation because the action of the Hamiltonian on physical states is annihilation. In Dirac's quantization of systems with general covariance, the restriction of our physical state is archived by demanding that the first class constraints in their quantum form must be satisfied, then we can use the tools of Loop Quantum Gravity by finding a set of quantum states for the theory as was performed in [20] using the spin foam models.

We will continue this section by computing the equations of motion obtained from the extended action, which are expressed by

$$
\begin{aligned}
\delta A_0{}^{IJ} : \dot{\Pi}^0{}_{IJ} &= -\frac{1}{2}\left[D_a \Pi^a{}_{IJ} - \eta_{abc}\left[\Pi^a{}_{IP}\Pi^{bc}{}_{QJ}\eta^{PQ} - \Pi^a{}_{JP}\Pi^{bc}{}_{QI}\eta^{PQ}\right]\right], \\
\delta \Pi^0{}_{IJ} : \dot{A}_0{}^{IJ} &= u_0{}^{IJ}, \\
\delta A_a{}^{IJ} : \dot{\Pi}^a{}_{IJ} &= \left[A_{0J}{}^F + u_J{}^F\right]\Pi^a{}_{IF} - \left[A_{0I}{}^F + u_I{}^F\right]\Pi^a{}_{JF} + \eta^{abc}\left[D_b B_{0cIJ} - D_b u_{cIJ}\right] \\
&+ 2\left[u_{bI}{}^F - B_{0bI}{}^F\right]\Pi^{ab}{}_{JF} - 2\left[u_{bJ}{}^F - B_{0bJ}{}^F\right]\Pi^{ab}{}_{IF}, \\
\delta \Pi^a{}_{IJ} : \dot{A}_a{}^{IJ} &= -D_a\left(\frac{1}{2}A_0{}^{IJ} + u^{IJ}\right) + \left(u_a{}^{IJ} - B_{0a}{}^{IJ}\right) + \frac{1}{2}\eta_{abc}\left[A_0{}^{IL}\Pi^{bc}{}_{QL}\eta^{JQ} - A_0{}^{JL}\Pi^{bc}{}_{QL}\eta^{IQ}\right] \\
&+ v_a{}^{IJ}, \\
\delta B_{0a}{}^{IJ} : \dot{\Pi}^{0a}{}_{IJ} &= -\left[\frac{1}{2}\eta^{abc}F_{bcIJ} - 2D_b\Pi^{ab}{}_{IJ}\right], \\
\delta \Pi^{0a}{}_{IJ} : \dot{B}_{0a}{}^{IJ} &= u_{0a}{}^{IJ}, \\
\delta B_{ab}{}^{IJ} : \dot{\Pi}^{ab}{}_{IJ} &= -\frac{1}{2}\eta^{abc}v_{cIJ}, \\
\delta \Pi^{ab}{}_{IJ} : \dot{B}_{ab}{}^{IJ} &= D_a\left(u_b{}^{IJ} - B_{0b}{}^{IJ}\right) - D_b\left(u_a{}^{IJ} - B_{0a}{}^{IJ}\right) + v_{ab}{}^{IJ}, \\
&- \frac{1}{2}\eta_{gab}\left[\Pi^g{}_{LK}A_0{}^{LJ}\eta^{PI} - \Pi^g{}_{LK}A_0{}^{LI}\eta^{PJ}\right], \\
\delta u_0{}^{IJ} : \gamma^0{}_{IJ} &= 0, \\
\delta u_{0a}{}^{IJ} : \gamma^{0a}{}_{IJ} &= 0, \\
\delta u^{IJ} : \gamma_{IJ} &= 0, \\
\delta u_a{}^{IJ} : \gamma^{0a}{}_{IJ} &= 0, \\
\delta v_a{}^{IJ} : \chi^a{}_{IJ} &= 0, \\
\delta v_{ab}{}^{IJ} : \chi^{ab}{}_{IJ} &= 0. \qquad (25)
\end{aligned}
$$

II. Gauge generator

By following with our analysis, we need to know the gauge transformations on the phase space of the theory under study. For this important step, we shall use Castellani's formalism which allows us to define the following gauge generator in terms of the first class constraints (15)

$$G = \int_\Sigma \left[D_0\varepsilon_0{}^{IJ}\gamma^0{}_{IJ} + D_0\varepsilon_{0a}{}^{IJ}\gamma^{0a}{}_{IJ} + \varepsilon^{IJ}\gamma_{IJ} + \varepsilon_a{}^{IJ}\gamma^{0a}{}_{IJ}\right] dx^3, \qquad (26)$$

thus, we find that the gauge transformations on the phase space are

$$\begin{aligned}
\delta_0 A_0{}^{IJ} &= D_0 \varepsilon_0{}^{IJ}, \\
\delta_0 A_a{}^{IJ} &= -D_a \varepsilon^{IJ}, \\
\delta_0 B_{0a}{}^{IJ} &= D_0 \varepsilon_{0a}{}^{IJ}, \\
\delta_0 B_{ab}{}^{IJ} &= \left[D_a \varepsilon_b{}^{IJ} - D_b \varepsilon_a{}^{IJ}\right] + \left[\varepsilon^{IF} B_{abF}{}^{J} - \varepsilon^{JF} B_{abF}{}^{I}\right], \\
\delta_0 \Pi^0{}_{IJ} &= 0, \\
\delta_0 \Pi^a{}_{IJ} &= \left[\Pi^a{}_{IL} \varepsilon_J{}^L - \Pi^a{}_{JL} \varepsilon_I{}^L\right] + \eta^{adc} D_d \varepsilon_{cIJ} + 2\left[\Pi^{ab}{}_{KI} \varepsilon_b{}^L{}_J - \Pi^{ab}{}_{KJ} \varepsilon_b{}^L{}_I\right], \\
\delta_0 \Pi^{0a}{}_I &= 0, \\
\delta_0 \Pi^{ab}{}_{IJ} &= -\left[\Pi^{ab}{}_{IF} \varepsilon^F{}_J - \Pi^{ab}{}_{JF} \varepsilon^F{}_I\right].
\end{aligned} \tag{27}$$

