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THE BARTH QUINTIC SURFACE HAS PICARD NUMBER 41

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Dedicated to Wolf Barth

1. Introduction

Quintic surfaces in $\mathbb{P}_3$ have been studied extensively by Barth and others, for instance with a view towards configurations of singularities or lines contained in them. This paper investigates a specific smooth quintic surface suggested by Barth for it contains the current record of 75 lines over $\mathbb{C}$ (see also [16]). In what follows the surface will be denoted by S_a. Our main incentive is to prove that over $\mathbb{C}$ the quintic S_a has Picard number 41 (Theorem 2.2). To the best of our knowledge this is the record Picard number for smooth quintics. In fact the surfaces with Picard number 43 or 45 exhibited in [10] involved several rational double point singularities. The previous record of 37 was attained by the Fermat quintic surface which also contains 75 lines (Remark 2.3).

This note is organised as follows. Section 2 reviews the surface S_a inside a pencil of quintics with an action of the symmetric group $\mathfrak{S}_5$. Sections 3 and 4 derive lower and upper bounds for the Picard number by exhibiting certain quotient surfaces (Godeaux and K3). As a by-product we prove the Tate conjecture for any non-degenerate member of the pencil of quintics (Proposition 4.8). Throughout we keep the exposition as characteristic free as possible. This also enables us to work out an explicit non-classical Godeaux surface (Proposition 3.1) compared to Miranda's implicit results in [8].

2. A pencil of $\mathfrak{S}_5$-invariant quintics in $\mathbb{P}_3$

In this note we consider certain surfaces that belong to the pencil of quintics

$$S_\lambda \quad : \quad \left\{ s_1 = \frac{5}{6}\lambda s_2 \cdot s_3 + s_5 = 0 \right\} \subset \mathbb{P}_4, \quad \lambda \in K\,, \tag{1}$$

where s_k stands for the symmetric polynomial

$$s_k := x_0^k + x_1^k + x_2^k + x_3^k + x_4^k \qquad (k \in \mathbb{N})$$

and K denotes an algebraically closed field of any characteristic. Mostly we will be concerned with the case $K = \mathbb{C}$, but our methods to investigate these surfaces will use reduction modulo different primes, and in fact we will also derive results exclusive to positive characteristic. The factor of 5/6 in front

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of λ might seem unnatural at first, but in fact it allows us to derive proper pencils in characteristics $2,3,5$ by substituting s_1 in the quintic polynomial and eliminating common factors. It should be understood that we always work with such a proper model of the pencil in the sequel.

The above pencil (albeit without the extra factor) was studied by Barth in order to find a quintic with three-divisible set of cusps ([4]) and smooth quintics with many lines ([5]). For the convenience of the reader we list below the facts from [4], [5] that we will use in the sequel. All of them can be verified by straightforward computation (possibly with help of a computer program), and related properties also appear in [16] (see Remark 2.4).

Observe that if we denote by B_{10} (resp. B_{15}) the curve in $\mathbb{P}_4$ given by $s_1 = s_5 = s_2 = 0$ (resp. $s_1 = s_5 = s_3 = 0$), then the base locus of the pencil in question is the curve $B_{10} \cup B_{15}$. One can check by direct computation that the curve B_{15} consists of the 15 lines

$$x_{i_1} = x_{i_2} + x_{i_3} = x_{i_4} + x_{i_5} = 0, \tag{2}$$

where i_1, i_2, i_3, i_4, i_5 are pairwise different, i.e. $\{i_1, i_2, i_3, i_4, i_5\} = \{0, 1, 2, 3, 4\}$. Similarly, the curve B_{10} is the union of the five conics C_{i_1} (smooth outside characteristic 2)

$$x_{i_1} = x_{i_2}^2 + x_{i_3}^2 + x_{i_4}^2 + x_{i_2}x_{i_3} + x_{i_2}x_{i_4} + x_{i_3}x_{i_4} = s_1 = 0. \tag{3}$$

Therefore, the plane $x_{i_1} = s_1 = 0$ meets the base locus along the three lines (2) and the conic (3). In particular, the four curves are the only components of intersection of the plane $x_{i_1} = s_1 = 0$ with an irreducible quintic S that belongs to the pencil.

