COMMUTING EXPONENTIALS IN DIMENSION AT MOST 3

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Abstract. Let A, B be two square complex matrices of dimension at most 3. We show that the following conditions are equivalent
i) There exists a finite subset $U \subset \mathbb{N}_{\geq 2}$ such that for every $t \in \mathbb{N} \setminus U$, $\exp(tA+B) = \exp(tA)\exp(B) = \exp(B)\exp(tA)$.
ii) The pair (A, B) has property L of Motzkin and Taussky and $\exp(A+B) = \exp(A)\exp(B) = \exp(B)\exp(A)$.

1. Introduction

Notation. *i)* We denote by $\mathbb{N}$ the set of positive integers and, if $n \in \mathbb{N}$, by $I_n, 0_n$ the identity matrix and the zero-matrix of dimension n.
ii) If X is a complex square matrix, then $s(X)$ refers to the spectrum of X.

Definition. *The $n \times n$ complex matrices A, B are said to be simultaneously triangularizable (ST) if there exists an $n \times n$ invertible matrix P such that $P^{-1}AP$ and $P^{-1}BP$ are upper triangular matrices.*

In [1], the author dealed with square matrices $A, B \in \mathcal{M}_n(\mathbb{C})$, $(n = 2$ or $3)$, satisfying

$$\text{(1)} \qquad \text{for every } t \in \mathbb{N}, \exp(tA+B) = \exp(tA)\exp(B) = \exp(B)\exp(tA)$$

The author concluded that these matrices are simultaneously triangularizable.
The result is true for $n = 2$. However it is false for $n = 3$. Indeed J.L. Tu communicated to the author the following counter-example

$$\text{(2)} \qquad A_0 = 2i\pi \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}, B_0 = 2i\pi \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & 1 & 0 \end{pmatrix}.$$

Clearly A_0, B_0 are not ST. However it is easy to see that, for every $t \in \mathbb{C}$, the eigenvalues of $tA_0 + B_0$ are the entries of its diagonal. Moreover, for every $t \in \mathbb{N}$, the eigenvalues of $tA_0 + B_0$ belong to $2i\pi\mathbb{Z}$ and are distinct. Therefore, for every $t \in \mathbb{N}$,

$$\exp(A_0) = \exp(B_0) = \exp(tA_0 + B_0) = I_3.$$

Definition. [6, Property L] *A pair $(A, B) \in \mathcal{M}_n(\mathbb{C})^2$ has property L if there exist orderings of the eigenvalues $(\lambda_i)_{i \leq n}, (\mu_i)_{i \leq n}$ of A, B such that for all $(x, y) \in \mathbb{C}^2$, $s(xA + yB) = (x\lambda_i + y\mu_i)_{i \leq n}$.*

Remark. *If A, B are ST, then the pair (A, B) has property L. The converse is false but for $n = 2$ (see [6]).*

Recently, in [8, Proposition 5], C. de Seguins Pazzis proved this result

Proposition 1. *Let $A, B \in \mathcal{M}_n(\mathbb{C})$ satisfying Condition (1). The pair (A, B) has property L.*

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We are interested in the converse of Proposition 1. We can wonder whether the conditions $e^Ae^B = e^Be^A = e^{A+B}$ and (A, B) has property L imply Condition (1). The answer is no as the following shows
Counter-example. The pair $(A_0, -2B_0)$ (cf Example (2)) has property L and $\exp(A_0) = \exp(-2B_0) = I_3$. Moreover, for $t \in \mathbb{N} \setminus \{2, 3, 4\}$, one has $\exp(tA_0 - 2B_0) = I_3$, and for $t \in \{2, 3, 4\}$, one has not. Therefore Condition (1) does not hold for this pair.
Thus, we weaken Condition (1) as follows
There exists a finite subset $U \subset \mathbb{N}_{n\geq 2}$ such that

$$\forall t \in \mathbb{N} \setminus U, \exp(tA + B) = \exp(tA)\exp(B) = \exp(B)\exp(tA). \tag{3}$$

In this paper, we show that, in dimension 2 or 3, the pair (A, B) satisfies Condition (3) if and only if $e^{A+B} = e^Ae^B = e^Be^A$ and (A, B) has property L.

2. Property L and Condition (3)

The following is a partial converse of Proposition 1.

