EXTENSION OF EULER LAGRANGE IDENTITY BY SUPERQUADRATIC POWER FUNCTIONS

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Abstract. Using convexity and superquadracity we extend in this paper Euler Lagrange identity, Bohr's inequalitiy and the triangle inequality.

1. Generalization of the triangle inequality via convexity

In [3] Theorem 1.1 inequalities related to the Euler Lagrange identity are proved on Banach space. Using the convexity of x^p $p \geq 1$, $x \geq 0$ we prove in this section a generalization of this theorem for complex numbers, for which Bohr's inequality is a special case. This gives us the tools to achieve the main result of Section 2. There we extend the result to the superquadratic functions x^p $p \geq 2$, $x \geq 0$ and obtain the Euler Lagrange identity as a special case.

Theorem 1. *Let x, y, a, b be complex numbers and let μ, ν, $\lambda \in \mathbb{R}\backslash 0$ then*

$$\frac{|x|^p}{\mu} + \frac{|y|^p}{\nu} \geq \frac{|ax+by|^p}{\lambda}$$

holds if

(i) $\mu > 0$, $\nu > 0$, $\lambda > 0$ and

$$|\lambda|^{1/(p-1)} \geq |\mu|^{1/(p-1)} |a|^q + |\nu|^{1/(p-1)} |b|^q ,$$

(ii) $\mu < 0$, $\nu > 0$, $\lambda < 0$ and

$$|\lambda|^{1/(p-1)} \leq |\mu|^{1/(p-1)} |a|^q - |\nu|^{1/(p-1)} |b|^q ,$$

(iii) $\mu > 0$, $\nu < 0$, $\lambda < 0$ and

$$|\lambda|^{1/(p-1)} \leq - |\mu|^{1/(p-1)} |a|^q + |\nu|^{1/(p-1)} |b|^q ,$$

where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Comment: Bohr's inequality

$$sx^p + ty^p \geq \frac{1}{(s-1)\, s^{p-2}} \left((s-1)\, x + y\right)^p \geq \frac{1}{2^{p-2}} \left((s-1)\, x + y\right)^p ,$$

when $1 < s \leq 2$, $\frac{1}{s} + \frac{1}{t} = 1$, $p > 1$ is a special case of Theorem 1 for $a = s-1$, $b = 1$, $\mu = \frac{1}{s}$, $\nu = \frac{1}{t}$, $\lambda = (s-1)\, s^{p-2}$ (see also [2]).

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We first prove a theorem similar Theorem 1.1 in [3] but by dealing with a general integer n instead of $n = 2$. Our proof is completely different than the proof in [3]. It relies on the convexity of $f(x) = x^p$, $p > 1$, $x \geq 0$.

Theorem 2. *Let x_i, a_i, $i = 1, ..., n$ be complex numbers and $p > 1$, $\frac{1}{q} + \frac{1}{p} = 1$. Case (i): If $\mu_i > 0$, $i = 1, ..., n$, $\lambda > 0$, then*

$$\sum_{i=1}^{n} \frac{|x_i|^p}{\mu_i} \geq \frac{|\sum a_i x_i|^p}{\lambda} \tag{1.1}$$

where

$$|\lambda|^{\frac{1}{p-1}} \geq \sum_{i=1}^{n} |\mu_i|^{\frac{1}{p-1}} |a_i|^q . \tag{1.2}$$

Case (ii): If $\mu_1 > 0$, $\mu_i < 0$, $i = 2, ..., n$, $\lambda > 0$, then

$$\sum_{i=1}^{n} \frac{|x_i|^p}{\mu_i} \leq \frac{|\sum a_i x_i|^p}{\lambda} \tag{1.3}$$

where

$$|\lambda|^{\frac{1}{p-1}} \leq |\mu_1|^{\frac{1}{p-1}} |a_1|^q - \sum_{i=2}^{n} |\mu_i|^{\frac{1}{p-1}} |a_i|^q . \tag{1.4}$$

Case (iii): If $\mu_1 < 0$, $\mu_i > 0$, $i = 2, ..., n$, $\lambda < 0$, then

$$\sum_{i=1}^{n} \frac{|x_i|^p}{\mu_i} \geq \frac{|\sum a_i x_i|^p}{\lambda}$$

where λ satisfies (1.4)