Lemma 2.1. *A general member of the pencil $\{S_\lambda\}$ is smooth.*

Proof. It suffices to show that the pencil contains a smooth quintic. The Jacobi criterion reveals that the special member at $\lambda = 0$ is smooth outside characteristics $3, 13, 17$. For the exceptional characteristics, the computations are greatly simplified by arguing with K3 quotients as in 4.1. Indeed the pencil of elliptic fibrations (12) is non-degenerate in any odd characteristic, so the argumentation from Lemma 4.3 and Corollary 4.4 applies, for instance to S_3 for $p = 13, 17$ and $S_{\sqrt{-1}}$ for $p = 3$ (confer Remark 4.5). □

For the special quintic with 75 lines we introduce the following notation:

$$a \quad := \quad -\frac{2}{b+2}, \qquad \text{where} \quad b^4 - b^3 + 1 = 0. \tag{4}$$

Throughout the paper S_a stands for the surface given by (1) with $\lambda = a$ (over $\mathbb{C}$ unless specified otherwise). In the sequel we shall often write S instead of S_λ when there is no ambiguity from the context.

Suppose that $K = \mathbb{C}$. Then, by [5] (see also [16, Table, p. 2070])

$$\text{the surface } S_\lambda \text{ is smooth exactly for } \lambda \neq -1, -\frac{3}{2}, -\frac{51}{50}, -\frac{13}{25}, -\frac{1}{2}. \tag{5}$$

In particular, the surface S_a is smooth over $\mathbb{C}$ (for positive characteristic see Corollary 4.4). One directly verifies that S_a contains the line

$$\text{span}(\{(1:-1:b:-b:0),((b-1):1:-(b-1):0:-1)\}), \tag{6}$$

with b given by (4). In fact, the $\mathfrak{S}_5$-action endows S_a with 60 lines obtained from (6) by virtue of the symmetries. Thus we already have a good portion of divisors on S_a. Our main result for this paper is:

Theorem 2.2. *Over $\mathbb{C}$, the quintic S_a has Picard number 41.*

Remark 2.3. To the best of our knowledge, the Picard number 41 of S_a gives a new record among smooth complex quintics. In comparison, 43 and 45 have so far only been realised by desingularisations of quintics with rational double point singularities in [10]. The previous record of 37 was attained by the Fermat quintic, so Theorem 2.2 also gives an alternative way to see that S_a and the Fermat quintic surface cannot be isomorphic over $\mathbb{C}$. In fact the surfaces differ also in another respect: the Fermat quintic has NS generated by lines (even over $\mathbb{Z}$ by [12]) while any basis of $\mathrm{NS}(S_a)$ includes some other divisor class that will be made visible in the proof of Lemma 3.2.

The proof of Theorem 2.2 proceeds in two steps. First we derive the lower bound $\rho(S_a) \geq 41$ by exhibiting a suitable quotient surface Q of S by a cyclic group of order 5 (a Godeaux surface studied in section 3). Then we establish the upper bound $\rho(S_a) \leq 41$ through a quotient surface X that is K3 in section 4. Here we use reduction modulo different primes and the Artin-Tate conjecture in a technique following van Luijk [15].

Remark 2.4. For $K = \mathbb{C}$ the pencil $\{S_\lambda\}$ has also been studied in [16]. By [16, Thm 1.2] the pencil in question contains (up to Galois conjugation) exactly three smooth surfaces that carry a line given by (6). Moreover, no quintic in the pencil (1) contains more than 75 lines and the surface S_a is the unique (up to Galois conjugation) element of the pencil which carries the maximal number of lines. We will not use that result in the sequel, but it certainly motivates our interest in the quintic S_a.

3. Lower bound – Godeaux quotient

In this section we derive the lower bound $\rho(S_a) \geq 41$ (Lemma 3.2). At first we exhibit a Godeaux surface Q that arises from S as a quotient by a cyclic group of order 5 acting without fixed points. Then a close examination of the 75 lines on S_a and their images under the quotient map implies the inequality in question.

