

A CONGRUENCE PROPERTY OF THE LOCAL LANGLANDS CORRESPONDENCE

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ABSTRACT. Let F be a non-Archimedean local field of residual characteristic p , and ℓ a prime number, $\ell \neq p$. We consider the Langlands correspondence, between irreducible, n -dimensional, smooth representations of the Weil group of F and irreducible cuspidal representations of $\mathrm{GL}_n(F)$. We use an explicit description of the correspondence from an earlier paper, and otherwise entirely elementary methods, to show that it respects the relationship of congruence modulo ℓ . The ℓ -modular correspondence thereby becomes as effective as the complex one.

Let ℓ be a prime number and let F be a non-Archimedean local field of residual characteristic $p \neq \ell$. We form the Weil group \mathcal{W}_F of F , relative to some separable algebraic closure \overline{F}/F . Let \mathbf{E} denote either the field \mathbf{C} of complex numbers or an algebraic closure $\overline{\mathbf{Q}}_\ell$ of the field \mathbf{Q}_ℓ of ℓ -adic numbers. For an integer $n \geq 1$, let $\mathcal{G}_n(F)_{\mathbf{E}}$ denote the set of equivalence classes of irreducible, smooth \mathbf{E} -representations of \mathcal{W}_F of dimension n . Likewise, let $\mathcal{A}_n(F)_{\mathbf{E}}$ be the set of equivalence classes of irreducible, smooth, cuspidal \mathbf{E} -representations of the group $G = \mathrm{GL}_n(F)$.

The Langlands correspondence gives a canonical bijection $\mathcal{G}_n(F)_{\mathbf{C}} \rightarrow \mathcal{A}_n(F)_{\mathbf{C}}$ which we denote $\sigma \mapsto {}^L\sigma$. By choosing a field isomorphism $\alpha : \mathbf{C} \rightarrow \overline{\mathbf{Q}}_\ell$, one may transport the Langlands correspondence to a bijection $\mathcal{G}_n(F)_{\overline{\mathbf{Q}}_\ell} \rightarrow \mathcal{A}_n(F)_{\overline{\mathbf{Q}}_\ell}$. We continue to denote this $\sigma \mapsto {}^L\sigma$, eliding the fact that it may also depend on the choice of α .

Let $\overline{\mathbf{F}}_\ell$ be an algebraic closure of the field \mathbf{F}_ℓ of ℓ elements. If $\sigma \in \mathcal{G}_n(F)_{\overline{\mathbf{Q}}_\ell}$ has determinant of finite order then $\sigma(\mathcal{W}_F)$ is finite and, following a standard technique from the representation theory of finite groups, one may attach to σ an isomorphism class $[\sigma]_\ell^{\mathrm{ss}}$ of n -dimensional, smooth, semisimple $\overline{\mathbf{F}}_\ell$ -representations of \mathcal{W}_F . Likewise, if $\pi \in \mathcal{A}_n(F)_{\overline{\mathbf{Q}}_\ell}$ has central character of finite order, a variation on the same technique attaches to π an irreducible cuspidal $\overline{\mathbf{F}}_\ell$ -representation $[\pi]_\ell$ of $\mathrm{GL}_n(F)$. For $\sigma_1, \sigma_2 \in \mathcal{G}_n(F)_{\overline{\mathbf{Q}}_\ell}$, with determinants of finite order, we say $\sigma_1 \equiv \sigma_2 \pmod{\ell}$ if $[\sigma_1]_\ell^{\mathrm{ss}} = [\sigma_2]_\ell^{\mathrm{ss}}$, and similarly on the other side. We prove:

Main Theorem. *If $\sigma_1, \sigma_2 \in \mathcal{G}_n(F)_{\overline{\mathbf{Q}}_\ell}$ have determinant characters $\det \sigma_1, \det \sigma_2$ of finite order, then*

$$\sigma_1 \equiv \sigma_2 \pmod{\ell} \iff {}^L\sigma_1 \equiv {}^L\sigma_2 \pmod{\ell}.$$

We use the classification of irreducible cuspidal representations of $\mathrm{GL}_n(F)$ in terms of simple types, as in [8]: although written there in terms of \mathbf{C} -representations, it applies unchanged to $\overline{\mathbf{Q}}_\ell$ -representations. Thanks to fundamental results of James [17] concerning cuspidal modular representations of finite general linear groups, the relevant classes of simple type behave extraordinarily well under reduction modulo ℓ . Only the elementary techniques available in [9] or [19], for example, are necessary to analyze them completely. Another elementary argument

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relates the congruence behaviour of types to that of cuspidal representations of $\mathrm{GL}_n(F)$. The Comparison Theorem of [6] then implies the desired result for the Langlands correspondence. To reflect this, we have structured the paper to give a brief introduction to the relevant results of [6] alongside an essentially complete and self-contained account of reduction modulo ℓ .

Some remarks are in order. First, the map $\sigma \mapsto {}^L\sigma$ on $\overline{\mathcal{Q}}_\ell$ -representations does depend on the choice of a field isomorphism $\alpha : \mathcal{C} \rightarrow \overline{\mathcal{Q}}_\ell$. If we replace α by another isomorphism $\beta : \mathcal{C} \rightarrow \overline{\mathcal{Q}}_\ell$, then ${}^L\sigma$ changes to ${}^L(\chi \otimes \sigma)$, where χ is an unramified character of \mathcal{W}_F , of order ≤ 2 , depending only on F , n and $\alpha^{-1}\beta$ [13] (7.4). The assertion of the theorem is therefore independent of the choice of α , and we may regard it as an arithmetic property of the complex Langlands correspondence. Next, the techniques and the result hold equally for representations $\sigma \in \mathcal{G}_n(F)_{\overline{\mathcal{Q}}_\ell}$ such that $\det \sigma$ takes its values in $\overline{\mathcal{Z}}_\ell^\times$, where $\overline{\mathcal{Z}}_\ell$ is the integral closure in $\overline{\mathcal{Q}}_\ell$ of the ring \mathcal{Z}_ℓ of ℓ -adic integers. However, the reduction from there to the finite case may be achieved by the trivial step of tensoring with an unramified character: we impose the restriction only because it facilitates direct appeal to standard results from finite group theory.

The ‘‘modulo ℓ ’’ Langlands correspondence already has a substantial literature. Of particular relevance here are the works [20], [21] of M.-F. Vignéras, laying the foundations of a theory of smooth $\overline{\mathcal{F}}_\ell$ -representations of p -adic reductive groups, and her paper [22] establishing the existence of a semisimple version of the correspondence. More recently, J.-F. Dat has posted a pre-print [10] in which he shows that the standard geometric model, realizing the Langlands correspondence for $\overline{\mathcal{Q}}_\ell$ -representations, admits an ℓ -integral model. On reduction modulo ℓ , this yields the Langlands correspondence for $\overline{\mathcal{F}}_\ell$ -representations.

The results of [10] are certainly broader than those here, although the core is the same. However, we feel that the brevity and simplicity of our arguments, presenting the $\overline{\mathcal{F}}_\ell$ -correspondence as an easy consequence of an explicit complex result, may be found appealing. While the arguments presented here are short and straightforward, the proof in its totality cannot be described as elementary or local in nature. It relies via [6] on the standard treatment of the Langlands correspondence [11], [12], [18], along with the classification theory of [8], and its relations with automorphic induction [14], [15], [16], as developed in [1], [2] and [5].

1. REPRESENTATIONS OF FINITE GROUPS

This section is preparatory in nature. We fix a prime number ℓ and recall the definition of the classical *decomposition map* modulo ℓ . This relates the representations of a finite group over fields of characteristic zero to its representations over fields of characteristic ℓ . Standard expositions (of which we use [9] and [19]) start from representations over local or global fields, but this will not be convenient for us. We therefore outline the translation to a simpler framework. We append a couple of specific results used repeatedly later in the paper.

1.1. Let ℓ be a prime number. Let $\overline{\mathcal{Q}}_\ell/\mathcal{Q}_\ell$ be an algebraic closure of the field \mathcal{Q}_ℓ of ℓ -adic numbers. Let $\overline{\mathcal{Z}}_\ell$ be the integral closure of \mathcal{Z}_ℓ in $\overline{\mathcal{Q}}_\ell$. Thus $\overline{\mathcal{Z}}_\ell$ is a local ring but, we recall, it is not Noetherian and its unique maximal ideal \mathfrak{P}_ℓ is not principal. Its residue class field $\overline{\mathcal{Z}}_\ell/\mathfrak{P}_\ell$ provides an algebraic closure $\overline{\mathcal{F}}_\ell$ of the field \mathcal{F}_ℓ of ℓ elements.

Definition 1. Let V be a finite-dimensional $\overline{\mathcal{Q}}_\ell$ -vector space. A $\overline{\mathcal{Z}}_\ell$ -lattice in V is a finitely generated $\overline{\mathcal{Z}}_\ell$ -submodule L of V which spans V over $\overline{\mathcal{Q}}_\ell$.

Proposition. Let V be a $\overline{\mathcal{Q}}_\ell$ -vector space of finite dimension m , and let L be a $\overline{\mathcal{Z}}_\ell$ -submodule of V . The module L is a $\overline{\mathcal{Z}}_\ell$ -lattice in V if and only if it is generated over $\overline{\mathcal{Z}}_\ell$ by a $\overline{\mathcal{Q}}_\ell$ -basis of V . In particular, any $\overline{\mathcal{Z}}_\ell$ -lattice in V is $\overline{\mathcal{Z}}_\ell$ -free of rank m .