Proof. Case (i): It is obvious that it is enough to prove this case of the theorem for a_i, $x_i \geq 0$, $i = 1, ..., n$ and show that here

$$\sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \geq \frac{\sum a_i x_i^p}{\lambda} \tag{1.5}$$

holds if

$$\lambda^{\frac{1}{p-1}} \geq \sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q. \tag{1.6}$$

Let us consider first a more general inequality than (1.5) where instead of the function $f(x) = x^p$, $p > 1$, $x \geq 0$, we deal with a positive strictly increasing convex function f on $(0, \infty)$ which satisfies $f^{-1}(AB) \geq f^{-1}(A) f^{-1}(B)$, $A, B > 0$. In this case we write

$$\sum_{i=1}^{n} \frac{f(x_i)}{\mu_i} = \sum_{i=1}^{n} Q_i f\left(f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)\right), \tag{1.7}$$

and then by the convexity of f we get

$$\sum_{i=1}^{n} Q_i f\left(f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)\right) \geq \left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} Q_i f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right). \tag{1.8}$$

As $f^{-1}(AB) \geq f^{-1}(A) f^{-1}(B)$ and f is increasing we get that

$$\left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} Q_i f^{-1}\left(\frac{f(x_i)}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right) \geq \left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} x_i Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right). \tag{1.9}$$

Therefore, from (1.7), (1.8) and (1.9) it is enough to solve the equality

$$\left(\sum_{j=1}^{n} Q_j\right) f\left(\frac{\sum_{i=1}^{n} x_i Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j}\right) = \frac{f\left(\sum_{i=1}^{n} a_i x_i\right)}{\overline{\lambda}},$$

in other words to solve

$$\frac{Q_i f^{-1}\left(\frac{1}{\mu_i Q_i}\right)}{\sum_{j=1}^{n} Q_j} = a_i, \qquad i = 1, ..., n \tag{1.10}$$

and then insert

$$\overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1} \tag{1.11}$$

in order for $\overline{\lambda}$ to satisfy for given $\mu_i > 0$ and $a_i \geq 0$, $i = 1, ..., n$ the inequality

$$\sum_{i=1}^{n} \frac{f(x_i)}{\mu_i} \geq \frac{f\left(\sum_{i=1}^{n} a_i x_i\right)}{\overline{\lambda}}. \tag{1.12}$$

Replacing $\overline{\lambda}$ by

$$\lambda > \overline{\lambda} = \left(\sum_{j=1}^{n} Q_j\right)^{-1} \tag{1.13}$$

inequality (1.12) holds too.

Now we return to deal with our function $f(x) = x^p$, $p > 1$, $x \geq 0$. This is a nonnegative increasing convex function for $x \geq 0$ and it satisfies $f^{-1}(AB) = f^{-1}(A) f^{-1}(B)$ for $A, B > 0$.

Returning to the proof of (1.5) under the condition (1.6) we obtain from (1.10) that

$$Q_i (\mu_i Q_i)^{-\frac{1}{p}} \left(\sum_{j=1}^{n} Q_j\right)^{-1} = a_i, \qquad i = 1, ..., n. \tag{1.14}$$

Solving (1.14) we get that

$$Q_i = \frac{\mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^p}, \quad i = 1, ..., n, \tag{1.15}$$

and from (1.11) that

$$\overline{\lambda} = \left(\sum_{i=1}^n Q_i\right)^{-1} = \left(\sum_{i=1}^n \mu_i^{\frac{1}{p-1}} a_i^q\right)^{p-1}. \tag{1.16}$$

Hence from (1.13), (1.5) and (1.6) are proved when a_i, $x_i \geq 0$, $i = 1, ..., n$ and therefore (1.1) and (1.2) are proved for the complex numbers x_i, a_i, $i = 1, ...n$.