Consider the automorphism $R : \mathbb{P}_4 \to \mathbb{P}_4$ defined as

$$R(x_0 : x_1 : x_2 : x_3 : x_4) := (x_4 : x_0 : x_1 : x_2 : x_3).$$

Outside characteristic 5, R has five fixed points: $(1 : \varepsilon_5^k : \varepsilon_5^{2k} : \varepsilon_5^{3k} : \varepsilon_5^{4k})$ where $k \in \{0, 1, 2, 3, 4\}$ and $\varepsilon_5 \neq 1$ is a root of unity of order five. Clearly each member of the pencil $\{S_\lambda\}$ is invariant under R, so we can restrict R to S_λ and compute the fixed points. One directly sees that $s_l(1,\dots,\varepsilon_5^{4k})=0$ for all $k\neq 0$ and $5\nmid l$, whereas $s_{5l}(1,\dots,\varepsilon_5^{4k})=s_l(1,\dots,1)=5$ for each k and l. In conclusion none of the fixed points of R belong to S_λ for any $\lambda\in K$. In characteristic 5, there is only one fixed point $(1:1:1:1:1)$ which is easily verified to lie outside any quintic S_λ.

The automorphism R generates a subgroup $\mathfrak{C}_5\subset\mathfrak{S}_5\subset \operatorname{Aut}(S)$. Assume that the quintic S is smooth (or replace it by the minimal desingularisation if it is non-degenerate, i.e. it has only rational double points as singularities), then

$$\text{the quotient surface } Q:=S/_{\mathfrak{C}_5} \text{ is smooth.} \tag{7}$$

We thus obtain a Godeaux surface. If $\operatorname{char}(K)\neq 5$, we can almost verbatim repeat the considerations of [3, Example 9.6.2] to show that Q is a minimal surface of general type with $\operatorname{Pic}^\tau(Q)\cong\mathbb{Z}/5\mathbb{Z}$ and the following invariants:

$$h^1(\mathcal{O}_Q)=h^2(\mathcal{O}_Q)=0, \text{ and } \mathrm{K}_Q^2=1. \tag{8}$$

In characteristic 5, however, the invariants differ as Q is a non-classical Godeaux surface with $\operatorname{Pic}^\tau(Q)\cong\mu_5$. Namely, because $\mathfrak{C}_5\cong\mathbb{Z}/5\mathbb{Z}$, one finds as in [8]

$$h^1(\mathcal{O}_Q)=h^2(\mathcal{O}_Q)=\mathrm{K}_Q^2=1.$$

Remember that S_0 has a smooth model in characteristic 5. As opposed to the implicit result of [8], this yields an explicit non-classical Godeaux surface in characteristic 5:

Proposition 3.1. *In characteristic five, Q_0 is a non-classical Godeaux surface.*

We shall now turn to the Picard group of the special quintic S_a. Our previous considerations put us in the position to derive a geometric lower bound for the Picard number. We state the result here only over $\mathbb{C}$. The argument goes through without essential modifications in positive characteristic as well, but there we will derive better bounds in relation with the Tate conjecture (see Remark 3.3 and Corollary 4.9).

Lemma 3.2. *Over $\mathbb{C}$ the following inequality holds*

$$\rho(S_a)\geq 41.$$

Proof. The surface S_a carries the 15 lines (2). Moreover, it contains the 60 lines obtained by action of symmetries on the line (6). Let M be the Gram matrix of the 75 lines in question. By direct computation we obtain

$$\operatorname{rank}(M)=40.$$

Observe that both $\mathcal{O}_{S_a}(3)$ and the divisors $3C_i$, where $i=0,\dots,4$, lie in the span of the 75 lines (see section 2).

Let $\pi: S_a\to Q_a$ be the quotient map and let L, L' be two of the 75 lines on S_a. From the equality

$$\pi(L).\pi(L')=\frac{1}{5}\left(\sum_{i=0}^{4}R^i(L)\right).\left(\sum_{i=0}^{4}R^i(L')\right)$$

we can compute the Gram matrix N of the 15 divisors on Q_a that are the images of the 75 lines on S_a under the quotient map π. A direct computation gives

$$\operatorname{rank}(N) = 8. \tag{9}$$

On the other hand, by (8) and Noether's formula, we compute the topological Euler number (or Euler-Poincaré characteristic) $e(Q_a) = 11$. Since we work over $\mathbb{C}$, inequality (8) and Lefschetz' theorem on $(1,1)$-classes yield

$$\rho(Q_a) = b_2(Q_a) = 9.$$

Thus by (9) there is an R-invariant divisor on S_a whose class in $\mathrm{NS}(S_a)$ is not contained in the $\mathbb{Q}$-span of the 75 lines. Therefore $\rho(S_a) \geq 41$. □