Proposition 2. *Assume that $A = diag(\lambda_1, \cdots, \lambda_n) \in \mathcal{M}_n(\mathbb{C})$ has n distinct eigenvalues in $2i\pi\mathbb{Z}$, that $B = [b_{ij}] \in \mathcal{M}_n(\mathbb{C})$ (where for every $i \leq n$, $b_{ii} \in 2i\pi\mathbb{Z}$) is diagonalizable and that the pair (A, B) has property L. Then the pair (A, B) satisfies Condition (3).*

Proof. Note that $e^A = I_n$. According to [6, Theorem 1], for every $t \in \mathbb{C}$,

$$s(tA + B) = (t\lambda_i + b_{ii})_{i\leq n}.$$

Thus $e^B = I_n$. For almost all $t \in \mathbb{N}$, $tA + B$ has n distinct eigenvalues in $2i\pi\mathbb{Z}$ and $\exp(tA + B) = I_n$. □

Definition. *i) Let $A \in \mathcal{M}_n(\mathbb{C})$. The spectrum of A is said to be $2i\pi$ congruence-free ($2i\pi$ CF) if, for all $\lambda, \mu \in s(A)$, $\lambda - \mu \notin 2i\pi\mathbb{Z}^*$.*
ii) For every $z \in \mathbb{C}$, $\Im(z)$ denotes its imaginary part.
iii) Let $\log : GL_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ be the (non continuous) primary matrix function (cf. [2]) associated to the principal branch of the logarithm, defined by $\Im(log(z)) \in (-\pi, \pi]$, for every $z \in \mathbb{C}^*$. Thus for every $X \in GL_n(\mathbb{C})$, $s(\log(X)) \subset \{z \in \mathbb{C} | \Im(z) \in (-\pi, \pi]\}$.

Lemma 1. *Let $A \in \mathcal{M}_n(\mathbb{C})$. There exists a unique pair (F, Δ) of square $n \times n$ matrices, that are polynomials in A, such that*

$$A = F + \Delta, e^F = e^A, e^\Delta = I_n \text{ and for all } \lambda \in s(F), \Im(\lambda) \in (-\pi, \pi].$$

Proof. Necessarily $F = \log(e^A)$. Let $f : x \in U \to e^x \in \mathbb{C}$ where U is a neighborhood of $s(F)$. Then f is a holomorphic function that is one to one on U and such that f' is not zero on U. According to [4, Theorem 2], F is a polynomial in $e^F = e^A$. Therefore F is a polynomial in A. Let $\Delta = A - F$. One has $AF = FA$ and $e^\Delta = e^Ae^{-F} = I_n$. □

Remark. *Note that $s(F)$ is $2i\pi$ CF, Δ is diagonalizable and $s(\Delta) \subset 2i\pi\mathbb{Z}$.*

The following two results concern the equation

$$e^{A+B} = e^Ae^B = e^Be^A$$

in dimension 3.

Proposition 3. *Let (A, B) be a pair of 3×3 complex matrices such that $e^{A+B} = e^Ae^B = e^Be^A$ and $AB \neq BA$. If $\mathbb{C}^3$ is an indecomposable $< A, B >$ module, then there exist a complex number σ and two 3×3 complex matrices Δ and F, that are polynomials in A, such that $A = \sigma I_3 + \Delta + F$ with $e^\Delta = I_3, F^2 = 0_3$. In the same way, $B = \tau I_3 + \Theta + G$ with $e^\Theta = I_3, G^2 = 0_3$. Moreover $FG = GF$.*

Remark. *It can be derived from* [5, Case (I) p. 165-166]. *However we give an alternative proof.*

Proof. According to [7], $s(A), s(B)$ are not $2i\pi$ CF. Moreover the equality
$$e^{A+B}e^{-A} = e^{-A}e^{A+B} = e^{B}$$
implies that $s(A+B)$ is not $2i\pi$ CF. By Lemma 1, one has $A = F + \Delta, B = G + \Theta$ where $e^F = e^A, e^G = e^B$. Thus $e^F e^G = e^G e^F$. According to [9], $FG = GF$. It remains to show that there exists a complex number σ such that $(F - \sigma I_3)^2 = 0_3$. Assume that the minimal polynomial of F has degree 3, that is, in the Jordan normal form of F, there is exactly one Jordan block associated to each eigenvalue of F. Obviously $s(A)$ is $2i\pi$ CF, and one obtains a contradiction. Since $s(A)$ is not $2i\pi$ CF, F has an eigenvalue with multiplicity at least 2 and its minimal polynomial is of degree at most two. We may assume that $s(F) = \{0, 0, *\}$. Up to similarity, F is one of the following three forms.
$$\begin{aligned} F &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix} \text{ where } \lambda \neq 0, \\ F &= 0_3 \\ \text{or } F &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$
In the last two cases, we are done. It remains to show that if $F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, where $\lambda \neq 0$, then we obtain a contradiction. Note that
$$e^{F+G} = e^F e^G = e^A e^B = e^{A+B}.$$
Therefore, if $s(F+G) \subset (-\pi, \pi]$, then $F + G = \log(e^{A+B})$. Clearly $F+G$ has also an eigenvalue with multiplicity at least 2 and its minimal polynomial is of degree at most two. The matrices F, G commute and, in the same way than for F, we can prove that G is similar to one of the previous three forms. We obtain three possible values

Case 1 : $G = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$. Then $\mathbb{C}^3$ is decomposable.