Proof. One implication is trivial. To prove the converse, let L be a $\overline{\mathcal{Z}}_\ell$ -lattice in V . Since L is finitely generated over $\overline{\mathcal{Z}}_\ell$, we may choose a $\overline{\mathcal{Z}}_\ell$ -generating set $\{x_1, x_2, \dots, x_r\}$ of L , of minimal cardinality. This set must surely span V over $\overline{\mathcal{Q}}_\ell$. Suppose, for a contradiction, that it is linearly dependent over $\overline{\mathcal{Q}}_\ell$, say $\sum_{i=1}^r a_i x_i = 0$, for elements $a_i \in \overline{\mathcal{Q}}_\ell$, not all zero. There is a finite field extension E/\mathcal{Q}_ℓ containing all a_i . Scaling by an element of E^\times , we may assume that all a_i lie

in the discrete valuation ring \mathfrak{o}_E in E , and that at least one of them, say a_1 , is a unit in \mathfrak{o}_E . Thus x_1 lies in the $\overline{\mathbb{Z}}_\ell$ -module generated by $\{x_2, x_3, \dots, x_r\}$, contrary to hypothesis. It follows that $\{x_1, x_2, \dots, x_r\}$ is a $\overline{\mathbb{Q}}_\ell$ -basis of V (and a $\overline{\mathbb{Z}}_\ell$ -basis of L). \square

Corollary. *Let V be a finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector space, let U be a subspace of V and let L be a $\overline{\mathbb{Z}}_\ell$ -lattice in V . The group $L \cap U$ is then a $\overline{\mathbb{Z}}_\ell$ -lattice in U .*

Proof. The image $L/L \cap U$ of L in V/U is a finitely generated $\overline{\mathbb{Z}}_\ell$ -module, spanning V/U over $\overline{\mathbb{Z}}_\ell$. It is, by the proposition, $\overline{\mathbb{Z}}_\ell$ -free of finite rank. Thus $L = L \cap U \oplus L'$, for a finitely generated free $\overline{\mathbb{Z}}_\ell$ -submodule L' of L , isomorphic to $L/L \cap U$. In particular, $L \cap U \cong L/L'$, so $L \cap U$ is finitely generated over $\overline{\mathbb{Z}}_\ell$. If W is the $\overline{\mathbb{Q}}_\ell$ -span of L' , then $V = U \oplus W$ and so $L \cap U$ spans U . Thus $L \cap U$ is a $\overline{\mathbb{Z}}_\ell$ -lattice in U , as required. \square

Definition 2. *Let V be a $\overline{\mathbb{Q}}_\ell$ -vector space, of possibly infinite dimension. A $\overline{\mathbb{Z}}_\ell$ -submodule L of V is a $\overline{\mathbb{Z}}_\ell$ -lattice in V if $L \cap U$ is a $\overline{\mathbb{Z}}_\ell$ -lattice in U , for every finite-dimensional subspace U of V .*

The corollary implies that the two definitions are consistent.

1.2. Let G be a finite group. If k is a field, we denote by $\mathfrak{G}_0(kG)$ the Grothendieck group of finite-dimensional representations of G over k , formed relative to exact sequences. Let \mathfrak{R} be the set of isomorphism classes of irreducible k -representations of G . The group $\mathfrak{G}_0(kG)$ is then free abelian on \mathfrak{R} . We may identify the set of formal sums $\sum_{R \in \mathfrak{R}} n_R R$, in which all n_R are non-negative, with the set of isomorphism classes of finite-dimensional *semisimple* k -representations of G .

We consider the category of finite-dimensional representations of G over the field $\overline{\mathbb{Q}}_\ell$. The fields $\overline{\mathbb{Q}}_\ell, \mathbb{C}$ are isomorphic, and the choice of an isomorphism $\mathbb{C} \rightarrow \overline{\mathbb{Q}}_\ell$ induces an equivalence between the categories of finite-dimensional representations of G over the two fields. We may so translate the basic theory of complex representations to the new context unchanged.

Let (ρ, U) be a representation of G over $\overline{\mathbb{Q}}_\ell$, of finite dimension m . A $\overline{\mathbb{Z}}_\ell G$ -lattice in U is a $\overline{\mathbb{Z}}_\ell$ -lattice in U which is stable under the action of $\rho(G)$. If L_0 is a $\overline{\mathbb{Z}}_\ell$ -lattice in U , then the $\overline{\mathbb{Z}}_\ell$ -module $\sum_{g \in G} \rho(g)L_0$ is G -stable and it is a $\overline{\mathbb{Z}}_\ell$ -lattice. Thus (ρ, U) admits a $\overline{\mathbb{Z}}_\ell G$ -lattice.

Let L be a $\overline{\mathbb{Z}}_\ell G$ -lattice in U . By 1.1 Proposition, the quotient space $\tilde{L} = L/\mathfrak{P}_\ell L$ is a vector space over $\overline{\mathbb{F}}_\ell$, of dimension m . The representation ρ induces a representation ρ_L of G on \tilde{L} . Let ρ_L^{ss} denote the image of ρ_L in the Grothendieck group $\mathfrak{G}_0(\overline{\mathbb{F}}_\ell G)$.

Proposition. *The class ρ_L^{ss} depends only on the isomorphism class of (ρ, U) , and not on the choice of $\overline{\mathbb{Z}}_\ell G$ -lattice L in U . In particular, if ρ_L is irreducible, then its isomorphism class is determined by that of (ρ, U) , independently of the choice of L .*

Proof. We perform a preliminary reduction. Let K/\mathbb{Q}_ℓ be a finite field extension, $K \subset \overline{\mathbb{Q}}_\ell$, such that K contains a primitive $|G|$ -th root of unity. Extension of scalars then induces an isomorphism $\mathfrak{G}_0(KG) \rightarrow \mathfrak{G}_0(\overline{\mathbb{Q}}_\ell G)$ ([19] Théorème 24, Corollaire). In particular, there is an irreducible K -representation (ρ_0, U_0) of G such that (ρ, U) is isomorphic to the representation of G on $\overline{\mathbb{Q}}_\ell \otimes_K U_0$ induced by ρ_0 . We henceforward identify U with $\overline{\mathbb{Q}}_\ell \otimes_K U_0$.

Let L be a $\overline{\mathbb{Z}}_\ell G$ -lattice in U . Let \mathfrak{o}_K be the discrete valuation ring in K and let L_0 be a G -stable \mathfrak{o}_K -lattice in U_0 . We set $\overline{L}_0 = \overline{\mathbb{Z}}_\ell \otimes_{\mathfrak{o}_K} L_0$. Thus \overline{L}_0 is a G -stable $\overline{\mathbb{Z}}_\ell$ -lattice in U . Expressing a $\overline{\mathbb{Z}}_\ell$ -basis of L in terms of an \mathfrak{o}_K -basis of L_0 , we see:

Lemma. *Let L be a $\overline{\mathbb{Z}}_\ell G$ -lattice in U . There is a finite field extension K'/K and an $\mathfrak{o}_{K'} G$ -lattice L'_0 in $K' \otimes_K U_0$ such that $L = \overline{\mathbb{Z}}_\ell \otimes_{\mathfrak{o}_{K'}} L'_0$.*

Let L_1, L_2 be $\overline{\mathbb{Z}}_\ell G$ -lattices in U . According to the lemma, we may choose the field K and $\mathfrak{o}_K G$ -lattices L_{i0} in U_0 such that $L_i = \overline{\mathbb{Z}}_\ell \otimes_{\mathfrak{o}_K} L_{i0}$, $i = 1, 2$. Let $\mathfrak{p}_K = \mathfrak{P}_\ell \cap K$ be the maximal ideal of \mathfrak{o}_K and write $\mathbb{k}_K = \mathfrak{o}_K/\mathfrak{p}_K$. By [9] (16.16) or [19] Théorème 32, the \mathbb{k}_K -representations $\rho_{L_{i0}}$ of G on $L_{i0}/\mathfrak{p}_K L_{i0}$, induced by ρ_0 , define the same element of $\mathfrak{G}_0(\mathbb{k}_K G)$. The representation ρ_{L_i}

is obtained from $\rho_{L_{i_0}}$ by extension of scalars from \mathbb{k}_K to $\overline{\mathbf{F}}_\ell$. Therefore $\rho_{L_1}^{\text{ss}} = \rho_{L_2}^{\text{ss}}$, as required. The final assertion follows. \square

The proposition shows that $\rho \mapsto \rho_L^{\text{ss}}$ induces a well-defined map $\mathfrak{G}_0(\overline{\mathbf{Q}}_\ell G) \rightarrow \mathfrak{G}_0(\overline{\mathbf{F}}_\ell G)$. We denote it

$$(1.2.1) \quad \begin{aligned} \mathfrak{G}_0(\overline{\mathbf{Q}}_\ell G) &\longrightarrow \mathfrak{G}_0(\overline{\mathbf{F}}_\ell G), \\ \rho &\longmapsto [\rho]_\ell^{\text{ss}}. \end{aligned}$$

Remark. The map (1.2.1) is the *decomposition map* of the literature, in which it is usually denoted d_ℓ . We often think of $[\rho]_\ell^{\text{ss}}$ as the isomorphism class of the semisimplification of the representation ρ_L , for some lattice L .

1.3. We continue this general discussion with a simple remark which will be useful later.

Proposition. *Let (ρ, V) be an irreducible $\overline{\mathbf{Q}}_\ell$ -representation of the finite group G , and suppose that $[\rho]_\ell^{\text{ss}}$ is irreducible. If L, L' are $\overline{\mathbf{Z}}_\ell G$ -lattices in V , there exists $a \in \overline{\mathbf{Q}}_\ell^\times$ such that $L' = aL$.*

Proof. As in the proof of 1.1 Proposition, there exists $a \in \overline{\mathbf{Q}}_\ell^\times$ such that $L \supset aL' \not\subset \mathfrak{P}_\ell L$. Replacing L' by aL' , we may assume $L \supset L' \not\subset \mathfrak{P}_\ell L$. The image of L' in $L/\mathfrak{P}_\ell L$ is then a non-zero G -subspace of the irreducible $\overline{\mathbf{F}}_\ell G$ -space $L/\mathfrak{P}_\ell L$. We deduce that $L = L' + \mathfrak{P}_\ell L$.