We can see that, the diffeomorphisms are not present in the previous gauge transformations; however it is well known that BF theory is diffeomorphism covariant. Thus, the next question that arises is: how can we recover the diffeomorphisms symmetry from the above gauge transformations?. The answer for this question can be found redefining the gauge parameters as $-\varepsilon_0{}^{IJ} = \varepsilon^{IJ} = -\xi^\rho A_\rho{}^{IJ}$ and $\varepsilon_\mu^{IJ} = -\xi^\rho B_{\mu\rho}{}^{IJ}$. In this manner the gauge transformations (27) take the following form

$$\begin{aligned}
A'_\mu{}^{IJ} &\rightarrow A_\mu{}^{IJ} + \mathcal{L}_\xi A_\mu{}^{IJ} + \xi^\rho F_{\mu\rho}{}^{IJ}, \\
B'_{\mu\nu}{}^{IJ} &\rightarrow B_{\mu\nu}{}^{IJ} + \mathcal{L}_\xi B_{\mu\nu}{}^{IJ} + \xi^\rho \left[D_\nu B_{\mu\rho}{}^{IJ} + D_\mu B_{\rho\nu}{}^{IJ} + D_\rho B_{\nu\mu}{}^{IJ}\right],
\end{aligned} \tag{28}$$

which correspond to diffeomorphisms. Therefore, the latter correspond to an internal symmetry of the theory. It is important to remark that all this information has not been reported in the literature with the details that we have developed, on the contrary, usually the people prefer to work on a smaller phase space context. Nevertheless, the price to pay for working on a smaller phase space is that we could not know all the relevant information of the theory, as the complete form of the constraints, and the complete form of the gauge transformations. In any case, given a theory whose symmetries we wish study one must; first, to develop a pure Dirac's method, and then with all the information at hand, we will able to reproduce those results that are obtained under the smaller phase context. All these ideas are being already applied to Plebański theories (see appendix), however the analysis is somewhat complicated but the right structure of the constraints will be reported in forthcoming works [16].

In the later section, we shall develop the Hamiltonian framework for a theory BF-like theory that emerges from a generalized Chern-Simons theory.

III. A PURE DIRAC'S METHOD FOR A GENERALIZED FOUR-DIMENSIONAL BF THEORY

For this section, the action under consideration is given by

$$S[A,\mathbf{B}] = \int_M \left[\mathbf{F}^{IJ}(A)\wedge\mathbf{F}\ _{IJ}(A) + 2k\mathbf{B}^{IJ}\wedge\mathbf{F}_{IJ}(A) + k^2\mathbf{B}^{IJ}\wedge\mathbf{B}_{IJ}\right]. \tag{29}$$

As commented in the introduction, the above action was obtained by using a generalized differential calculus, taking the generalized exterior derivative to a generalized Chern-Simons form [8]. We are able to observe that the first term in (29) corresponds to second Chern-class, the second one is a pure BF term and the third one is identified as a cosmological-like term [9], where k is a constant [8]. The Hamiltonian study for the action (29) has not been reported in the literature. In this manner, the action (29) is a good example for applying a pure Dirac's method just as in above section. With the present analysis, we will be able to identify the full symmetries of the theory; we could think, however, that the action (29) being the coupling of topological terms, then the complete theory is topological as well. Nevertheless, the answer is not trivial because in the literature we can find examples where the coupling of topological theories is not topological any more, since there exist physical degrees of freedom [15]. Therefore we need to perform the Hamiltonian analysis to know the symmetries of the theory. For this aim, we will proceed just as in above section developing a pure Hamiltonian framework.

By taking the variation of (29) respect to our set of dynamical variables (A,B), the equations of motion are given by

$$\begin{aligned} D(F+kB) &= 0, \\ k(F+kB) &= 0, \end{aligned} \tag{30}$$

from Bianchi's identities $DF=0$ we can see that the second equation implies the first one. Thus, with that learned in earlier sections we expect for this theory reducibility conditions among the constraints, just as in BF theory.

By performing the 3+1 decomposition for all the terms of the action (29) we obtain

$$\begin{aligned} \mathbf{F}^{IJ}(A)\wedge\mathbf{F}\ _{IJ}(A) &= \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}F^{IJ}{}_{\alpha\beta}F_{IJ\mu\nu}dx^4 = \eta^{abc}F_{bc}{}^{IJ}\left(\dot{A}_{aIJ}-D_aA_{0IJ}\right), \\ \mathbf{B}^{IJ}\wedge\mathbf{F}_{IJ}(A) &= \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}B^{IJ}{}_{\alpha\beta}F_{IJ\mu\nu}dx^4 = \frac{1}{2}\eta^{abc}B_{0a}{}^{IJ}F_{bcIJ} + \frac{1}{2}\eta^{abc}B_{bc}{}^{IJ}\left(\dot{A}_{aIJ}-D_aA_{0IJ}\right), \\ \mathbf{B}^{IJ}\wedge\mathbf{B}_{IJ} &= \frac{1}{4}\epsilon^{\alpha\beta\mu\nu}B^{IJ}{}_{\alpha\beta}B_{IJ\mu\nu}dx^4 = \eta^{abc}B_{0a}{}^{IJ}B_{bc}{}^{IJ}. \end{aligned} \tag{31}$$

In this way, the action principle takes the following form

$$\begin{aligned} S[A,\mathbf{B}] &= \int_R\int_\Sigma [\eta^{abc}F_{bc}{}^{IJ}\left(\dot{A}_{aIJ}-D_aA_{0IJ}\right) + k\eta^{abc}B_{0a}{}^{IJ}F_{bcIJ} + k\eta^{abc}B_{bc}{}^{IJ}\left(\dot{A}_{aIJ}-D_aA_{0IJ}\right) \\ &+ k^2\eta^{abc}B_{0a}{}^{IJ}B_{bc}{}^{IJ}]dx^3dt, \end{aligned} \tag{32}$$

and the Lagrangian density is given by

$$\begin{aligned} \mathcal{L} &= \eta^{abc}F_{bc}{}^{IJ}\left(\dot{A}_{aIJ}-D_aA_{0IJ}\right) + k\eta^{abc}B_{0a}{}^{IJ}F_{bcIJ} + k\eta^{abc}B_{bc}{}^{IJ}\left(\dot{A}_{aIJ}-D_aA_{0IJ}\right) \\ &+ k^2\eta^{abc}B_{0a}{}^{IJ}B_{bc}{}^{IJ}. \end{aligned} \tag{33}$$