Case (ii): If $\mu_1 > 0$, $\mu_i < 0$, $i = 2, ..., n$ and $\lambda > 0$ we rewrite (1.3) as

$$\frac{\left|\sum_{i=2}^n a_i x_i\right|^p}{|\lambda|} + \sum_{i=1}^n \frac{|x_i|^p}{|\mu_i|} \geq \frac{|x_1|^p}{|\mu_1|}. \tag{1.17}$$

Let us make the substitutions

$$\begin{aligned} |\mu_i| &= \nu_i, \qquad i = 2, ..., n, \qquad |\mu_i| = \Lambda, \qquad |\lambda| = \nu, \\ z_1 &= \sum_{i=1}^n a_i x_i, \qquad z_i = x_i, \qquad i = 2, ..., n, \end{aligned}$$

and

$$x_1 = \frac{1}{a_1} z_1 + \sum_{i=2}^n \left(\frac{-a_i}{a_1}\right) z_i = \sum_{i=1}^n C_i z_i.$$

Inequality (1.17) becomes

$$\sum_{i=1}^n \frac{|z_i|^p}{\nu_i} \geq \frac{\left|\sum_{i=1}^n C_i z_i\right|^p}{\Lambda}.$$

Therefore from Case (i) we get that

$$\Lambda^{\frac{1}{p-1}} \geq \sum_{i=1}^n \nu_i^{\frac{1}{p-1}} |C_i|^q.$$

In other words (1.3) holds when

$$|\mu_1|^{\frac{1}{p-1}} \geq \frac{|\lambda|^{\frac{1}{p-1}}}{|a_1|^q} + \sum_{i=2}^n |\mu_i|^{\frac{1}{p-1}} \left|\frac{a_i}{a_1}\right|^q,$$

which is the same as (1.4).

The proof of Case (iii) follows immediately from Case (ii).

This completes the proof of the theorem. □

Corollary 1. *For $n = 2$ we get Theorem 1 which is Theorem 1.1 in* [3] *for complex numbers x_i, a_i, $i = 1, ...n$.*

2. Extension of Euler Lagrange type identity

Now we extend the Euler Lagrange type inequalities by introducing the set of superquadratic functions and its basic properties. Euler Lagrange identity is a special case of this extension.

A function f :$[0,b) \to \mathbb{R}$ is superquadratic provided that for all $x \in [0,b)$ there exists a constant $C_f(x) \in \mathbb{R}$ such that the inequality

$$f(y) \geq f(x) + C_f(x)(y-x) + f(|y-x)| \,, \tag{2.1}$$

holds for all $y \in [0,b)$, ([1, Definition 2.1]). The function $f : [0,b) \to \mathbb{R}$ is subquadratic if $-f$ is supequadratic.

According to [1, Theorem 2.2] the inequality

$$\begin{aligned} & f\left(\int h(s)d\mu(s)\right) \\ \leq \; & \int f(h(s)) - f\left(\left|h(s) - \int h(s)d\mu(s)\right|\right) d\mu(s) \end{aligned} \tag{2.2}$$

holds for all probability measures μ and all nonnegative μ-integrable h, if and only if f is superquadratic.

The discrete version of (2.2) is

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i \left(f\left(x_i\right) - f\left(\left| x_i - \sum_{j=1}^{n} \alpha_j x_j \right|\right)\right), \tag{2.3}$$

$x_i \in [0,b), \quad \alpha_i \geq 0, \quad 1 = i, ..., n, \quad \sum_{i=1}^{n} \alpha_i = 1.$

The power functions $f(x) = x^p$, $x \geq 0$, are convex and superquadratic for $p \geq 2$, and convex and subquadratic for $1 \leq p \leq 2$. Inequalities (2.1), (2.2) and (2.3) reduce to inequalities for the function $f(x) = x^2$.

Now we use (2.3) in order to get the Euler Lagrange type inequality.