We show that this condition implies $L' = L$, whence the result will follow. We choose a $\overline{\mathbf{Z}}_\ell$ -basis \mathcal{B} of L and a $\overline{\mathbf{Z}}_\ell$ -basis \mathcal{B}' of L' . The ‘‘transition matrix’’, relating \mathcal{B}' to \mathcal{B} , then has coefficients in $\overline{\mathbf{Z}}_\ell$, and its reduction modulo \mathfrak{P}_ℓ is invertible over $\overline{\mathbf{F}}_\ell$. The matrix is therefore invertible over $\overline{\mathbf{Z}}_\ell$, whence $L' = L$ as required. \square

1.4. We record some special properties of p -groups, where p is a prime number other than ℓ .

Proposition. *Let p be a prime number, $p \neq \ell$. Let G be a finite p -group. Any $\overline{\mathbf{F}}_\ell$ -representation of G is semisimple.*

If ρ_1, ρ_2 are irreducible $\overline{\mathbf{Q}}_\ell$ -representations of G , then

- (1) *the $\overline{\mathbf{F}}_\ell$ -representations $[\rho_1]_\ell^{\text{ss}}, [\rho_2]_\ell^{\text{ss}}$ are irreducible, and*
- (2) *$[\rho_1]_\ell^{\text{ss}} = [\rho_2]_\ell^{\text{ss}}$ if and only if $\rho_1 \cong \rho_2$.*

Proof. The first assertion is Maschke’s Theorem. The others are equivalent to [19] Proposition 43. \square

1.5. For our next result, we are given a finite group G and a subgroup H . We define the induction functors $\text{Ind}_H^G, c\text{-Ind}_H^G$ following exactly the formulation for complex representations of locally profinite groups. In this situation, they are the same so we get two versions of Frobenius Reciprocity: if ρ is an $\overline{\mathbf{F}}_\ell$ -representation of G , and τ an $\overline{\mathbf{F}}_\ell$ -representation of H , then

$$\begin{aligned} \text{Hom}_G(\rho, \text{Ind}_H^G \tau) &\cong \text{Hom}_H(\rho, \tau), \\ \text{Hom}_G(\text{Ind}_H^G \tau, \rho) &\cong \text{Hom}_H(\tau, \rho). \end{aligned}$$

Proposition. *Let p be a prime number, $p \neq \ell$. Let G be a finite group, and let P be a normal p -subgroup of G . Let α be an irreducible $\overline{\mathbf{F}}_\ell$ -representation of P , and let H be the group of $g \in G$ such that $\alpha^g \cong \alpha$. Suppose there exists a representation ρ of H such that $\rho|_P \cong \alpha$.*

- (1) *Let τ be an irreducible $\overline{\mathbf{F}}_\ell$ -representation of H , trivial on P . The representation*

$$\nu_\tau = \text{Ind}_H^G \rho \otimes \tau$$

is irreducible.

- (2) *The map $\tau \mapsto \nu_\tau$ is a bijection between the set of isomorphism classes of irreducible $\overline{\mathbf{F}}_\ell$ -representations τ of H such that $\tau|_P$ is trivial, and the set of isomorphism classes of irreducible representations ν of G such that $\text{Hom}_P(\alpha, \nu) \neq 0$.*

Proof. In (1), set $\dim \tau = d_\tau$, and consider the semisimple representation $\nu_\tau|_P$. Since P is normal in G , it is the direct sum of representations α^g , $g \in H \backslash G$, each occurring with multiplicity d_τ . If $\check{\rho}$ denotes the contragredient of ρ , it follows that the space of P -fixed points in $\check{\rho} \otimes \nu_\tau$ has dimension d_τ . This space, moreover, carries a representation τ' of H , trivial on P . To identify the representation τ' , we observe that

$$\mathrm{Hom}_H(\tau, \tau') = \mathrm{Hom}_H(\tau, \check{\rho} \otimes \nu_\tau) \cong \mathrm{Hom}_H(\rho \otimes \tau, \nu_\tau) \cong \mathrm{Hom}_G(\nu_\tau, \nu_\tau) \neq 0.$$

Since $\dim \tau' = d_\tau = \dim \tau$, we conclude that τ' is equivalent to τ . We further deduce that $\mathrm{Hom}_G(\nu_\tau, \nu_\tau) = \overline{\mathbf{F}}_\ell$.

Next, let μ be an irreducible G sub-representation of ν_τ . The restriction of μ to P is again a direct sum of representations α^g , $g \in H \backslash G$, each occurring with the same multiplicity. The space of P -fixed points in the representation $\check{\rho} \otimes \mu$ is therefore nonzero, and carries a representation μ_0 of H , trivial on P . This representation μ_0 appears naturally as a sub-representation of τ' and, since $\tau' \cong \tau$ is irreducible, we get $\mu_0 \cong \tau$. Therefore

$$\mathrm{Hom}_G(\nu_\tau, \mu) \cong \mathrm{Hom}_H(\tau, \check{\rho} \otimes \mu) \cong \overline{\mathbf{F}}_\ell.$$

The space $\mathrm{Hom}_G(\nu_\tau, \mu)$ is a subspace of $\mathrm{Hom}_G(\nu_\tau, \nu_\tau)$, while

$$\mathrm{Hom}_G(\nu_\tau, \nu_\tau) \cong \mathrm{Hom}_H(\tau, \check{\rho} \otimes \nu_\tau) \cong \overline{\mathbf{F}}_\ell.$$

The identity endomorphism of ν_τ therefore lies in $\mathrm{Hom}_G(\nu_\tau, \mu)$, whence $\mu = \nu_\tau$. In particular, ν_τ is irreducible, as required for (1). Since τ appears as the natural representation of H on the space of P -fixed points in $\check{\rho} \otimes \nu_\tau$, we have also proved (2). \square

Remark. In the proposition, the hypothesis $P \triangleleft G$ is essential. Given that, the result and its proof are valid for representations of G over any algebraically closed field of characteristic other than p .

2. LINEAR GROUPS OVER LOCAL FIELDS

Again, ℓ is a prime number. From now on, we work with a non-Archimedean local field F with residue class field \mathbb{k}_F of characteristic $p \neq \ell$. We mildly generalize the machinery of §1 to admissible $\overline{\mathbf{Q}}_\ell$ -representations of the locally profinite group $G = \mathrm{GL}_n(F)$.

2.1. Let (π, V) be an *admissible* $\overline{\mathbf{Q}}_\ell$ -representation of G , and let L be a $\overline{\mathbf{Z}}_\ell$ -lattice in V , in the sense of 1.1 Definition 2. In particular, if K is a compact open subgroup of G , the space V^K of $\pi(K)$ -fixed points in V has finite dimension and $L \cap V^K$ is a $\overline{\mathbf{Z}}_\ell$ -lattice in V^K .

We say that L is a $\overline{\mathbf{Z}}_\ell G$ -lattice if $\pi(g)L = L$, for all $g \in G$. In such a case, the lattice $L \cap V^K$ is the set L^K of $\pi(K)$ -fixed points in L . More generally, let ρ be an irreducible, smooth $\overline{\mathbf{Q}}_\ell$ -representation of K . The isotypic subspace V^ρ of V has finite dimension, and so $L^\rho = L \cap V^\rho$ is a $\overline{\mathbf{Z}}_\ell K$ -lattice in V^ρ .

Proposition. *Let (π, V) be an admissible $\overline{\mathbf{Q}}_\ell$ -representation of G , and let L be a $\overline{\mathbf{Z}}_\ell G$ -lattice in V . Let K be an open pro- p subgroup of G , and let \widehat{K} denote the set of equivalence classes of irreducible smooth $\overline{\mathbf{Q}}_\ell$ -representations of K . For $\rho \in \widehat{K}$, the set L^ρ is a $\overline{\mathbf{Z}}_\ell K$ -lattice in V^ρ and $L = \bigoplus_{\rho \in \widehat{K}} L^\rho$.*

Proof. The first assertion has already been remarked. Let K' be an open normal subgroup of K . We view $V^{K'}$ as providing a representation of the finite group K/K' , and then $L^{K'}$ is a $\overline{\mathbf{Z}}_\ell[K/K']$ -lattice in $V^{K'}$. Moreover, $V^{K'} = \bigoplus_{\rho} V^\rho$, where ρ ranges over those elements of \widehat{K} with $K' \subset \mathrm{Ker} \rho$. For such ρ , there is a primitive central idempotent e_ρ of the group algebra $\overline{\mathbf{Q}}_\ell[K/K']$ such that $V^\rho = e_\rho V^{K'}$. The integer $(K:K')$ is a power of p , so it lies in $\overline{\mathbf{Z}}_\ell^\times$. Therefore $e_\rho \in \overline{\mathbf{Z}}_\ell[K/K']$, whence $L^{K'} = \bigoplus_{\rho} L^\rho$, with ρ ranging as before. Allowing K' to range

over a descending sequence of open normal subgroups of K with trivial intersection, the result follows. \square

Remark. We have not asserted the existence of a $\overline{\mathbb{Z}}_\ell G$ -lattice in an arbitrary admissible representation of G . In the one case we need, we will give a direct construction. For a more general discussion of lattices, see [20] I, §9.