Remark 3.3. a) In positive characteristics where S_a is smooth, exactly the same argument goes through after lifting Q_a to $\mathbb{C}$ which implies the analogous (in)equalities (or use reduction modulo $\mathfrak{p}$). Those characteristics where S_a attains singularities require some extra care.
b) K3 quotients and the Tate conjecture will allow us to derive better, and in fact precise estimates for the Picard numbers $\rho(S_a \otimes \bar{\mathbb{F}}_p)$ regardless of the (rational double point) singularities (Corollary 4.9).

Remark 3.4. Alternatively, one could argue with the induced action of R on the holomorphic 2-form over $\mathbb{C}$. As we will infer in (10), this implies that the transcendental lattice $T(S)$ generally has 4-divisible rank. Consequently $\rho(S) \equiv 1 \mod 4$, so that for S_a our lower bound $\rho \geq 40$ coming from the lines on S_a automatically improves to the bound of Lemma 3.2. In our eyes, the previous proof has the advantages of a constructive nature and relative independence of the characteristic (as sketched in Remark 3.3.a)).

4. Upper bound – quotient K3 surface

4.1. In order to complete the proof of Theorem 2.2 it remains to establish the upper bound $\rho(S_a) \leq 41$ over $\mathbb{C}$. Here we shall crucially use the $\mathfrak{S}_5$ action on the pencil S. Consider the transcendental part $T(S)$ of $H^2(S)$ obtained as the orthogonal complement of $\mathrm{NS}(S)$ with respect to the intersection pairing and understood as Hodge structure or as a Galois representation. From the operation of $\mathfrak{S}_5$ on the regular 2-forms on S, we infer the splitting

$$T(S) = V^4. \tag{10}$$

Here V will be made visible on a K3 quotient X of S that we exhibit below.

Recall the special member S_a. Since a quintic S has $b_2(S) = 53$, we know by Lemma 3.2 that $T(S_a)$ has rank at most 12. On the other hand, $\operatorname{rank}(T(S_a)) \geq 8$ by Lefschetz' theorem. In view of the splitting $T(S_a) = V_a^4$ there are only two possibilities remaining: rank 8 or 12. Our goal is to prove that the latter alternative holds:

Lemma 4.1. *On S_a over $\mathbb{C}$, the transcendental part $T(S_a)$ has rank 12.*

We shall prove the lemma by constructing a suitable K3 quotient X_a of S_a. Before going into the details, we comment briefly on other possible approaches. In a similar situation of a surface with $\mathfrak{S}_5$ action in [13], the authors alluded to modularity in order to rule out the small rank alternative. This line of argument does not apply here since S_a is not defined over $\mathbb{Q}$. Instead one can use the Artin-Tate conjecture to compare square classes of discriminants of reductions modulo different primes. For S_a, however, this approach would always result in perfect 4th powers due to the splitting (10). This is the main reason to switch to a quotient surface of S_a that is a K3 surface (or any other surface of geometric genus 1).

In order to prove Lemma 4.1 our first aim is to construct a quotient surface of S that has geometric genus 1. The easiest way to achieve this builds on an involution interchanging exactly two homogeneous coordinates, say x_0, x_1, as one easily verifies that this fixes exactly one holomorphic 2-form on S up to scaling. The involution has six isolated fixed points on S, yielding A_1 singularities on the quotient surface.

For the quotient surface X, we introduce the invariant coordinates

$$u = x_0 x_1, \quad v = x_0 + x_1.$$

Then the quotient is birationally given in weighted projective space $\mathbb{P}[1,1,1,1,2]$. Expressing x_4 through s_1 and setting affinely $v = 1$, we obtain the equation

$$\begin{aligned}(x_3+1)(x_2+1)(x_2+x_3)(x_2^2+x_2x_3+x_2+x_3+1+x_3^2)(\lambda+1)\\ = (\lambda x_2x_3 - \lambda + \lambda x_2^2x_3 + \lambda x_2x_3^2 - 1)u + (\lambda+1)u^2\end{aligned}$$

This realises X as the minimal resolution of a double sextic with rational double point singularities over the affine x_2, x_3-plane, so X is a K3 surface. By construction, X has the Hodge structure

$$T(X) = V, \tag{11}$$

and the corresponding equality holds for Galois representations on specific members.