We can see that this theory is also background independent and has the same number of dynamical variables as a pure BF theory. The momenta $\left(\Pi^{\alpha}{}_{IJ}, \Pi^{\alpha\beta}{}_{IJ}\right)$ canonically conjugate to $\left(A^{IJ}{}_{\alpha}, B_{\alpha\beta}{}^{IJ}\right)$, are given by

$$\Pi^{\alpha}{}_{IJ} = \frac{\delta L}{\delta \dot{A}^{IJ}{}_{\alpha}}, \quad \Pi^{\alpha\beta}{}_{IJ} = \frac{\delta L}{\delta \dot{B}_{\alpha\beta}{}^{IJ}}. \tag{34}$$

The matrix elements of the Hessian

$$\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu(A_\alpha{}^{IJ}))\partial(\partial_\mu(A_\beta{}^{IJ}))}, \qquad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu(A_\alpha{}^{IJ}))\partial(\partial_\mu(B_{\rho\nu}{}^{IJ}))}, \qquad \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu(B_{\rho\nu}{}^{IJ}))\partial(\partial_\mu(B_{\gamma\sigma}{}^{IJ}))}, \tag{35}$$

are identically zero, the rank of the Hessian is zero, thus, we expect 60 primary constraints. From the definition of the momenta (34), we identify the following 60 primary constraints

$$\begin{aligned} \phi^{0}{}_{IJ} &: \Pi^{0}{}_{IJ} \approx 0, \\ \phi^{a}{}_{IJ} &: \Pi^{a}{}_{IJ} - \eta^{abc}\left(F_{bcIJ} + kB_{bcIJ}\right) \approx 0, \\ \phi^{0a}{}_{IJ} &: \Pi^{0a}{}_{IJ} \approx 0, \\ \phi^{ab}{}_{IJ} &: \Pi^{ab}{}_{IJ} \approx 0. \end{aligned} \tag{36}$$

We can appreciate that, with respect to the primary constraints for a pure BF theory, the primary constraints (36) present an extra term (F_{bcIJ}) because of the presence of the second Chern class. By using the definition of the momenta (34), we find that the canonical Hamiltonian takes the form

$$H_c = -\int dx^3 \left[A^{IJ}{}_{0} D_a \Pi^{a}{}_{IJ} + kB_{0a}{}^{IJ}\Pi^{a}{}_{IJ}\right], \tag{37}$$

the canonical Hamiltonian and the addition of primary constraints allow us to identify the primary Hamiltonian

$$H_P = H_c + \int dx^3 \left[\lambda^{IJ}{}_{0}\phi_{IJ}{}^{0} + \lambda^{IJ}{}_{a}\phi_{IJ}{}^{a} + \lambda_{0a}{}^{IJ}\phi^{0a}{}_{IJ} + \lambda_{ab}{}^{IJ}\phi^{ab}{}_{IJ}\right]. \tag{38}$$

The non-vanishing fundamental Poisson brackets for the theory under study are given by

$$\begin{aligned} \{A^{IJ}{}_{\mu}(x^0,x), \Pi^{\nu}{}_{KL}(x^0,y)\} &= \delta^{\nu}{}_{\mu}\frac{1}{2}\left(\delta^{I}{}_{K}\delta^{J}{}_{L} - \delta^{I}{}_{L}\delta^{J}{}_{K}\right)\delta^3(x-y), \\ \{B_{\alpha\beta}{}^{IJ}(x^0,x), \Pi^{\mu\nu}{}_{KL}(x^0,y)\} &= \frac{1}{4}\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right)\left(\delta^{I}{}_{K}\delta^{J}{}_{L} - \delta^{I}{}_{L}\delta^{J}{}_{K}\right)\delta^3(x-y). \end{aligned} \tag{39}$$

Just as in the above section, we need to identify if the theory presents secondary constraints. For this aim, we compute the 60×60 matrix whose entries are the Poisson brackets among the primary constraints (36), the nonzero brackets are given by

$$\{\phi^{a}{}_{IJ}(x), \phi^{bc}{}_{KL}(y)\} = \frac{k}{2}\eta^{abc}(\eta_{IL}\eta_{JK} - \eta_{IK}\eta_{JL})\delta^3(x,y), \tag{40}$$

this matrix has rank$= 36$ and 24 linearly independent null-vectors. The null vectors and consistency conditions imply 24 secondary constraints. From the temporal evolution of the constraints (36) and the 24 null vectors arise the following 24 secondary constraints

$$\begin{aligned} \dot{\phi}^{0}{}_{IJ} &= \{\phi^{0}{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi_{IJ} := D_a\Pi^{a}{}_{IJ} \approx 0. \\ \dot{\phi}^{0a}{}_{IJ} &= \{\phi^{0a}{}_{IJ}(x), H_P\} \approx 0 \quad \Rightarrow \quad \psi^{0a}{}_{IJ} := k\Pi^{a}{}_{IJ} \approx 0, \end{aligned} \tag{41}$$

and the rank fix the following 36 Lagrange multipliers

$$
\begin{aligned}
\dot{\phi}^a{}_{IJ} = \{\phi^a{}_{IJ}(x), H_P\} \approx 0 \quad &\Rightarrow \quad [\Pi^a{}_{JL}\eta_{KI} - \Pi^a{}_{IL}\eta_{KJ}]\, A_0{}^{KL} - \eta^{abc} k D_b B_{0cIJ} \\
&- \left[\eta^{abc}F_{bcKJ}\eta_{IL} + \eta^{abc}F_{bcIK}\eta_{LJ}\right] A_0{}^{KL} - \eta^{abc}k\lambda_{bcIJ} \approx 0, \\
\dot{\phi}^{ab}{}_{IJ} = \{\phi^{ab}{}_{IJ}(x), H_P\} \approx 0 \quad &\Rightarrow \quad \eta^{abc}\lambda_{cIJ} \approx 0.
\end{aligned} \tag{42}
$$