2.2. We need one more simple property.

Proposition. *Let (π, V) be an admissible $\overline{\mathbb{Q}}_\ell$ -representation of G , admitting a $\overline{\mathbb{Z}}_\ell G$ -lattice L . Let $\tilde{L} = L/\mathfrak{P}_\ell L$ and let π_L denote the natural $\overline{\mathbb{F}}_\ell$ -representation of G on \tilde{L} . Let K be an open pro- p subgroup of G .*

- (1) *If $q : L \rightarrow \tilde{L}$ is the quotient map, then $q(L^K) = \tilde{L}^K$. In particular, the representation π_L is admissible.*
- (2) *Let ρ be an irreducible smooth $\overline{\mathbb{Q}}_\ell$ -representation of K . If $\tilde{\rho}$ denotes the irreducible $\overline{\mathbb{F}}_\ell$ -representation $[\rho]_\ell^{\text{ss}}$, then $q(L^\rho) = \tilde{L}^{\tilde{\rho}}$.*

Proof. Let $\tilde{v} \in L^K$ and choose $v \in L$ such that $q(v) = \tilde{v}$. There then exists an open normal subgroup K' of K such that $v \in L^{K'}$. The element $v' = (K:K')^{-1} \sum_{x \in K/K'} \pi(x)v$ lies in L^K and satisfies $q(v') = \tilde{v}$. This proves part (1), and part (2) now follows from 2.1 Proposition. \square

3. INDUCED CUSPIDAL REPRESENTATIONS

We consider irreducible *cuspidal* representations of the group $G = \text{GL}_n(F)$ over the field $\overline{\mathbb{Q}}_\ell$. The classification and structure theory for complex cuspidal representations of G , laid out in [8] and further developed in the first nine sections of [1], are purely algebraic in nature: we may apply them unchanged to $\overline{\mathbb{Q}}_\ell$ -representations via a field isomorphism $\overline{\mathbb{Q}}_\ell \rightarrow \mathbb{C}$. We investigate the behaviour of these structures under reduction modulo ℓ .

We start by reviewing the background, relying as much as possible on the summary given in [6] (especially sections 2.1 and 2.2).

3.1. Let θ be an m -simple character in G . Thus, by definition, θ is either the trivial character of $U_{\mathfrak{m}}^1$, where \mathfrak{m} is a maximal order in $A = M_n(F)$, or else θ is a simple character attached to a simple stratum $[\mathfrak{a}, \beta]$ in A , in which \mathfrak{a} is maximal among the hereditary \mathfrak{o}_F -orders in A that are stable under conjugation by the group $F[\beta]^\times$ of non-zero elements of the field $F[\beta]$. In particular, θ is a character of the compact open subgroup $H_\theta^1 = H^1(\beta, \mathfrak{a})$ of G , and H_θ^1 is a pro- p group.

In all cases, we let \mathbf{J}_θ be the G -normalizer of θ , we let J_θ^0 be the unique maximal compact subgroup of \mathbf{J}_θ and J_θ^1 the pro- p radical of J_θ^0 . An *extended maximal simple type* over θ is an irreducible representation of \mathbf{J}_θ which contains θ and is intertwined only by elements of \mathbf{J}_θ . Let $\mathcal{T}(\theta)$ be the set of equivalence classes of extended maximal simple types over θ . We remark (cf. 3.2 below) that a representation Λ of \mathbf{J}_θ lies in $\mathcal{T}(\theta)$ if and only if $(J_\theta^0, \Lambda|_{J_\theta^0})$ is a *maximal simple type* in G , in the sense of [8]. We summarize the main points.

Proposition.

- (1) *Let (π, V) be an irreducible cuspidal representation of G .*
 - (a) *The representation π contains a simple character θ . The character θ is m -simple and is unique up to G -conjugacy.*
 - (b) *The natural representation Λ_π of \mathbf{J}_θ on V^θ is irreducible and lies in $\mathcal{T}(\theta)$. In particular, $\pi \cong c\text{-Ind}_{\mathbf{J}_\theta}^G \Lambda_\pi$.*
- (2) *Let θ be an m -simple character in G . The map $\Lambda \mapsto c\text{-Ind}_{\mathbf{J}_\theta}^G \Lambda$, $\Lambda \in \mathcal{T}(\theta)$ is a bijection between $\mathcal{T}(\theta)$ and the set of equivalence classes of irreducible cuspidal representations of G which contain θ .*

3.2. Let θ be an \mathfrak{m} -simple character in G . We describe the elements of $\mathcal{T}(\theta)$ more explicitly.

Suppose first that θ is the trivial character of $U_{\mathfrak{m}}^1$, for a maximal \mathfrak{o}_F -order \mathfrak{m} in A . In this case, $\mathbf{J}_\theta = F^\times J_\theta^0$, $J_\theta^0 = U_{\mathfrak{m}}$ and $J_\theta^1 = H_\theta^1 = U_{\mathfrak{m}}^1$. We have $J_\theta^0/J_\theta^1 = U_{\mathfrak{m}}/U_{\mathfrak{m}}^1 \cong \mathrm{GL}_n(\mathbb{k}_F)$. A representation Λ of \mathbf{J}_θ then lies in $\mathcal{T}(\theta)$ if and only if $\Lambda|_{J_\theta^0}$ is the inflation of an irreducible *cuspidal* representation of $J_\theta^0/J_\theta^1 \cong \mathrm{GL}_n(\mathbb{k}_F)$. Any two maximal orders in A are G -conjugate, so the set of G -conjugacy classes of elements of $\mathcal{T}(\theta)$ effectively depends only on the dimension n .

We therefore assume θ is non-trivial, attached to a simple stratum $[\mathfrak{a}, \beta]$. Denoting the field $F[\beta]$ by P , we here have $\mathbf{J}_\theta = P^\times J_\theta^0$. If C denotes the A -centralizer of P , the set of elements of G which intertwine θ is $I_G(\theta) = J_\theta^1 C^\times J_\theta^1$.

We define two sets of irreducible representations of \mathbf{J}_θ . Let η_θ be the unique irreducible representation of J_θ^1 containing θ . Let $\mathcal{H}(\theta)$ denote the set of equivalence classes of representations κ of \mathbf{J}_θ such that $\kappa|_{J_\theta^1} \cong \eta_\theta$ and such that κ is intertwined by every element of $I_G(\theta)$.

Let $\mathfrak{c} = \mathfrak{a} \cap C$. Since θ is \mathfrak{m} -simple, \mathfrak{c} is a maximal \mathfrak{o}_P -order in C and so isomorphic to $M_n(\mathfrak{o}_P)$, $n = m[P:F]$. We have $J_\theta^0 = U_{\mathfrak{c}} J_\theta^1$, and $U_{\mathfrak{c}} \cap J_\theta^1 = U_{\mathfrak{c}}^1$. We define $\mathcal{T}_0(\theta)$ to be the set of equivalence classes of irreducible representations λ of \mathbf{J}_θ such that λ is trivial on J_θ^1 and $\lambda|_{J_\theta^0}$ is inflated from an irreducible cuspidal representation of $J_\theta^0/J_\theta^1 \cong U_{\mathfrak{c}}/U_{\mathfrak{c}}^1$.

Remark. If we write $\theta_{\mathfrak{c}}$ for the trivial character of $U_{\mathfrak{c}}^1$, then $\theta_{\mathfrak{c}}$ is a trivial \mathfrak{m} -simple character in C^\times , and the map $\lambda \mapsto \lambda|_{P^\times U_{\mathfrak{c}}}$ gives a bijection $\mathcal{T}_0(\theta) \rightarrow \mathcal{T}(\theta_{\mathfrak{c}})$.

Proposition. *Let $\kappa \in \mathcal{H}(\theta)$, $\lambda \in \mathcal{T}_0(\theta)$. The representation $\kappa \otimes \lambda$ lies in $\mathcal{T}(\theta)$, and the map $\mathcal{T}_0(\theta) \rightarrow \mathcal{T}(\theta)$, $\lambda \mapsto \kappa \otimes \lambda$, is a bijection.*

Proof. See 3.6 Proposition of [6]. \square

3.3. We use these facts to investigate the reduction properties of cuspidal representations.

Let θ be an \mathfrak{m} -simple character in G , and let $\Lambda \in \mathcal{T}(\theta)$. The restriction of Λ to F^\times is a multiple of a character ω_Λ . The group $\Lambda(\mathbf{J}_\theta)$ is then finite if and only if $\omega_\Lambda(F^\times)$ is finite. So, when ω_Λ has finite order, the representation Λ admits a $\overline{\mathbf{Z}}_\ell \mathbf{J}_\theta$ -lattice.

Proposition. *Let $(\mathbf{J}_\theta, \Lambda, W) \in \mathcal{T}(\theta)$, suppose that ω_Λ has finite order, and let L be a $\overline{\mathbf{Z}}_\ell \mathbf{J}_\theta$ -lattice in W .*

- (1) *The $\overline{\mathbf{F}}_\ell$ -representation Λ_L of \mathbf{J}_θ on $\tilde{L} = L/\mathfrak{P}_\ell L$ is irreducible.*
- (2) *The isomorphism class of Λ_L is independent of the choice of L .*
- (3) *If L' is a $\overline{\mathbf{Z}}_\ell \mathbf{J}_\theta$ -lattice in W , there exists $a \in \overline{\mathbf{Q}}_\ell^\times$ such that $L' = aL$.*

Proof. We observe that (1) implies (2) (by 1.2 Proposition) and (2) implies (3) (by 1.3 Proposition).

To prove (1), we consider first the case where the \mathfrak{m} -simple character θ is trivial. Thus, we may assume, θ is the trivial character of $U_{\mathfrak{m}}^1$, where $\mathfrak{m} = M_n(\mathfrak{o}_F)$. The group \mathbf{J}_θ is $F^\times \mathrm{GL}_n(\mathfrak{o}_F)$ and, by definition, the restriction of Λ to $\mathrm{GL}_n(\mathfrak{o}_F)$ is the inflation of an irreducible cuspidal representation of $\mathrm{GL}_n(\mathbb{k}_F)$. The assertion (1) is thus an instance of [17] Theorem 3.6.

We therefore assume θ to be non-trivial, attached to a simple stratum $[\mathfrak{a}, \beta]$, as in 3.2. In particular, $\Lambda = \kappa \otimes \lambda$, for representations $\kappa \in \mathcal{H}(\theta)$ and $\lambda \in \mathcal{T}_0(\theta)$. We may choose κ such that $\kappa(F^\times)$ is finite [6] 3.2 Corollary; in this case, $\lambda(F^\times)$ is also finite.