We can also interpret X as an elliptic surface over the affine x_3 line, say; this fibration has four obvious sections given by $x_2 = -1$ and $x_2 = -x_3$. Converting to Weierstrass form, we directly find a 2-torsion section; translation to $(0,0)$ yields the following equation in the standard coordinates $x = x_2, t = x_3$:

$$X: \qquad u^2 = x(x^2 + A(t)x + B(t)) \tag{12}$$

$$\begin{aligned} A &= \lambda^2t^4 - (4+8\lambda+2\lambda^2)t^3 - (24\lambda+12+11\lambda^2)t^2 \\ &\qquad -4(2\lambda+3)(1+\lambda)t - 4(1+\lambda)^2 \\ B &= 16t(t+1)(1+\lambda)^2[(2\lambda+1)(t^4+t^3)+(3\lambda+2)t^2+(2\lambda+2)t+1+\lambda]\end{aligned}$$

The discriminant reveals generally 6 singular fibres of type I_2 in Kodaira's notation and a split-multiplicative fibre of type I_4 at ∞.

4.2. **Special fibre X_a.** We shall now specialise to the fibre S_a and its K3 quotient X_a where a is given by (4) as before. Here we know that the Hodge structure V_a has rank 2 or 3, so X_a has Picard number 19 or 20 over $\mathbb{C}$ by (11).

Proposition 4.2. *The complex K3 surface X_a has Picard number 19.*

In order to prove the proposition, we assume on the contrary that $\rho(X_a) = 20$ and establish a contradiction by reducing modulo different good primes and comparing the square classes of the discriminants of the Néron-Severi lattices by virtue of the Artin-Tate conjecture. This method was pioneered in [15], refined in [6] and applied in a similar context in [9].

To be on the safe side when applying the reduction method, we compute the primes of bad reduction for X_a. This is easily achieved thanks to the elliptic fibration which specialises from the pencil X_λ. On X_a it attains 8 singular fibres of type I_2, each of them defined over the ground field $k(a)$ (in addition to the I_4 fibre at ∞). For the bad primes it suffices to study the degeneration of this fibration upon reduction mod p.

Lemma 4.3. *X_a has good reduction outside characteristics* $\{2, 3, 5, 11, 17, 433\}$.

Proof. It is an easy exercise using the discriminant to verify that the given elliptic fibration is non-degenerate outside the above characteristics and $83, 151$. In the latter two characteristics (and exactly for the $\mathbb{F}_p$-rational root of (4)), the fibration degenerates by merging fibres of type I_1 and I_2 to a fibre of type III. In other words, the two nodes of the I_2 fibre come together without reduction causing a singularity. Thus X_a has good reduction at all primes dividing 83 and 151, and the lemma follows. □

On the quintic S_a, the above characteristics (except for 2) are also visible in terms of singularities: the $\mathbb{F}_p$-rational root of (4) equals some exceptional value for λ from (5). We can easily show that there are no other bad characteristics:

Corollary 4.4. *The quintic S_a has good reduction outside characteristics* $\{3, 5, 11, 17, 433\}$.

Proof. If S_a has bad reduction at some prime $\mathfrak{p}$, then the ℓ-adic étale cohomology is ramified. Note that S_a has enough symmetries over the ground field k so that all of $H^2_{\text{ét}}(S_a \otimes \bar{k}, \mathbb{Q}_\ell)$ is governed by K3 quotients as in 4.1 via pull-back. Actually the quotients always lead to one and the same K3 surface X_a. But then $\mathfrak{p}$, being a prime of bad reduction for S_a, divides some prime from Lemma 4.3. In order to rule out $\mathfrak{p} \mid 2$, note that any root a of (4) reduces to zero modulo $\mathfrak{p}$. Since S_0 is smooth outside characteristics $3, 13, 17$ by the Jacobi criterion, this suffices to conclude the proof. □

Remark 4.5. The same argument goes through for other members of the pencil $\{S_\lambda\}$ (as we will exploit in the proof of Proposition 4.8). In particular, the non-degeneracy of the pencil of elliptic fibrations (12) on X (visible from the discriminant) in any odd characteristic suffices to prove that the general member of the pencil is smooth (Lemma 2.1).