From consistency of secondary constraints, does not emerge more constraints. In this way, with all the constraints at hand we need to identify which ones correspond to first and second class. For this aim, we compute the Poisson brackets between the primary and secondary constraints which are given in the following 84×84 matrix whose nonzero brackets are given by

$$
\begin{aligned}
\{\phi^a{}_{IJ}(x), \phi^{bc}{}_{KL}(y)\} &= \frac{k}{2}\eta^{abc}(\eta_{IL}\eta_{JK} - \eta_{IK}\eta_{JL})\delta^3(x,y), \\
\{\phi^a{}_{IJ}(x), \Psi_{KL}(y)\} &= \frac{1}{2}\left(\Pi^a{}_{JL}\eta_{IK} - \Pi^a{}_{IL}\eta_{KJ} + \Pi^a{}_{KJ}\eta_{IL} - \Pi^a{}_{KI}\eta_{LJ}\right)\delta^3(x-y), \\
\{\phi^a{}_{IJ}(x), \Psi^b{}_{KL}(y)\} &= k\eta^{abc}\Big[(\eta_{IK}\eta_{JL} - \eta_{KJ}\eta_{IL})\partial_c + (A_{JLc}\eta_{IK} - A_{ILc}\eta_{KJ}) \\
&- (A_{KI}\eta_{LJ} - A_{KJc}\eta_{LI})\Big]\delta^3(x-y), \\
\{\Psi_{IJ}(x), \Psi_{KL}(y)\} &= -\frac{1}{2}(\Psi_{LJ}\eta_{IK} - \Psi_{KJ}\eta_{IL} + \Psi_{IL}\eta_{KJ} - \Psi_{IK}\eta_{LJ})\delta^3(x-y) \approx 0, \\
\{\Psi_{IJ}(x), \Psi^a{}_{KL}(y)\} &= -\frac{k}{2}\left(\Pi^a{}_{JL}\eta_{IK} - \Pi^a{}_{IL}\eta_{KJ} + \Pi^a{}_{KJ}\eta_{IL} - \Pi^a{}_{KI}\eta_{LJ}\right)\delta^3(x-y),
\end{aligned} \tag{43}
$$

After long calculations, we observe that this matrix has a rank= 36 and 48 null-vectors. In this manner, by using the null vectors one finds the following 48 first class constraints

$$
\begin{aligned}
\gamma^0{}_{IJ} &: \Pi^0{}_{IJ}, \\
\gamma^{0a}{}_{IJ} &: \Pi^{0a}{}_{IJ}, \\
\gamma_{IJ} &: D_a\Pi^a{}_{IJ} - \left[\Pi^{ab}{}_{JK}B_{abI}{}^K - \Pi^{ab}{}_{IK}B_{abJ}{}^K\right], \\
\gamma^a{}_{IJ} &: k\Pi^a{}_{IJ} + D_b\Pi^{ba}{}_{IJ},
\end{aligned} \tag{44}
$$

where the third constraint can be identified as the Gauss constraint for this generalized BF theory, corresponding to the generator of $SO(3,1)$ transformations. Again, we observe that by means of the null vectors the form of the secondary constraints has been changed and becomes to be of first class; this fact is important because we do not need fix by hand the constraints to convert them in first class.

From the rank we can identify the following 36 second class constraints

$$
\begin{aligned}
\chi^a{}_{IJ} &: \Pi^a{}_{IJ} - \eta^{abc}\left(F_{bcIJ} + kB_{bcIJ}\right), \\
\chi^{ab}{}_{IJ} &: \Pi^{ab}{}_{IJ}.
\end{aligned} \tag{45}
$$

So, we see that does exist a clear difference among pure BF theory and this generalized theory in the set of the first and second class constraints. In fact, the corresponding $\gamma^a{}_{IJ}$ constraints and second class constraints $\chi^a{}_{IJ}$ are quite different, however this result is expected due to presence of second Chern class and the cosmological-like terms in the action.

An other important point to observe is that the constraints (44) are not all independent because of Bianchis identity $DF = 0$, that now implies

$$D_a\gamma^a{}_{IJ} - k\gamma_{IJ} - k\left[\chi^{ab}{}_{JK}B_{abI}{}^K - \chi^{ab}{}_{IK}B_{abJ}{}^K\right] - \left[\chi^{ab}{}_{IK}F_{abJ}{}^K - \chi^{ab}{}_{KJ}F_{abI}{}^K\right] = 0, \quad (46)$$

which correspond to 6 reducibility conditions for the theory. Thus, from the γ^a_{IJ}=18 first class constraints we identify that [18-6]= 12 are independent, just as for a pure BF theory. Therefore, we are able to procedure and calculate the physical degrees of freedom as follows; the theory presents 120 canonical variables, [48-6]=42 independent first class constraints and 36 independent second class constraints. This information allows us to conclude that this generalized four-dimensional BF theory is devoid of physical degrees of freedom and corresponds to a topological field theory as well. It is important to comment that this program allowed us to know the full structure of the constraints, and now it is straightforward to know the results that can be obtained by considering as dynamical variables those occurring with time derivative in the action principle.