Let λ (resp. κ) act on the vector space W_1 (resp. W_2), and let L_i be a $\overline{\mathbf{Z}}_\ell \mathbf{J}_\theta$ -lattice in W_i . As in the first case of the proof, the representation λ_{L_1} is irreducible. Let η_θ be the unique irreducible representation of J_θ^1 containing θ . Since J_θ^1 is a pro- p group, the representation $(\eta_\theta)_{L_2}$ is irreducible (1.4 Proposition). As $\kappa|_{J_\theta^1} \cong \eta_\theta$, the representation κ_{L_2} is also irreducible. The representation Λ acts on $W = W_1 \otimes W_2$, and $L = L_1 \otimes L_2$ provides a $\overline{\mathbf{Z}}_\ell \mathbf{J}_\theta$ -lattice in W . The representation $\Lambda_L = \lambda_{L_1} \otimes \kappa_{L_2}$ is irreducible, by 1.5 Proposition. \square

3.4. We continue in the same situation, with an m -simple character θ in $G = \mathrm{GL}_n(F)$, and an extended maximal simple type $\Lambda \in \mathcal{T}(\theta)$ such that ω_Λ has finite order. The representation $(\pi, V) = c\text{-Ind}_{\mathbf{J}_\theta}^G \Lambda$ is irreducible, cuspidal and the central character $\omega_\pi = \omega_\Lambda$ has finite order. We let Λ act on the $\overline{\mathbf{Q}}_\ell$ -space W , and choose a $\overline{\mathbf{Z}}_\ell \mathbf{J}_\theta$ -lattice L in W . The space V consists of all functions $f : G \rightarrow W$ which are compactly supported modulo the centre of G and satisfy $f(jg) = \Lambda(j)f(g)$, for $j \in \mathbf{J}_\theta$, $g \in G$. The group G acts on V by right translation. We define

$$(3.4.1) \quad L_* = \{f \in V : f(G) \subset L\}.$$

Surely, L_* is a $\pi(G)$ -stable $\overline{\mathbf{Z}}_\ell$ -submodule of V .

Lemma. *The $\overline{\mathbf{Z}}_\ell$ -module L_* is a $\overline{\mathbf{Z}}_\ell G$ -lattice in V .*

Proof. It is enough to show that L_*^K is a $\overline{\mathbf{Z}}_\ell$ -lattice in V^K , for any open, pro- p subgroup K of G . We follow the procedure of [4] 3.5 to write down a basis of the (finite-dimensional) space V^K . First, we choose a set X of coset representatives for $\mathbf{J}_\theta \backslash G/K$. For each $x \in X$, we choose a basis \mathcal{B}_x of the space of $xKx^{-1} \cap \mathbf{J}_\theta$ -fixed points in W . For each $x \in X$ and each $w \in \mathcal{B}_x$, there is a unique function $f_{x,w} \in V^K$ with support $\mathbf{J}_\theta xK$ and such that $f_{x,w}(x) = w$. The set

$$\mathcal{B}(K) = \{f_{x,w} : x \in X, w \in \mathcal{B}_x\}$$

is then a basis of V^K . For each $x \in X$, we may choose \mathcal{B}_x to be a $\overline{\mathbf{Z}}_\ell$ -basis of the lattice $L^{xKx^{-1} \cap \mathbf{J}_\theta}$. The corresponding set $\mathcal{B}(K)$ is then a $\overline{\mathbf{Z}}_\ell$ -basis of L_*^K . \square

We form the $\overline{\mathbf{F}}_\ell$ -representation $(\pi_{L_*}, L_*/\mathfrak{P}_\ell L_*)$ of G : this is admissible by 2.2 Proposition.

Proposition. *The representation $(\pi_{L_*}, L_*/\mathfrak{P}_\ell L_*)$ is irreducible, and equivalent to the representation $c\text{-Ind}_{\mathbf{J}_\theta}^G \Lambda_L$.*

Proof. Let $f \in L_*$; we define a function $f_L : G \rightarrow \tilde{L}$ by composing f with the canonical map $L \rightarrow \tilde{L}$. If $f \in \mathfrak{P}_\ell L_*$, then $f(G) \subset \mathfrak{P}_\ell L$ and $f_L = 0$. We thus obtain a G -homomorphism $L_*/\mathfrak{P}_\ell L_* \rightarrow c\text{-Ind } \Lambda_L$. If K is an open pro- p subgroup of G , we follow the procedure of the lemma to construct a $\overline{\mathbf{Z}}_\ell$ -basis $\mathcal{B}(K)$ of L_*^K . The set $\{b_L : b \in \mathcal{B}(K)\}$ then provides an $\overline{\mathbf{F}}_\ell$ -basis of the space $(c\text{-Ind } \Lambda_L)^K$. It follows that the map

$$\begin{aligned} L_* &\longrightarrow c\text{-Ind}_{\mathbf{J}_\theta}^G \Lambda_L, \\ f &\longmapsto f_L, \end{aligned}$$

induces an $\overline{\mathbf{F}}_\ell G$ -isomorphism $L_*/\mathfrak{P}_\ell L_* \cong c\text{-Ind } \Lambda_L$.

We identify L with the sublattice of L_* consisting of functions with support \mathbf{J}_θ . The lattice L is also the θ -isotypic sublattice $L_*^\theta = L_* \cap V^\theta$ of L_* . The character θ has finite order, which is a power of p . Let $\mu_p(\overline{\mathbf{Q}}_\ell)$ denote the group of roots of unity in $\overline{\mathbf{Q}}_\ell$ of p -power order. Thus $\mu_p(\overline{\mathbf{Q}}_\ell) \subset \overline{\mathbf{Z}}_\ell^\times$ and, under the reduction map $\overline{\mathbf{Z}}_\ell^\times \rightarrow \overline{\mathbf{F}}_\ell^\times$, $\mu_p(\overline{\mathbf{Q}}_\ell)$ is mapped isomorphically to the group $\mu_p(\overline{\mathbf{F}}_\ell)$ of p -power roots of unity in $\overline{\mathbf{F}}_\ell^\times$. Consequently, there is no need to distinguish between θ and $[\theta]_\ell^{\mathrm{ss}}$, its reduction mod. ℓ . With this convention, 2.2 Proposition implies that the canonical map $L_* \rightarrow L_*/\mathfrak{P}_\ell L_*$ identifies $L_*^\theta/\mathfrak{P}_\ell L_*^\theta$ with $(L_*/\mathfrak{P}_\ell L_*)^\theta$, that is, with $\tilde{L} \subset L_*/\mathfrak{P}_\ell L_*$.

Let U be a non-zero G -subspace of $c\text{-Ind } \Lambda_L$. We apply Frobenius Reciprocity,

$$0 \neq \mathrm{Hom}_G(U, c\text{-Ind } \Lambda_L) \subset \mathrm{Hom}_G(U, \mathrm{Ind } \Lambda_L) \cong \mathrm{Hom}_{\mathbf{J}_\theta}(U, \Lambda_L).$$

In particular, $U^\theta \neq 0$. However, the isotypic space $(c\text{-Ind } \Lambda_L)^\theta = \tilde{L}$ is irreducible over \mathbf{J}_θ , so $U \supset \tilde{L}$. Since \tilde{L} generates $c\text{-Ind } \Lambda_L$ over G , we have $U = c\text{-Ind } \Lambda_L$, as required. \square

Remark. The standard argument, as in for example [4] 11.4, shows that the irreducible $\overline{\mathbf{F}}_\ell$ -representation π_{L_*} is *cuspidal*, in that its matrix coefficients are compactly supported modulo F^\times . As in [20] II 2.3, 2.7, this is equivalent to π_{L_*} not being equivalent to a sub-representation of any proper parabolically induced representation.

3.5. We consider general $\overline{\mathbb{Z}}_\ell G$ -lattices in an irreducible cuspidal representation.

Proposition. *Let (π, V) be an irreducible, cuspidal $\overline{\mathbb{Q}}_\ell$ -representation of G , such that ω_π has finite order.*

- (1) *The representation (π, V) admits a $\overline{\mathbb{Z}}_\ell G$ -lattice.*
- (2) *Let M be a $\overline{\mathbb{Z}}_\ell G$ -lattice in (π, V) . The natural representation π_M of G on $M/\mathfrak{P}_\ell M$ is irreducible. The isomorphism class of π_M depends only on that of π , and not on M .*

Proof. The representation (π, V) contains an m -simple character θ . Let Λ denote the natural representation of \mathbf{J}_θ on V^θ , so that $\pi \cong c\text{-Ind}_{\mathbf{J}_\theta}^G \Lambda$. As in 3.4 Proposition, we choose a $\overline{\mathbb{Z}}_\ell \mathbf{J}_\theta$ -lattice L in V^θ . According to that result, the set L_* of (3.4.1) provides a $\overline{\mathbb{Z}}_\ell G$ -lattice in V , as required for part (1).

For part (2), let M be a $\overline{\mathbb{Z}}_\ell G$ -lattice in V . The $\overline{\mathbb{Z}}_\ell$ -module $M^\theta = M \cap V^\theta$ provides a $\overline{\mathbb{Z}}_\ell \mathbf{J}_\theta$ -lattice in Λ (2.1 Proposition). It follows from 1.3 Proposition that $M^\theta = aL$, for some $a \in \overline{\mathbb{Q}}_\ell^\times$. Scaling, therefore, we may assume that $M^\theta = L = L_*^\theta$. Since L_* is generated by L as $\overline{\mathbb{Z}}_\ell G$ -lattice, we have $L_* \subset M$.