Case 2 : $G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. One has $F+G = \log(e^{A+B})$ but its minimal polynomial is of degree 3, that is a contradiction.

Case 3 : $G = \begin{pmatrix} \nu & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ where $\nu \neq 0$. We have $F+G = \log(e^{A+B})$ and necessarily $\nu = \lambda$. Moreover $s(F+G)$ is $2i\pi$ CF and $e^{F+G} = e^{F+G+\Delta+\Theta}$. According to [3], $F+G$ and $\Delta + \Theta$ commute. We conclude that Δ and Θ are diagonal matrices and that $AB = BA$. That is a contradiction. □

Proposition 4. *Let* (A, B) *be a pair of* 3×3 *complex matrices such that*
$$e^{A+B} = e^A e^B = e^B e^A,$$
$AB \neq BA$ *and such that* $\mathbb{C}^3$ *is an indecomposable* $< A, B >$ *module. Then the pair* (A, B) *has the property*

(∗) *The Jordan-Chevalley decompositions of* $A, B, A+B$ *are in the form*

$$A = (\sigma I_3 + \Delta) + F, \tag{4}$$
$$B = (\tau I_3 + \Theta) + G, \tag{5}$$
$$A+B = ((\sigma+\tau)I_3 + \Delta + \Theta) + (F+G) \tag{6}$$

with the following equalities

$$\begin{aligned} F^2 &= G^2 = FG = GF = 0_3, \\ e^{\Delta} &= e^{\Theta} = e^{\Delta+\Theta} = I_3 \\ \text{and } [F,\Theta] &= [\Delta, G]. \end{aligned}$$

Conversely, if the pair (A, B) *has property* (∗)*, then* $e^{A+B} = e^A e^B = e^B e^A$*.*

Proof. We use the notations and results of Proposition 3. Note that $\sigma I_3 + \Delta$ is diagonalizable, F is nilpotent and these matrices are polynomials in A. Thus (4) and (5) are the Jordan-Chevalley decompositions of A, B. Moreover

$$\begin{aligned} e^A &= e^{\sigma}(I_3 + F), \\ e^B &= e^{\tau}(I_3 + G), \\ \text{and } e^{A+B} &= e^{\sigma+\tau}(I_3 + F + G + FG) \end{aligned}$$

with $FG = GF$. Thus $F + G + FG$ is nilpotent. According to the proof of Proposition 3, $A + B = (\omega I_3 + \Sigma) + O$ with $O\Sigma = \Sigma O, e^{\Sigma} = I_3, O^2 = 0_3$. One has $e^{A+B} = e^{\omega}(I_3 + O)$ and then $e^{\omega} = e^{\sigma+\tau}, O = F + G + FG$. Finally $O^2 = 0_3$ implies that $FG = 0_3$ and (6) is the Jordan-Chevalley decomposition of $A + B$. Since $\Delta + \Theta$ and $F + G$ commute, one has $[F, \Theta] = [\Delta, G]$. Obviously $e^{\Delta+\Theta} = I_3$. The last assertion is clear. □

Our main result, in dimension two, is as follows

Theorem 1. *Let* (A, B) *be a pair of* 2×2 *complex matrices. Then* (A, B) *satisfies Condition (3) if and only if* $e^{A+B} = e^A e^B = e^B e^A$ *and* (A, B) *has property L.*

Proof. (⇒) There exists $t_0 \in \mathbb{N}$ such that Condition (3) holds for every $t \geq t_0$. According to Proposition 1, the pair $(t_0 A, B)$ has property L and (A, B) too.
(⇐) Suppose $AB \neq BA$. According to [7], $s(A)$ and $s(B)$ are not $2i\pi$ CF and, since $n = 2$, A, B are diagonalizable. An homothety can be added to A or B and we may assume $A = \begin{pmatrix} 2i\pi\lambda & 0 \\ 0 & 0 \end{pmatrix}$, $s(B) = \{2i\pi\mu, 0\}$, where $\lambda, \mu \in \mathbb{Z}^*$. Again since $n = 2$, A and B are ST, that is, they have a common eigenvector. Thus we may assume $B = \begin{pmatrix} 2i\pi\mu & 1 \\ 0 & 0 \end{pmatrix}$ (eventually replacing λ with $-\lambda$ or μ with $-\mu$). Note that $e^A e^B = e^{A+B}$ if and only if $\lambda + \mu \neq 0$. If $t \in \mathbb{N}$, we obtain

$$e^{tA} e^B = e^B e^{tA} = e^{tA+B},$$

but eventually if $t = -\mu/\lambda$. □

Remark. *The pair* $A = i\pi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, B = \pi \begin{pmatrix} -11i & 6 \\ 16 & 11i \end{pmatrix}$ *satisfies the condition* $e^{A+B} = e^A e^B = e^B e^A$ *but has not property L.*

We prove our main result in dimension 3.