Now, we will calculate the algebra of the constraints. Smearing the constraints with test fields

$$\begin{aligned}
\phi_1 &:= \gamma^0{}_{IJ}\left[\mathbf{A}\right] = \int dx^3\mathbf{A}^{IJ}\left[\Pi^0{}_{IJ}\right],\\
\phi_2 &:= \gamma^{0a}{}_{IJ}\left[\mathbf{B}\right] = \int dx^3\mathbf{B}_a{}^{IJ}\left[\Pi^{0a}{}_{IJ}\right],\\
\phi_3 &:= \gamma_{IJ}\left[\mathbf{C}\right] = \int dx^3\mathbf{C}_a{}^{IJ}\left[D_a\Pi^a{}_{IJ} - \left[\Pi^{ab}{}_{JK}B_{abI}{}^K - \Pi^{ab}{}_{IK}B_{abJ}{}^K\right]\right],\\
\phi_4 &:= \gamma^a{}_{IJ}\left[\mathbf{D}\right] = \int dx^3\mathbf{D}_{0a}{}^{IJ}\left[\Pi^a{}_{IJ} + D_b\Pi^{ba}{}_{IJ}\right],\\
\phi_5 &:= \chi^a{}_{IJ}\left[\mathbf{H}\right] = \int dx^3\mathbf{H}_a{}^{IJ}\left[\Pi^a{}_{IJ} - \eta^{abc}\left(F_{bcIJ} + kB_{bcIJ}\right)\right],\\
\phi_6 &:= \chi^{ab}{}_{IJ}\left[\mathbf{G}\right] = \int dx^3\mathbf{G}_{ab}{}^{IJ}\left[\Pi^{ab}{}_{IJ}\right].
\end{aligned} \quad (47)$$

the non-zero brackets of the constraints are given by

$$\begin{aligned}
\left\{\phi_3\left[\mathbf{C}^{IJ}\right], \phi_3\left[\mathbf{C}'^{KL}\right]\right\} &= \int dx^3\left[\mathbf{C}^{IK}\mathbf{C}'_K{}^J - \mathbf{C}^{JK}\mathbf{C}'_K{}^I\right]\gamma_{IJ} \approx 0,\\
\left\{\phi_3\left[\mathbf{C}^{IJ}\right], \phi_4\left[\mathbf{D}_{0a}{}^{KL}\right]\right\} &= \int dx^3\left[\mathbf{C}^{IK}\mathbf{D}_{0aK}{}^J - \mathbf{C}^{JK}\mathbf{D}_{0aK}{}^I\right]\gamma^{0a}{}_{IJ} \approx 0,\\
\left\{\phi_3\left[\mathbf{C}^{IJ}\right], \phi_5\left[\mathbf{H}_a{}^{KL}\right]\right\} &= \int dx^3\left[\mathbf{C}^{IK}\mathbf{H}_{aK}{}^J - \mathbf{C}^{JK}\mathbf{H}_{aK}{}^I\right]\chi^a{}_{IJ} \approx 0,\\
\left\{\phi_5\left[\mathbf{H}_a{}^{IJ}\right], \phi_6\left[\mathbf{G}'_{ab}{}^{KL}\right]\right\} &= -\eta^{abc}k\int dx^3\left[\mathbf{H}_{aKH}\mathbf{G}_{bc}{}^{KH}\right],
\end{aligned} \quad (48)$$

where we can see that the constraints (47) correspond to a set of first and second class constraints respectively.

The constraint algebra (48) allows us to identify the Dirac bracket for the theory, and we observe that the matrix whose elements are only the Poisson brackets among second class constraints is given by

$$C_{\alpha\beta} = \begin{pmatrix} 0 & \frac{k}{2}\eta^{abc}(\eta_{IL}\eta_{JK} - \eta_{IK}\eta_{JL})\delta^3(x,y) \\ -\frac{k}{2}\eta^{abc}(\eta_{IL}\eta_{JK} - \eta_{IK}\eta_{JL})\delta^3(x,y) & 0 \end{pmatrix}. \quad (49)$$

In this manner, the Dirac bracket among two functionals A, B is expressed by

$$\{A(x), B(y)\}_D = \{A(x), B(y)\}_P + \int dudv \{A(x), \zeta^\alpha(u)\} C^{-1}_{\alpha\beta}(u,v) \{\zeta^\beta(v), B(y)\}, \tag{50}$$

where $\{A(x), B(y)\}_P$ is the usual Poisson bracket between the functionals A, B, $\zeta^\alpha(u) = (\chi_{IJ}{}^a, \chi_{IJ}{}^{ab})$ with $C^{-1}_{\alpha\beta}(u,v)$ being the inverse of (49) which is straightforward to obtain. We will use in future works Dirac's bracket performing a canonical quantization scheme, since in this paper we are only focused on a classical description of the theories under study.
Just as in above section, with the identification of the constraints as first and second class and by using the Lagrange multipliers (42), the constraints (44) and (45), we find that extended action has the following form

$$\begin{aligned} S_E \ & \left[A_\alpha{}^{IJ}, \Pi^\alpha{}_{IJ}, B_{\mu\nu}{}^{IJ}, \Pi^{\mu\nu}{}_{IJ}, u_0{}^{IJ}, u_{0a}{}^{IJ}, u^{IJ}, u_a{}^{IJ}, v_a{}^{IJ}, v_{ab}{}^{IJ}\right] = \int \Big\{ \dot{A}_\alpha{}^{IJ} \Pi^\alpha{}_{IJ} + \dot{B}_{0a}{}^{IJ} \Pi^{0a}{}_{IJ} \\ + \ & \dot{B}_{ab}{}^{IJ} \Pi^{ab}{}_{IJ} - H - u_0{}^{IJ} \gamma^0{}_{IJ} - u_{0a}{}^{IJ} \gamma^{0a}{}_{IJ} - u^{IJ} \gamma_{IJ} - u_a{}^{IJ} \gamma^{0a}{}_{IJ} - v_a{}^{IJ} \chi^a{}_{IJ} - v_{ab}{}^{IJ} \chi^{ab}{}_{IJ} \Big\} dx^4, \end{aligned} \tag{51}$$

and here H is a linear combination of first class constraints

$$H = A_0{}^{IJ} \left[D_a \Pi^a{}_{IJ} - \left[\Pi^{ab}{}_{JK} B_{abI}{}^K - \Pi^{ab}{}_{IK} B_{abJ}{}^K \right] \right] - B_{0a}{}^{IJ} \left[k \Pi^a{}_{IJ} + D_b \Pi^{ba}{}_{IJ} \right], \tag{52}$$

and $u_0{}^{IJ}, u_{0a}{}^{IJ}, u^{IJ}, u_a{}^{IJ}, v_a{}^{IJ}, v_{ab}{}^{IJ}$ are the Lagrange multipliers enforcing the constraints.
From (48) we similarly identify the extended Hamiltonian

$$H_E = H - u_0{}^{IJ} \gamma^0{}_{IJ} - u_{0a}{}^{IJ} \gamma^{0a}{}_{IJ} - u^{IJ} \gamma_{IJ} - u_a{}^{IJ} \gamma^{0a}{}_{IJ}. \tag{53}$$

Again, it is remarkable to observe that the extended Hamiltonian (53) is a linear combination of first class constraints reflecting the general covariance of theory, thus, we can not construct the Shcrodinger equation because of the action of the Hamiltonian on physical states is annihilation [16].