Suppose, for a contradiction, that $M \neq L_*$. There then exists an open, pro- p subgroup K of \mathbf{J}_θ such that $M^K \supsetneq L_*^K$. We choose $x \in \overline{\mathbb{Q}}_\ell^\times$ such that, if M_1 denotes the lattice xM , then M_1^K is contained in L_*^K but not in $\mathfrak{P}_\ell L_*^K$. Let M_2 be the $\overline{\mathbb{Z}}_\ell G$ -lattice generated by M_1^K . Thus $\mathfrak{P}_\ell L_* \not\supset M_2 \subset L_*$, but $M_2^\theta = M_1^\theta$ is contained in $\mathfrak{P}_\ell L$. The image of M_2 in $L_*/\mathfrak{P}_\ell L_*$ is then a proper G -subspace, which is impossible. This proves (2). \square

Remark. We have also shown that the cuspidal representation (π, V) admits only one $\overline{\mathbb{Z}}_\ell G$ -lattice, up to scaling by elements of $\overline{\mathbb{Q}}_\ell^\times$.

3.6. Let $\mathcal{A}_n(F)$ denote the set of equivalence classes of irreducible, cuspidal, $\overline{\mathbb{Q}}_\ell$ -representations of $G = \text{GL}_n(F)$, and let $\mathcal{A}_n(F)^{\text{fin}}$ be the set of $(\pi, V) \in \mathcal{A}_n(F)$ such that ω_π has finite order. For $(\pi, V) \in \mathcal{A}_n(F)^{\text{fin}}$, we define $[\pi]_\ell$ to be the isomorphism class of the natural representation π_M of G on $M/\mathfrak{P}_\ell M$, for a $\overline{\mathbb{Z}}_\ell G$ -lattice M in V . As in 3.5 Proposition, $[\pi]_\ell$ is irreducible and independent of the choice of M . We may therefore make the following definition.

Definition. *Let $\pi_1, \pi_2 \in \mathcal{A}_n(F)^{\text{fin}}$; say $\pi_1 \equiv \pi_2 \pmod{\ell}$ if $[\pi_1]_\ell = [\pi_2]_\ell$.*

Remark. The group $\text{Aut } \overline{\mathbb{Q}}_\ell$ of field automorphisms of $\overline{\mathbb{Q}}_\ell$ acts on $\mathcal{A}_n(F)^{\text{fin}}$: we denote this action by $\alpha : \pi \mapsto \pi^\alpha$. This action is compatible with $\overline{\mathbb{Z}}_\ell$ -structures, and we conclude that, if $\alpha \in \text{Aut } \overline{\mathbb{Q}}_\ell$ and $\pi_i \in \mathcal{A}_n(F)^{\text{fin}}$, then $\pi_1^\alpha \equiv \pi_2^\alpha \pmod{\ell}$ if and only if $\pi_1 \equiv \pi_2 \pmod{\ell}$. Consequently, we can transfer the congruence relation to complex representations as follows. Let π_1, π_2 be irreducible, cuspidal \mathbf{C} -representations of G , each with central character of finite order. By composing with a field isomorphism $\gamma : \mathbf{C} \rightarrow \overline{\mathbb{Q}}_\ell$, we obtain representations $\pi_1^\gamma, \pi_2^\gamma \in \mathcal{A}_n(F)^{\text{fin}}$, and we may say that $\pi_1 \equiv \pi_2 \pmod{\ell}$ if $\pi_1^\gamma \equiv \pi_2^\gamma \pmod{\ell}$. This relation on complex cuspidal representations of G is then independent of the choice of γ .

4. MAIN THEOREM

We prove the Main Theorem of the introduction. As there, \mathcal{W}_F denotes the Weil group of a separable algebraic closure \overline{F}/F of F .

4.1. Let $\mathcal{G}_n(F)$ denote the set of equivalence classes of irreducible smooth $\overline{\mathbb{Q}}_\ell$ -representations of \mathcal{W}_F of dimension n , and $\mathcal{G}_n^{\text{fin}}(F)$ the set of $\sigma \in \mathcal{G}_n(F)$ such that $\det \sigma$ has finite order. For $\sigma \in \mathcal{G}_n^{\text{fin}}(F)$, the group $\sigma(\mathcal{W}_F)$ is finite and σ is effectively a representation of the Galois group of \overline{F}/F (cf. [4] 28.6). We may use the method of 1.1 to attach to σ an isomorphism class $[\sigma]_\ell^{\text{ss}}$ of smooth, n -dimensional semisimple representations of \mathcal{W}_F . For $\sigma_1, \sigma_2 \in \mathcal{G}_n^{\text{fin}}(F)$, we say that $\sigma_1 \equiv \sigma_2 \pmod{\ell}$ if $[\sigma_1]_\ell^{\text{ss}} = [\sigma_2]_\ell^{\text{ss}}$.

4.2. We recall some results from §1 of [6]. Let \mathcal{P}_F be the wild inertia subgroup of \mathcal{W}_F , and let $\widehat{\mathcal{P}}_F$ denote the set of equivalence classes of irreducible, smooth $\overline{\mathbb{Q}}_\ell$ -representations of \mathcal{P}_F . The group \mathcal{W}_F acts on $\widehat{\mathcal{P}}_F$ by conjugation.

Lemma 1. *Let $\alpha \in \widehat{\mathcal{P}}_F$.*

- (1) *The \mathcal{W}_F -isotropy group of α is of the form \mathcal{W}_E , for a finite, tamely ramified field extension E/F .*
- (2) *There exists a smooth representation ρ of \mathcal{W}_E such that $\rho|_{\mathcal{P}_F} \cong \alpha$.*

Proof. Parts (1) and (2) are given by [6] 1.2 Proposition and [6] 1.3 Proposition respectively. \square

In part (1) of Lemma 1, we use the notation $E = Z_F(\alpha)$.

Let σ be an irreducible, smooth $\overline{\mathcal{Q}}_\ell$ -representation of \mathcal{W}_F . The restriction $\sigma|_{\mathcal{P}_F}$ is then a direct sum of representations $\alpha \in \widehat{\mathcal{P}}_F$, any two of which are \mathcal{W}_F -conjugate. In other words, σ determines a unique \mathcal{W}_F -orbit $r_F^1(\sigma) \in \mathcal{W}_F \backslash \widehat{\mathcal{P}}_F$.

Lemma 2. *Let $\sigma_1, \sigma_2 \in \mathcal{G}_n^{\text{fin}}(F)$. If $\sigma_1 \equiv \sigma_2 \pmod{\ell}$, then $r_F^1(\sigma_1) = r_F^1(\sigma_2)$.*

Proof. Since \mathcal{P}_F is a pro- p group, the result follows from 1.4 Proposition. \square

If $m \geq 1$ is an integer and $\alpha \in \widehat{\mathcal{P}}_F$, we define $\mathcal{G}_m(F; \alpha)$ to be the set of equivalence classes of irreducible smooth representations of \mathcal{W}_F containing α with multiplicity m . For example, if $\mathbf{1}$ denotes the trivial representation of \mathcal{P}_F , then $\mathcal{G}_m(F; \mathbf{1})$ is the set of equivalence classes of irreducible, m -dimensional, *tamely ramified* representations of \mathcal{W}_F . In general, we write $\mathcal{G}_m^{\text{fin}}(F; \alpha)$ for the set of $\sigma \in \mathcal{G}_m(F; \alpha)$ such that $\det \sigma$ has finite order.

Notation. What we here call $\mathcal{G}_m(F; \alpha)$ is denoted in [6] by $\mathcal{G}_m(F; \mathcal{O})$, where $\mathcal{O} = \mathcal{O}_F(\alpha)$ is the \mathcal{W}_F -orbit of α . If E/F is a finite separable extension of F with $E \subset \bar{F}$, we denote by $\text{Ind}_{E/F}$ the smooth induction functor $\text{Ind}_{\mathcal{W}_E}^{\mathcal{W}_F}$.

Lemma 3. *Let $\alpha \in \widehat{\mathcal{P}}_F$, let $E = Z_F(\alpha)$, and let ρ be an irreducible smooth representation of \mathcal{W}_E such that $\rho|_{\mathcal{P}_F} \cong \alpha$. Let $m \geq 1$ be an integer. If $\tau \in \mathcal{G}_m(E; \mathbf{1})$, the representation $\Sigma_\rho(\tau) = \text{Ind}_{E/F} \rho \otimes \tau$ is irreducible, and the map*

$$\Sigma_\rho : \mathcal{G}_m(E; \mathbf{1}) \longrightarrow \mathcal{G}_m(F; \alpha)$$

is a bijection. If $\det \rho$ has finite order, then $\Sigma_\rho(\mathcal{G}_m^{\text{fin}}(E; \mathbf{1})) = \mathcal{G}_m^{\text{fin}}(F; \alpha)$.

Proof. The first two assertions combine 1.3 Proposition and 1.4 Theorem of [6]. For the final one, we recall that

$$\det \Sigma_\rho(\tau) = \delta \cdot (\det \rho \otimes \tau)|_{F^\times} = \delta \cdot \det \tau^{\dim \rho} \cdot \det \rho^{\dim \tau}|_{F^\times},$$

for a certain tamely ramified character δ of F^\times of order ≤ 2 : if $d_{E/F} = \det \text{Ind}_{E/F} 1_E$, then δ is $d_{E/F}^{\dim \rho \otimes \tau}$. The assertion follows. \square

Proposition. *In the context of Lemma 3, suppose that $\det \rho$ has finite order. If $\tau_1, \tau_2 \in \mathcal{G}_m^{\text{fin}}(E; \mathbf{1})$, then*

$$\Sigma_\rho(\tau_1) \equiv \Sigma_\rho(\tau_2) \pmod{\ell} \iff \tau_1 \equiv \tau_2 \pmod{\ell}.$$

Proof. Set $\sigma_i = \Sigma_\rho(\tau_i)$. Since \mathcal{P}_F is a pro- p group, the representation $[\alpha]_\ell^{\text{ss}}$ is irreducible. Therefore the representation $\tilde{\rho} = [\rho]_\ell^{\text{ss}}$ is irreducible. If μ is an irreducible $\overline{\mathcal{F}}_\ell$ -representation of \mathcal{W}_E , trivial on \mathcal{P}_F , 1.5 Proposition asserts that $\nu_\mu = \text{Ind}_{E/F} \tilde{\rho} \otimes \mu$ is irreducible, and that ν_μ determines μ . This implies first the relation $[\sigma_i]_\ell^{\text{ss}} = \text{Ind}_{E/F} \tilde{\rho} \otimes [\tau_i]_\ell^{\text{ss}}$, $i = 1, 2$, and then the result. \square

4.3. We follow a parallel path on the other side. For this, it will be convenient to use the language of endo-classes of simple characters: the theory is developed in [1], but the summary in [6] 2.2 will be adequate.