Theorem 2. *Let* (A, B) *be a pair of* 3×3 *complex matrices. Then* (A, B) *satisfies Condition (3) if and only if* $e^{A+B} = e^A e^B = e^B e^A$ *and* (A, B) *has property L.*

Proof. ($\Rightarrow$) Use the same argument than in the proof of the necessary condition of Theorem 1.
($\Leftarrow$) Assume that the pair (A, B) has property L, $AB \neq BA$ and

$$e^{A+B} = e^A e^B = e^B e^A.$$

• If $\mathbb{C}^3$ is a decomposable $< A, B >$ module, we conclude using Theorem 1.
• Now $\mathbb{C}^3$ is an indecomposable $< A, B >$ module.
i) The pair (A, B) has property $(*)$. Using notations of Proposition 4, we obtain for every $t \in \mathbb{N}$,

$$\begin{aligned} e^{tA} &= e^{t\sigma}(I_3 + tF), \\ e^{tA}e^B &= e^B e^{tA} = e^{t\sigma+\tau}(I_3 + tF + G), \\ e^{tA+B} &= e^{t\sigma+\tau}e^{t\Delta+\Theta}(I_3 + tF + G). \end{aligned}$$

Thus $e^{tA+B} = e^{tA}e^B = e^B e^{tA}$ if and only if $e^{t\Delta+\Theta} = I_3$.
ii) The pair $(\Delta + F, \Theta + G)$ has property L. We consider the associated orderings $s(\Delta + F) = s(\Delta) = (\lambda_i)_{i\leq 3}$ and $s(\Theta + G) = s(\Theta) = (\mu_i)_{i\leq 3}$. If $t \in \mathbb{C}$, one has

$$s(t(\Delta + F) + \Theta + G) = s((t\Delta + \Theta) + (tF + G)) = (t\lambda_i + \mu_i)_{i\leq 3}.$$

Since $t\Delta+\Theta$ commute with the nilpotent matrix $tF+G$, $s(t\Delta+\Theta) = (t\lambda_i+\mu_i)_{i\leq 3}$ and the pair (Δ, Θ) has property L.
iii) Since $s(\Delta) \subset 2i\pi\mathbb{Z}, s(\Theta) \subset 2i\pi\mathbb{Z}$, if $t \in \mathbb{N}$, then $s(t\Delta + \Theta) \subset 2i\pi\mathbb{Z}$. Thus it remains to prove that, for almost all $t \in \mathbb{N}$, $t\Delta + \Theta$ is diagonalizable. If Δ and Θ commute, we are done.
We assume that Δ and Θ do not commute. Suppose that, for an infinite number of values of $t \in \mathbb{N}$, $t\Delta + \Theta$ is not diagonalizable. Then, for theses values of t, $(t\lambda_i+\mu_i)_{i\leq 3}$ contains at least two equal elements. Thus, for instance, for an infinite number of values of t, $t\lambda_1 + \mu_1 = t\lambda_2 + \mu_2$. This implies that $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$ and we may assume that these eigenvalues are 0. Therefore the associated orderings are $s(\Delta) = \{0, 0, \lambda\}$ where $\lambda \in 2i\pi\mathbb{Z}^*$ and $s(\Theta) = \{0, 0, \mu\}$ where $\mu \in 2i\pi\mathbb{Z}^*$. We may assume that $\Delta = \text{diag}(0, 0, \lambda)$. According to [6, Theorem 1],

$$\Theta = \begin{pmatrix} W & \begin{pmatrix} u \\ v \end{pmatrix} \\ \begin{pmatrix} p & q \end{pmatrix} & \mu \end{pmatrix}$$

where W is a nilpotent 2×2 matrix and u, v, p, q are complex numbers. We know that Θ and $\Delta + \Theta$ are diagonalizable, that is, their rank is 1 and $\lambda + \mu \neq 0$. It remains to show that, except for a finite number of values of $t \in \mathbb{N}$, $\text{rank}(tA+B) = 1$ and $t\lambda + \mu \neq 0$.
Case 1. $W = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Therefore $\text{rank}(\Theta) = 1$ implies $p = v = 0, \mu = qu$. Then $\text{rank}(\Delta + \Theta) = 1$ implies $\lambda = 0$, a contradiction.
Case 2. $W = 0_3$. Therefore $\text{rank}(\Theta) = \text{rank}(\Delta + \Theta) = 1$ implies that

$$pu = pv = qu = qv = 0.$$

The previous condition implies that $\text{rank}(t\Delta + \Theta) = 1$, but for $t = -\mu/\lambda$. □

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