The equations of motion obtained from the extended action are expressed by

$$
\begin{aligned}
\delta A_0{}^{IJ} : \dot{\Pi}^0{}_{IJ} &= -\left[D_a\Pi^a{}_{IJ} - \left[\Pi^{ab}{}_{JK}B_{abI}{}^K - \Pi^{ab}{}_{IK}B_{abJ}{}^K\right]\right],\\
\delta\Pi^0{}_{IJ} : \dot{A}_0{}^{IJ} &= u_0{}^{IJ},\\
\delta A_a{}^{IJ} : \dot{\Pi}^a{}_{IJ} &= 2A_0{}^{FH}\left[\Pi^a{}_{JH}\eta_{FI} - \Pi^a{}_{IH}\eta_{FJ}\right] - 2u^{FH}\left[\Pi^a{}_{JH}\eta_{FI} - \Pi^a{}_{IH}\eta_{FJ}\right]\\
&+ 2B_{0b}{}^{FH}\left[\Pi^{ab}{}_{JH}\eta_{FI} - \Pi^{ab}{}_{IH}\eta_{FJ}\right] - 2u_b{}^{FH}\left[\Pi^{ab}{}_{JH}\eta_{FI} - \Pi^{ab}{}_{IH}\eta_{FJ}\right] + 2\eta^{abc}D_c v_{bIJ},\\
\delta\Pi^a{}_{IJ} : \dot{A}_a{}^{IJ} &= -D_a\left(A_0{}^{IJ} + u^{IJ}\right) + k\left(u_a{}^{IJ} - B_{0a}{}^{IJ}\right) + v_a{}^{IJ},\\
\delta B_{0a}{}^{IJ} : \dot{\Pi}^{0a}{}_{IJ} &= \left[k\Pi^a{}_{IJ} + D_b\Pi^{ba}{}_{IJ}\right],\\
\delta\Pi^{0a}{}_{IJ} : \dot{B}_{0a}{}^{IJ} &= u_{0a}{}^{IJ},\\
\delta B_{ab}{}^{IJ} : \dot{\Pi}^{ab}{}_{IJ} &= -k\eta^{abc}v_{cIJ} + \left[A_{0I}{}^H\Pi^{ab}{}_{HJ} - A_{0J}{}^H\Pi^{ab}{}_{HI}\right] + \left[u_I{}^H\Pi^{ab}{}_{HJ} - u_J{}^H\Pi^{ab}{}_{HI}\right],\\
\delta\Pi^{ab}{}_{IJ} : \dot{B}_{ab}{}^{IJ} &= -D_a u_b{}^{IJ} + D_b u_a{}^{IJ} + v_{ab}{}^{IJ} - \left[A_0{}^{FI}B_{abF}{}^J - A_0{}^{FJ}B_{abF}{}^I\right],\\
&- \left[u^{FI}B_{abF}{}^J - u^{FJ}B_{abF}{}^I\right],\\
\delta u_0{}^{IJ} : \gamma^0{}_{IJ} &= 0,\\
\delta u_{0a}{}^{IJ} : \gamma^{0a}{}_{IJ} &= 0,\\
\delta u^{IJ} : \gamma_{IJ} &= 0,\\
\delta u_a{}^{IJ} : \gamma^{0a}{}_{IJ} &= 0,\\
\delta v_a{}^{IJ} : \chi^a{}_{IJ} &= 0,\\
\delta v_{ab}{}^{IJ} : \chi^{ab}{}_{IJ} &= 0.
\end{aligned}
\tag{54}
$$

V. Gauge generator

By following with our analysis, we need to know the gauge transformations of the theory. For this important step we shall use Castellani's formalism [11], by defining the following gauge generator in terms of the first class constraints (44)

$$
G = \int_\Sigma \left[D_0\varepsilon_0{}^{IJ}\gamma^0{}_{IJ} + D_0\varepsilon_{0a}{}^{IJ}\gamma^{0a}{}_{IJ} + \varepsilon^{IJ}\gamma_{IJ} + \varepsilon_a{}^{IJ}\gamma^{0a}{}_{IJ}\right] dx^3, \tag{55}
$$

thus, we find the following gauge transformations on the phase space

$$
\begin{aligned}
\delta_0 A_0{}^{IJ} &= D_0\varepsilon_0{}^{IJ},\\
\delta_0 A_a{}^{IJ} &= -D_a\varepsilon^{IJ} + k\varepsilon_a{}^{IJ},\\
\delta_0 B_{0a}{}^{IJ} &= D_0\varepsilon_{0a}{}^{IJ},\\
\delta_0 B_{ab}{}^{IJ} &= \left[D_a\varepsilon_b{}^{IJ} - D_b\varepsilon_a{}^{IJ}\right] + \left[\varepsilon^{IF}B_{abF}{}^J - \varepsilon^{JF}B_{abF}{}^I\right],\\
\delta_0\Pi^0{}_{IJ} &= 0,\\
\delta_0\Pi^a{}_{IJ} &= \left[\Pi^a{}_{IL}\varepsilon_J{}^L - \Pi^a{}_{JL}\varepsilon_I{}^L\right] + \left[\Pi^{ab}{}_{KI}\varepsilon_b{}^L{}_J - \Pi^{ab}{}_{KJ}\varepsilon_b{}^L{}_I\right],\\
\delta_0\Pi^{0a}{}_I &= 0,\\
\delta_0\Pi^{ab}{}_{IJ} &= -\left[\Pi^{ab}{}_{IF}\varepsilon^F{}_J - \Pi^{ab}{}_{JF}\varepsilon^F{}_I\right].
\end{aligned}
\tag{56}
$$