If $\pi \in \mathcal{A}_n(F)$, the simple characters contained in π lie in a single conjugacy class (3.1), and hence in a single endo-class which we denote $\vartheta(\pi)$.

Lemma. *Let $\pi_1, \pi_2 \in \mathcal{A}_n^{\text{fin}}(F)$. If $\pi_1 \equiv \pi_2 \pmod{\ell}$, then $\vartheta(\pi_1) = \vartheta(\pi_2)$.*

Proof. Let θ_i be an m -simple character occurring in π_i . The irreducible representation $[\pi_i]_\ell$ contains the $\overline{\mathbf{F}}_\ell$ -valued character $[\theta_i]_\ell$ of H_θ^1 , obtained by composing θ with the reduction map $\overline{\mathbf{Z}}_\ell^\times \rightarrow \overline{\mathbf{F}}_\ell^\times$ (2.2 Proposition). The isomorphism $[\pi_1]_\ell = [\pi_2]_\ell$ implies that the $[\theta_i]_\ell$ intertwine in G in the usual sense: there exists $g \in G$ such that the characters $[\theta_1]_\ell^g, [\theta_2]_\ell$ agree on $(H_{\theta_1}^1)^g \cap H_{\theta_2}^1$. Since the values of the θ_i are p -power roots of unity, the characters θ_1^g, θ_2 must agree on $(H_{\theta_1}^1)^g \cap H_{\theta_2}^1$. In other words, the θ_i intertwine in G . By the Intertwining Theorem of [7], they are endo-equivalent and $\vartheta(\pi_1) = \vartheta(\pi_2)$ as required. \square

Let $\mathcal{E}(F)$ denote the set of endo-equivalence classes of $\overline{\mathbf{Q}}_\ell$ -valued simple characters over F and, for $\Theta \in \mathcal{E}(F)$, let $\deg \Theta$ be the degree of Θ (cf. [6] (2.2.2)).

Let $\Theta \in \mathcal{E}(F)$ have degree dividing n , and set $m = n/\deg \Theta$. Let $\mathcal{A}_m(F; \Theta)$ be the set of $\pi \in \mathcal{A}_n(F)$ such that $\vartheta(\pi) = \Theta$. Choosing an m -simple character θ in G of endo-class Θ , a representation $\pi \in \mathcal{A}_n(F)$ lies in $\mathcal{A}_m(F; \Theta)$ if and only if it contains θ . We attach θ to a simple stratum $[\mathfrak{a}, \beta]$, and set $P = F[\beta]$.

For the next result, we consider a pair of representations $\pi_1, \pi_2 \in \mathcal{A}_m^{\text{fin}}(F; \Theta) = \mathcal{A}_n^{\text{fin}}(F) \cap \mathcal{A}_m(F; \Theta)$. Each π_i contains θ , so we let Λ_i be the natural representation of \mathbf{J}_θ on the θ -isotypic subspace of π_i . As in 3.2, we choose $\kappa \in \mathcal{H}(\theta)$ with $\kappa(F^\times)$ finite, and write $\Lambda_i = \kappa \otimes \lambda_i$, $\lambda_i \in \mathcal{J}_0(\theta)$. The representations $[\Lambda_i]_\ell^{\text{ss}}$ and $[\lambda_i]_\ell^{\text{ss}}$ are all irreducible, by 3.3 Proposition.

Proposition. *The representations π_i satisfy $\pi_1 \equiv \pi_2 \pmod{\ell}$ if and only if $[\lambda_1]_\ell^{\text{ss}} = [\lambda_2]_\ell^{\text{ss}}$.*

Proof. We have $[\Lambda_i]_\ell^{\text{ss}} = [\kappa]_\ell^{\text{ss}} \otimes [\lambda_i]_\ell^{\text{ss}}$. Consequently, if $[\lambda_1]_\ell^{\text{ss}} = [\lambda_2]_\ell^{\text{ss}}$, then $[\Lambda_1]_\ell^{\text{ss}} \cong [\Lambda_2]_\ell^{\text{ss}}$ and 3.4 Proposition implies $\pi_1 \equiv \pi_2 \pmod{\ell}$.

Conversely, suppose that $\pi_1 \equiv \pi_2 \pmod{\ell}$. The representation $[\Lambda_i]_\ell^{\text{ss}}$ is the natural representation of \mathbf{J}_θ on the $[\theta]_\ell$ -isotypic subspace of $[\pi_i]_\ell$, so $[\Lambda_1]_\ell^{\text{ss}} = [\Lambda_2]_\ell^{\text{ss}}$. The relation $[\Lambda_i]_\ell^{\text{ss}} = [\kappa]_\ell^{\text{ss}} \otimes [\lambda_i]_\ell^{\text{ss}}$ implies $[\lambda_1]_\ell^{\text{ss}} = [\lambda_2]_\ell^{\text{ss}}$, by 1.5 Proposition. \square

4.4. We prove the Main Theorem in a special case. It will be notationally convenient to use a base field E and consider representations of $\text{GL}_m(E)$, for some $m \geq 1$. Also, if K/F is any finite field extension, we denote by $X_1(K)$ the group of tamely ramified characters of K^\times .

Let $\mathbf{1}$ denote the trivial representation of \mathcal{P}_E and let $\mathbf{0}_E \in \mathcal{E}(E)$ be the endo-class of trivial simple characters. The Langlands correspondence then induces a bijection $\mathcal{G}_m(E; \mathbf{1}) \rightarrow \mathcal{A}_m(E; \mathbf{0}_E)$: this is well known but may be regarded as an instance of [6] 6.4 Corollary. We have to prove:

Proposition. *If $\sigma_1, \sigma_2 \in \mathcal{G}_m^{\text{fin}}(E; \mathbf{1})$, then $\sigma_1 \equiv \sigma_2 \pmod{\ell}$ if and only if ${}^L\sigma_1 \equiv {}^L\sigma_2 \pmod{\ell}$.*

Proof. We recall standard parametrizations of the sets $\mathcal{G}_m(E; \mathbf{1})$, $\mathcal{A}_m(E; \mathbf{0}_E)$. Let E_m/E be an unramified extension of degree m , and set $\Delta = \text{Gal}(E_m/E)$. Let $X_1(E_m)^{\Delta\text{-reg}}$ denote the set of Δ -regular characters $\chi \in X_1(E_m)$. For $\chi \in X_1(E_m)^{\Delta\text{-reg}}$, the representation $\sigma_\chi = \text{Ind}_{E_m/E} \chi$ is irreducible. It lies in $\mathcal{G}_m(E; \mathbf{1})$ and depends only on the Δ -orbit of χ . Indeed, the map

$$\begin{aligned} \Delta \backslash X_1(E_m)^{\Delta\text{-reg}} &\longrightarrow \mathcal{G}_m(E; \mathbf{1}), \\ \chi &\longmapsto \sigma_\chi, \end{aligned}$$

is a bijection. Under this map, elements of $\mathcal{G}_m^{\text{fin}}(E; \mathbf{1})$ correspond to characters of finite order.

On the other hand, let θ_0 denote the trivial character of $U_{\mathfrak{m}}^1$, where $\mathfrak{m} = M_m(\mathfrak{o}_E)$. Following [3] 2.2, we have a canonical bijection

$$\begin{aligned} \Delta \backslash X_1(E_m)^{\Delta\text{-reg}} &\longrightarrow \mathcal{T}(\theta_0), \\ \chi &\longmapsto \gamma_E(\chi), \end{aligned}$$

extending the Green parametrization. The set $\mathcal{T}(\theta_0)$ is in canonical bijection with $\mathcal{A}_m(E; \mathbf{0}_E)$ (as in 3.1 Proposition), so we have a further bijection

$$\begin{aligned} \Delta \backslash X_1(E_m)^{\Delta\text{-reg}} &\longrightarrow \mathcal{A}_m(E; \mathbf{0}_E), \\ \chi &\longmapsto \Gamma_E(\chi). \end{aligned}$$

We examine the behaviour of these constructions relative to congruences. A character $\chi \in X_1(E_m)$, of finite order, may be written uniquely in the form $\chi = \chi_{\text{reg}} \chi_{\text{sing}}$, where χ_{reg} has order relatively prime to ℓ and χ_{sing} has order a power of ℓ .

Lemma. *Let $\chi, \xi \in X_1(E_m)^{\Delta\text{-reg}}$ have finite orders. The following conditions are equivalent:*

- (1) χ_{reg} is Δ -conjugate to ξ_{reg} ;
- (2) $\sigma_\chi \equiv \sigma_\xi \pmod{\ell}$;
- (3) $[\gamma_E(\chi)]_\ell^{\text{ss}} = [\gamma_E(\xi)]_\ell^{\text{ss}}$;
- (4) $\Gamma_E(\chi) \equiv \Gamma_E(\xi) \pmod{\ell}$.

Proof. Let $\tilde{\chi}$ be the character obtained by composing χ with the reduction map $\overline{\mathbf{Z}}_\ell^\times \rightarrow \overline{\mathbf{F}}_\ell^\times$. In particular, $\tilde{\chi} = (\chi_{\text{reg}})^\sim$, so $\tilde{\chi}$ determines χ_{reg} completely. The restriction of $[\sigma_\chi]_\ell^{\text{ss}}$ to \mathcal{W}_{E_m} is $\sum_{\delta \in \Delta} \tilde{\chi}^\delta$. The same applies to σ_ξ , so (2) \Rightarrow (1).