4.3. **Proof of Proposition 4.2.** As a preparation we recall the Lefschetz fixed point formula for X_a. Over some finite field $\mathbb{F}_q$ ($q = p^e, p$ prime) containing a root a from (4), it returns with some auxiliary prime ℓ

$$\#X_a(\mathbb{F}_q) = 1 + \text{tr Frob}_q^*(H^2_{\text{ét}}(X_a \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)) + q^2.$$

On divisors, Frob_q^* has eigenvalues ζq for roots of unity ζ. In particular, the trace on the algebraic subspace inside $H^2_{\text{ét}}(X_a \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ spanned by $\text{NS}(X_a \otimes \bar{\mathbb{F}}_p)$ via the cycle class map equals an integer multiple of q. Presently $\rho(X_a \otimes \bar{\mathbb{F}}_p) = 20$ or 22 by assumption, since $\rho = 21$ is ruled out by [1]. By the previous argument, any non-congruence

$$\#X_a(\mathbb{F}_q) \not\equiv 1 \mod q \tag{13}$$

implies $\rho(X_a \otimes \bar{\mathbb{F}}_q) \leq 20$. This congruence is easily verified at specific primes; for instance, Table 1 shows $\rho(X_a \otimes \bar{\mathbb{F}}_p) \leq 20$ for $p = 19, 23$ and the respective choice of solution to (4) in $\mathbb{F}_p$. Thus our assumption implies equality, and in fact the validity of the Tate conjecture for X_a over any finite extension of $\mathbb{F}_{19}$ and $\mathbb{F}_{23}$ (alternatively one can use the elliptic fibration with section on X_a and appeal to [2]).

Since $\rho(X_a \otimes \bar{\mathbb{F}}_p) = 20$, the characteristic polynomial of Frob_q^* on $H^2_{\text{ét}}(X_a \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$ factors into a product of cyclotomic polynomials (shifted by q) and a single quadratic factor

$$\mu_q(T) = T^2 - a_q T + q^2. \tag{14}$$

Here we are concerned exclusively with the case $a_q \not\equiv 0 \mod q$, for instance for $q = p = 19$ or 23 as alluded to above. Moreover $a_q \in \{-2q, \dots, 2q\}$ by the Weil conjectures. Thus the parity of $\#X_a(\mathbb{F}_q)$ modulo q predicts four possibilities for the trace a_q without any further knowledge about the Galois action on divisors. (In fact the Galois action cannot be overly complicated since S_a contains numerous non-trivial divisor classes over $\mathbb{F}_q$, such as all components of the 8 I_2 fibres and the I_4 fibre at ∞ and the infinite section inherited from the generic member.)

Eventually, we want to apply the Artin-Tate conjecture [14] to X_a; it is equivalent to the Tate conjecture by [7], so it holds in the present situation. There is a little complication in mimicing the technique from [15]: the Artin-Tate conjecture for $X_a/\mathbb{F}_q$ allows us to read off the square class of the discriminant of $\text{NS}(X_a \otimes \bar{\mathbb{F}}_p)$ from the characteristic polynomial $\mu_q(T)$ a priori only if $\text{NS}(X_a \otimes \bar{\mathbb{F}}_p)$ is actually defined over $\mathbb{F}_q$, i.e. generated by divisors defined over $\mathbb{F}_q$. Presently this need not hold over $\mathbb{F}_p$. However, as $\mu_q(T)$ is quadratic, there is a simple way to circumvent this problem and avoid computing explicitly the minimal extension $\mathbb{F}_q$ where $\text{NS}(X_a)$ is defined. For this purpose we introduce the following auxiliary general result.