We have seen that the extended Hamiltonian for this theory was linear combination of first class constraints, so diffeomorphisms are the gauge transformations, however the diffeomorphisms are not present in above transformations. Furthermore, it is well-known that BF and Pontryagin theory are diffeomorphisms covariant and we expect that (29) being a coupled theory of topological terms this important symmetry is not lost. Thus, the next question that arises is; how can we recover diffeomorphisms symmetry from the former gauge transformations?. The answer can be found if we redefine now the gauge parameters as $-{\varepsilon_0}^{IJ} = \varepsilon^{IJ} = -\xi^\rho {A_\rho}^{IJ}$ and $\varepsilon_\mu^{IJ} = -\xi^\rho {B_{\mu\rho}}^{IJ}$, in this manner the gauge transformations (56) take the form

$$
\begin{aligned}
{A'_\mu}^{IJ} &\rightarrow {A_\mu}^{IJ} + \mathcal{L}_\xi {A_\mu}^{IJ} + \xi^\rho \left[{F_{\mu\rho}}^{IJ} + k{B_{\mu\rho}}^{IJ} \right], \\
{B'_{\mu\nu}}^{IJ} &\rightarrow {B_{\mu\nu}}^{IJ} + \mathcal{L}_\xi {B_{\mu\nu}}^{IJ} + \xi^\rho \left[D_\nu {B_{\mu\rho}}^{IJ} + D_\mu {B_{\rho\nu}}^{IJ} + D_\rho {B_{\nu\mu}}^{IJ} \right],
\end{aligned} \tag{57}
$$

which correspond to diffeomorphisms. Therefore, the latter are an internal symmetry of the theory. It is important to remark, that this analysis has not been reported in the literature with the details that are displayed here. Hence, we have performed a complete local study for the action (29) that will be useful to study global symmetries by using for instance, the Atiyah-Singer theorem [19], then we can use the tools developed in Loop Quantum Gravity to quantize this theory; we need to remember that Loop Quantum Gravity is a canonical approach where diffeomorphisms covariant theories are quantized without perturbative methods.

IV. CONCLUSIONS AND PROSPECTS

The Hamiltonian analysis for four-dimensional BF theories has been performed. By considering the complete set of dynamical variables that define these theories, we have obtained all the symmetries, the constraints, gauge transformations, the counting of degrees of freedom and the extended Hamiltonian. For the case of a pure BF theory, the present work has extended and completed the results reported in [9], where the study was performed on a smaller phase space context, allowing us to know the complete structure of the constraints and the algebra associated. Respect to the four-dimensional generalized BF theory, and despite of there are additional terms in the action such as the second Chern class and the cosmological-like term quadratic in B field, we were able to know the principal symmetries of the theory. The analysis allowed us to conclude that the topological structure of the theory as well as diffeomorphisms covariance was preserved.

With the present work, we have at hand a better classical description of the theories studied, thus the approach developed along the paper is an alternative way to perform a pure Hamiltonian framework for any theory under investigation. We can see alternative approaches in [10, 12], where in the former Dirac's study for Plebański's and the later for Palatini theories were performed;

however that study was not a pure Hamiltonian analysis as the present work; for instance, in [10] alternative variables in the fields were used to carry out the analysis and the final structure of the constraints was fixed by hand. In this sense, we expect that our approach will be an good alternative way to study the symmetries of Plebanski actions (see appendix below), expecting to obtain a better description [16].

We would finish with some remarks. Topological field theories juts as the theories studied here, are characterized by being devoid of local degrees of freedom. That is, the theories are susceptible only to global degrees of freedom associated with non-trivial topologies of the manifold on which they are defined and the topologies of the gauge bundle [1, 6, 19]. Hence, in this paper we have analyzed local symmetries of the theories under study, however, we would emphasize that our results have been useful to analyze the moduli space of a four-dimmensional BF theory for a general base manifold [19]. In fact, in [19] the studio of global symmetries for a BF theory by employing as the main tool the Atiyah-Singer theorem has been performed with base manifolds as S^4, $K3$, $E(n)$, S_d, etc., from which the dimension of the moduli space has been calculated, showing that there exist global degrees of freedom as expected. Therefore, BF theory has been characterized both globall and locally providing all necessary elements to make progress in the quantization; these subjects will be reported in forthcoming works.

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V. APPENDIX A

In this appendix we shall develop the standard way to carry out the canonical analysis to Plebański theory. Plebański's theory is given [21]

$$S[B, A, \Phi] = \frac{2i}{\kappa} \int \left(B_i \wedge F^i + \Phi_{ij} B^i \wedge B^j \right) dx^4, \tag{58}$$

where F is the two form strength of a $SU(2)$ complex one-form $A = A_i t^i$, being t^i the generators of gauge group, $B = B^i t_i$ is a Lie algebra valued two-form, and Φ is a zero-form. Now if we decompose Φ into its trace and its traceless part, namely $\phi_{ij} = \Phi^{ij} - \frac{1}{3}\phi\delta_{ij}$, with $\phi = \Phi_{ij}\delta^{ij}$ which is related with the cosmological constant [18], we get

$$S[B, A, \phi] = \frac{2i}{\kappa} \int \left(B_i \wedge F^i + \phi_{ij} B^i \wedge B^j - \frac{1}{3} \phi B^i \wedge B_i \right) dx^4 \tag{59}$$