Theorem 3. *Let* $x_i \geq 0$, $a_i \geq 0$, $\mu_i > 0$, $i = 1, ..., n$, $p \geq 2$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then*

$$\begin{aligned} \sum_{i=1}^{n} \frac{x_i^p}{\mu_i} \quad \geq \quad & \frac{\left(\sum a_i x_i\right)^p}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \\ & + \frac{\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p}} \left(\left| \left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \sum_{j=1}^{n} a_j x_j \right|\right)^p . \end{aligned} \tag{2.4}$$

If $1 < p \leq 2$ *the inverse of (2.4) holds.*

Proof. In Theorem 2 we showed that for $x_i \geq 0$, $a_i \geq 0$, $\mu_i > 0$, $i = 1, ..., n$. inequalities (1.5) and (1.6) hold. There

$$\sum_{i=1}^{n} Q_i \left(A_i\right)^p = \sum_{i=1}^{n} \frac{|x_i|^p}{\mu_i} \tag{2.5}$$

where

$$Q_i = \frac{\mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^p}, \qquad i = 1, ..., n, \tag{2.6}$$

$$A_i = \frac{1}{(a_i\mu_i)^{\frac{1}{p-1}}} \left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i, \qquad i = 1, , ..., n \tag{2.7}$$

and

$$\frac{\sum_{i=1}^n Q_i A_i}{\sum_{j=1}^n Q_j} = \sum_{i=1}^n a_i x_i. \tag{2.8}$$

Therefore, as $f(x) = x^p$, $p \geq 2$, $x \geq 0$ is superquadratic, (2.3) becomes by inserting (2.6)-(2.8)

$$\begin{aligned} &\sum_{i=1}^n Q_i (A_i)^p \qquad (2.9) \\ = \quad & \frac{\sum_{i=1}^n \mu_i^{\frac{1}{p-1}} a_i^q \left(\left(\frac{1}{a_i\mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i \right)^p}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \\ \geq \quad & \frac{\left(\sum_{i=1}^n a_i x_i\right)^p}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^{p-1}} \\ & + \frac{\sum_{i=1}^n \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \left(\left| \left(\frac{1}{a_i\mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^n \mu_j^{\frac{1}{p-1}} a_j^q\right) x_i - \sum_{i=1}^n a_j x_j \right| \right)^p. \end{aligned}$$

Hence from (2.5) and (2.9) we get that (2.4) holds.

If $1 < p \leq 2$ then $f(x) = x^p$, $x \geq 0$ is a subquadratic function, therefore the reverse of (2.4) holds. □

Corollary 2. *In case n=2 we get that*

$$\begin{aligned} \frac{x^p}{\mu} + \frac{y^p}{\nu} \quad \geq \quad & \frac{(ax+by)^p}{\left(\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q\right)^{p-1}} \qquad (2.10) \\ & + \mu^{\frac{1}{p-1}} a^q \left(\left| \left(\frac{1}{a\mu}\right)^{\frac{1}{p-1}} x - \frac{ax+by}{\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q} \right| \right)^p \\ & + \nu^{\frac{1}{p-1}} b^q \left(\left| \left(\frac{1}{\nu b}\right)^{\frac{1}{p-1}} y - \frac{ax+by}{\mu^{\frac{1}{p-1}} a^q + \nu^{\frac{1}{p-1}} b^q} \right| \right)^p \end{aligned}$$

In particular if $f(x) = x^2$, $n = 2$ as Inequality (2.4) reduces to equality we get from (2.10) that

$$\frac{x^2}{\mu} + \frac{y^2}{\nu} = \frac{(ax+by)^2}{\mu a^2 + \nu b^2} + \frac{(\nu b x - a\mu y)^2}{\mu\nu\left(\mu a^2 + \nu b^2\right)},$$

which is Euler Lagrange type identity.

Corollary 3. *From Theorem 3 as $f(x) = x^p$, $1 < p \leq 2$ is both subquadratic and convex, we get that*

$$\begin{aligned} 0 \quad &\leq \quad \sum_{i=1}^{n} \frac{x_i^p}{\mu_i} - \frac{\left(\sum a_i x_i\right)^p}{\left(\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q\right)^{p-1}} \\ &\leq \frac{\sum_{i=1}^{n} \mu_i^{\frac{1}{p-1}} a_i^q}{\left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q\right)^p} \left(\left| \left(\frac{1}{a_i \mu_i}\right)^{\frac{1}{p-1}} \left(\sum_{j=1}^{n} \mu_j^{\frac{1}{p-1}} a_j^q \right) x_i - \sum_{j=1}^{n} a_j x_j \right| \right)^p . \end{aligned}$$

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