The decomposition map $* \mapsto [*]_\ell^{\text{ss}}$ commutes with induction ([9] §21B). Identifying $\tilde{\chi}$ with $[\chi]_\ell^{\text{ss}}$, we therefore have $[\sigma_\chi]_\ell^{\text{ss}} = \text{Ind}_{E_m/E} [\chi]_\ell^{\text{ss}}$. That is, (1) \Rightarrow (2).

The equivalence of (3) and (4) is a case of 3.4 Proposition, so we consider the relation between (1) and (3). The representations $\gamma_E(\chi)$, $\gamma_E(\xi)$ agree on E^\times if and only if the characters $\chi|_{E^\times}$, $\xi|_{E^\times}$ are equal. The same applies to their reductions modulo ℓ . Further, $\gamma_E(\chi)$, $\gamma_E(\xi)$ are equivalent on $\text{GL}_m(\mathfrak{o}_E)$ if and only if the restrictions $\chi|_{U_{E_m}}$, $\xi|_{U_{E_m}}$ are Δ -conjugate. Appealing to [17] Theorem 3.6, the reductions modulo ℓ of these restrictions are equivalent if and only if $\tilde{\chi}|_{U_{E_m}}$ is Δ -conjugate to $\tilde{\xi}|_{U_{E_m}}$. Thus (1) is equivalent to (3). \square

To conclude the proof, we need only recall (from, for example, 2.4 Theorem 2 of [5]) that

$${}^L\sigma_\chi = \Gamma_E(\omega\chi), \quad \chi \in X_1(E_m)^{\Delta\text{-reg}},$$

where ω is the unramified character of E_m^\times such that

$$\omega(x) = (-1)^{(m-1)v_{E_m}(x)},$$

v_{E_m} being the normalized valuation $E_m^\times \rightarrow \mathbf{Z}$. \square

4.5. We return to $\Theta \in \mathcal{E}(F)$, θ and m , as in 4.3. We choose a simple stratum $[\mathfrak{a}, \beta]$ in $M_n(F)$ to which θ is attached and set $P = F[\beta]$. We let E/F be the maximal tamely ramified sub-extension of P/F . We choose $\kappa \in \mathcal{H}(\theta)$ and use it to define a bijection $\mathcal{A}_m(E; \mathbf{0}_E) \rightarrow \mathcal{A}_m(F; \Theta)$.

The lemma of 3.3 yields a bijection

$$\Pi_\kappa^P : \mathcal{A}_m(P; \mathbf{0}_P) \longrightarrow \mathcal{A}_m(F; \Theta),$$

as follows. Let G_P be the G -centralizer of P^\times and let θ_P denote the trivial character of $J_\theta^1 \cap G_P$. Thus θ_P is a trivial m -simple character in G_P and, if $\mu \in \mathcal{A}_m(P; \mathbf{0}_P)$, then μ contains a unique $\lambda_P \in \mathcal{T}(\theta_P)$. In the notation of 3.3, there is a unique representation $\lambda \in \mathcal{T}_0(\theta)$ such that $\lambda|_{J_\theta \cap G_P} = \lambda_P$. The tensor product $\lambda \otimes \kappa$ then lies in $\mathcal{T}(\theta)$ and we set

$$\Pi_\kappa^P(\mu) = c\text{-Ind}_{J_\theta}^G \lambda \otimes \kappa.$$

Next, let P_m/P be unramified of degree m , and set $\Delta = \text{Gal}(P_m/P)$. As in 4.4, we have the canonical bijection $\Gamma_P : \Delta \backslash X_1(P_m)^{\Delta\text{-reg}} \rightarrow \mathcal{A}_m(P; \mathbf{0}_P)$. Let E_m/F be the maximal tamely ramified sub-extension of P_m/F , so that E_m/E is unramified of degree m . We identify Δ with $\text{Gal}(E_m/E)$ by restriction. The extension P_m/E_m is totally wildly ramified, so composition with the field norm N_{P_m/E_m} induces a Δ -isomorphism $X_1(E_m) \rightarrow X_1(P_m)$. This in turn induces a bijection

$$\beta_{P/E} : \Delta \backslash X_1(E_m)^{\Delta\text{-reg}} \longrightarrow \Delta \backslash X_1(P_m)^{\Delta\text{-reg}}.$$

We so obtain a bijection

$$\begin{aligned} \mathfrak{b}_{P/E} : \mathcal{A}_m(E; \mathbf{0}_E) &\longrightarrow \mathcal{A}_m(P; \mathbf{0}_P), \\ \mu &\longmapsto \Gamma_P \circ \beta_{P/E} \circ \Gamma_E^{-1}(\mu). \end{aligned}$$

We now define

$$\begin{aligned} \Pi_\kappa : \mathcal{A}_m(E; \mathbf{0}_E) &\longrightarrow \mathcal{A}_m(F; \Theta), \\ \mu &\longmapsto \Pi_\kappa^P(\mathfrak{b}_{P/E} \mu). \end{aligned}$$

The map Π_κ is indeed a bijection and, if we choose κ so that $\kappa(F^\times)$ is finite, then Π_κ maps $\mathcal{A}_m^{\text{fin}}(E; \mathbf{0}_E)$ onto $\mathcal{A}_m^{\text{fin}}(F; \Theta)$.

Proposition. *Let $\kappa \in \mathcal{H}(\theta)$, and suppose that $\kappa(F^\times)$ is finite. If $\mu_1, \mu_2 \in \mathcal{A}_m^{\text{fin}}(E; \mathbf{0}_E)$, then*

$$\mu_1 \equiv \mu_2 \pmod{\ell} \iff \Pi_\kappa(\mu_1) \equiv \Pi_\kappa(\mu_2) \pmod{\ell}.$$

Proof. The proposition is just a re-statement of 4.3 Proposition. \square

4.6. We prove the Main Theorem in the general case. Let $\sigma_1, \sigma_2 \in \mathcal{G}_m^{\text{fin}}(F)$, and set $\pi_i = {}^L\sigma_i \in \mathcal{A}_n(F)^{\text{fin}}$, $i = 1, 2$.

We dispose of some trivial cases. If $r_F^1(\sigma_1) \neq r_F^1(\sigma_2)$, then $\sigma_1 \not\equiv \sigma_2 \pmod{\ell}$, by 4.2 Lemma 2. This implies, via 6.1 Theorem of [6], that $\vartheta(\pi_1) \neq \vartheta(\pi_2)$, whence $\pi_1 \not\equiv \pi_2 \pmod{\ell}$ (4.3 Lemma). This argument is reversible, so we need only consider the case $r_F^1(\sigma_1) = r_F^1(\sigma_2)$. In other words, there exists $\alpha \in \widehat{\mathcal{P}}_F$ such that $\sigma_1, \sigma_2 \in \mathcal{G}_m^{\text{fin}}(F; \alpha)$, for some integer $m \geq 1$. In light of 4.4, we need only treat the case where α is not trivial. We set $E = Z_F(\alpha)$.

According to 6.4 Corollary of [6], the set ${}^L(\mathcal{G}_m(F; \alpha))$ is of the form $\mathcal{A}_m(F; \Theta)$, for a unique $\Theta \in \mathcal{E}(F)$. In other words, $\pi_i \in \mathcal{A}_m^{\text{fin}}(F; \Theta)$, $i = 1, 2$. As in 4.5, we choose an m -simple character θ in $G = \text{GL}_n(F)$ of endo-class Θ , and attach it to a simple stratum $[\mathfrak{a}, \beta]$ with $P = F[\beta]$. According to the Tame Parameter Theorem of [6] 6.3, the maximal tamely ramified sub-extension of P/F is isomorphic to E/F . We fix such an isomorphism, and henceforward view E as a subfield of P . We choose a smooth representation ρ of \mathcal{W}_E such that $\rho|_{\mathcal{P}_F} \cong \alpha$ (as we may, by 4.2 Lemma 1) and also a representation $\kappa \in \mathcal{H}(\theta)$. We have a (non-commutative) diagram of bijections,

$$(4.6.1) \quad \begin{array}{ccc} \mathcal{G}_m(E; \mathbf{1}) & \xrightarrow{\Sigma_\rho} & \mathcal{G}_m(F; \alpha) \\ \downarrow & & \downarrow \\ \mathcal{A}_m(E; \mathbf{0}_E) & \xrightarrow{\Pi_\kappa} & \mathcal{A}_m(F; \Theta) \end{array}$$

in which the vertical arrows are given by the Langlands correspondence.

Lemma. *Let ρ be a smooth representation of \mathcal{W}_E such that $\rho|_{\mathcal{P}_F} \cong \alpha$. There exists a unique $\kappa = \kappa(\rho) \in \mathcal{H}(\theta)$ such that the diagram (4.6.1) commutes. If $\det \rho$ has finite order, then $\kappa(F^\times)$ is finite.*

Proof. The first assertion is a restatement of the Comparison Theorem of [6], while the second is immediate. \square

We take ρ, κ as in the lemma, with $\det \rho$ of finite order. For $\sigma_i \in \mathcal{G}_m^{\text{fin}}(F; \alpha)$, we write $\sigma_i = \Sigma_\rho(\tau_i)$, $\tau_i \in \mathcal{G}_m(E; \mathbf{1})$, $i = 1, 2$. The condition $\sigma_1 \equiv \sigma_2 \pmod{\ell}$ is equivalent to $\tau_1 \equiv \tau_2 \pmod{\ell}$, by 4.2 Proposition. This, in turn, is equivalent to ${}^L\tau_1 \equiv {}^L\tau_2 \pmod{\ell}$, by 4.4 Proposition. The lemma informs us that $\pi_i = \Pi_\kappa({}^L\tau_i)$, $i = 1, 2$, while 4.5 Proposition asserts that $\pi_1 \equiv \pi_2 \pmod{\ell}$ if and only if ${}^L\tau_1 \equiv {}^L\tau_2 \pmod{\ell}$. This completes the proof of the Main Theorem. \square

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