So, by performing the 3+1 decomposition in (59) we obtain

$$S[B,A,\phi]=\frac{2i}{\kappa}\int\left(\dot{A}^i_a(\eta^{abc}B_{ibc})+A^i_0D_a(\eta^{abc}B_{ibc})+\eta^{abc}F^i_{ab}B_{i0a}+\eta^{abc}\phi_{ij}B^i_{0a}B^j_{bc}-\frac{\phi}{3}\eta^{abc}B^i_{0a}B_{ibc}\right),\tag{60}$$

where $D_a\lambda^i_b=\partial_a\lambda^i_b+\varepsilon^i{}_{jk}A^j_a\lambda^k_b$ and $\eta^{abc}=\epsilon^{0abc}$, $a,b,c=1,2,3$. From (60) we identify the following Lagrangian density

$$\mathcal{L}=\frac{2i}{\kappa}\left(\dot{A}^i_a(\eta^{abc}B_{ibc})+A^i_0D_a(\eta^{abc}B_{ibc})+\eta^{abc}F^i_{bc}B_{i0a}+\eta^{abc}\phi_{ij}B^i_{0a}B^j_{bc}-\frac{\phi}{3}\eta^{abc}B^i_{0a}B_{ibc}\right),\tag{61}$$

remembering that in a smaller face space context, we will consider as dynamical variables those that occur in the action with time derivative, so the momenta P^a_i canonically conjugate to A^i_a are given by

$$P^a_i=\frac{\delta\mathcal{L}}{\delta A^i_a}=\frac{2i}{\kappa}(\eta^{abc}B_{ibc}),\tag{62}$$

thus, the variation respect to the fields A^i_0, B^i_{0a} and ϕ_{ij} reads

$$\begin{aligned}D_aP^a_i&\approx 0,&(63)\\ \eta^{abc}F^i_{bc}+\phi^{ij}P^a_j-\frac{\phi}{3}P^{ia}&\approx 0,&(64)\\ B^{(i}_{0a}P^{aj)}-\frac{\delta^{ij}}{3}B^k_{0a}P^a_k&\approx 0.&(65)\end{aligned}$$

We observe from the later equations that it is necessary to eliminate the B^i_{0a} and ϕ_{ij} variables in order to identify the structure of the constraints. The B^i_{0a} and ϕ_{ij} variables are Lagrange multipliers, however, is not easy to identified them by using the smaller phase space approach.

On the other hand, it is important to mention that the action (60) has a close relation with a pure BF theory and the action is not quadratic in the B_{0i} variables [10, 18], however the action worked out in [10] is quadratic in B_{0i} field and one needs involve extra variables to perform the hamiltonian analysis, however the canonical analysis reported in [10] becomes to be so much complicated.

Nevertheless, in our approach the canonical analysis of the first term of the action (59) has been performed, with those results and coupling the $\phi_{ij}B^i\wedge B^j$ and $\phi B^i\wedge B_i$ terms in our developments, we will procedure by using a pure Hamiltonian framework to perform the canonical analysis of the action (59).

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[1] M. Montesinos and A. Perez Phys.Rev.D77:104020,2008.
[2] G. T. Horowitz, Commun. Math.Phys.125 (1989) 417.
[3] G. T. Horowitz and M. Srednicki, Commun. Math.Phys.130 (1990) 83.
[4] J. F. Plebanski, *On the separation of Einstenian substructures*, J. Math. Phys. 18 (1977) 2511.
[5] Derek K. Wise, *MacDowell Mansouri Gravity and Cartan Geometry*, gr-qc/0611154; S. W. MacDowell and F. Mansouri, Unified geometric theory of gravity and supergravity, Phys. Rev. Lett. 38 (1977), 739742. Erratum, ibid. 38 (1977), 1376; F. Mansouri, Phys. Rev. D16, 2456 (1977).
[6] Freidel L and Starodubtsev A *Quantum Gravity in terms of topological observables*, arXiv: hep-th/0501191.

[7] M. Martellini, M. Zeni, Phys. Lett. B 401 (1997) 62; hep-th/9610090.
[8] H. Ying, Y. Ling, R. Suan, Y. Zhong, Phys.Rev. D66 (2002) 064017.
[9] M. Mondragon, M. Montesinos, J.Math.Phys. 47 (2006) 022301.
[10] E.Buffenoir, M.Henneaux, K.Noui, Ph.Roche, Class.Quant.Grav. 21 (2004) 5203-5220.
[11] D. M. Gitman and I.V.Tyutin, *Quantization of fields with constraints*. (Berlin, Germany: Springer. (Springer series in nuclear and particle physics, (1990)); A. Hanson, T. Regge and C. Teitelboim. *Constrained Hamiltonian Systems*, (Accademia Nazionale dei Lincei, Roma, (1978)); J. Govaerts, *Hamiltonian quantization and constrained dynamics*,Leuven, Belgium: Univ. Pr. (1991) 371 p. (Leuven notes in mathematical and theoretical physics, B4).
[12] P. Peldan, Class.Quant.Grav.11:1087-1132,1994.
[13] A. Escalante, Phys. Lett B: 676, (2009), p. 105 - 111; A. Escalante, Int. Jour. of Theo. Phys. 48: 24862498 DOI 10.1007/s10773-009-0035-9, 2009.
[14] S. Deser, R. Jackiw and S. Templeton, Annals of Physics 281, 409-449 (2000).

[15] A. Escalante and J. Berra, *A pure Dirac's method for Maxwell and Yang-Mills theories expressed as a constrained BF theory* , submitted to Annals of Physics (2011).
[16] A. Escalante and J. Berra, *A pure Hamiltonian analysis for Pleblański theory* , in preparation (2011).
[17] J.M.F. Labastida and , Carlos Lozano. *Lectures in topological quantum field theory*, hep-th/9709192.
[18] S. Alexandrov and K. Krasnov, Class.Quant.Grav.26:055005,(2009).
[19] R. Cartas-Fuentevilla, A. Escalante and J. Berra-Montiel *Dimension of the moduli space of BF field theories*, submitted to J. of Geom. and Phys. (2011).
[20] J. C. Baez, Lect.Notes Phys.543:25-94,(2000).
[21] C. Ramirez and E. Rosales, *Supergroup formulation of Plebanski theory of gravity* , AIP Conf.Proc.1116:458-460,2009.