Lemma 4.6. *Let $X/\mathbb{F}_q$ be a K3 surface with geometric Picard number 20. Consider the characteristic polynomial $\mu_q(T)$ as above. Let $d \in \mathbb{Z}$ such that $\mu_q(T)$ splits in $\mathbb{Q}(\sqrt{d})$. Then the square class of the discriminant of* $\text{NS}(X \otimes \bar{\mathbb{F}}_q)$ *is given by d.*

Proof. Denote the roots of $\mu_q(T)$ by $\alpha, \bar{\alpha}$. We will need that α does not equal q times a root of unity. Equivalently the Tate conjecture holds for X, as checked for X_a in conjunction with (13). For arbitrary X, assume to the contrary that α takes the shape q times a root of unity. Then X has infinite height, so it is supersingular in Artin's sense. On the other hand, X admits an elliptic fibration, induced by a divisor class with square zero (this holds for any K3 surface with $\rho \geq 5$ since then NS represents 0). But then $\rho = 22$ by [1, Thm. 1.7], giving the required contradiction.

Next we claim that the splitting field of $\mu_q(T)$ is stable under extension. To see this, we compute $\mu_{q^e}(T) = (T - \alpha^e)(T - \bar{\alpha}^e)$ for any $e \in \mathbb{N}$. Then we use that $\alpha^e \notin \mathbb{Q}$ by the above considerations.

As a consequence we can assume that q is chosen in such a way that $\mathrm{NS}(X \otimes \bar{\mathbb{F}}_q)$ is already defined over $\mathbb{F}_{q^2}$, so that $D = \mathrm{disc}(\mathrm{NS}(X \otimes \bar{\mathbb{F}}_q)) = \mathrm{disc}(\mathrm{NS}(X \otimes \mathbb{F}_{q^2}))$. Note that $D < 0$ by the Hodge index theorem. The Artin-Tate conjecture [14] then predicts that the square class of $-D$ is given by $\mu_{q^2}(T)$ evaluated at $T = q^2$ up to a factor of q:

$$2q^2 - a_{q^2} = -M^2 D. \tag{15}$$

Here M^2 is the size of the Brauer group of X over $\mathbb{F}_{q^2}$. Generally we have $a_{q^2} = a_q^2 - 2q^2$, so (15) simplifies as

$$4q^2 - a_q^2 = -M^2 D. \tag{16}$$

But this implies that the splitting field of $\mu_q(T)$ is exactly $\mathbb{Q}(\sqrt{D})$. □

Remark 4.7. As in [9] one can also deduce that q splits into two principal ideals in $\mathbb{Q}(\sqrt{D})$. In other words, if $q = p^e$, then the prime factors of p have order dividing e in the class group $\mathrm{Cl}(\mathbb{Q}(\sqrt{D}))$ which gives a severe restriction on e.

Now let us return to our specific K3 surface X_a. Counting points over $\mathbb{F}_p$ for $p = 19, 23$ we infer from Table 1 that $\rho(X_a \otimes \bar{\mathbb{F}}_p) = 20$ at both primes by the congruence argument from (13).

p	$\#X_a(\mathbb{F}_p)$	a_p	D
19	676	29	-67
		10	-21
		-9	$-29 \cdot 47$
		-28	$-3 \cdot 5 \cdot 11$
23	924	26	-10
		3	-43
		-20	$-3 \cdot 11 \cdot 13$
		-43	$-3 \cdot 89$

TABLE 1. Possible discriminants of $\mathrm{NS}(X_a \otimes \bar{\mathbb{F}}_p)$

Recall the original assumption $\rho(X_a \otimes \bar{\mathbb{Q}}) = 20$. This implies that reduction mod p induces specialisation embeddings of finite index

$$\mathrm{NS}(X_a \otimes \bar{\mathbb{Q}}) \hookrightarrow \mathrm{NS}(X_a \otimes \bar{\mathbb{F}}_p) \quad \text{for } \ p = 19, 23.$$

In consequence, the square classes of all three Néron-Severi lattices under consideration coincide. But then by Table 1 this is impossible for $p = 19$ and 23 thanks to Lemma 4.6 since no two possibilities for D match. Hence we reach the desired contradiction. This concludes the proof of Proposition 4.2. □

4.4. **Proof of Lemma 4.1 and Theorem 2.2.** From Proposition 4.2 together with the splitting (10) we directly deduce Lemma 4.1. Theorem 2.2 follows immediately in conjunction with Lemma 3.2. □

4.5. **Remark on the Tate conjecture for the pencil $\{S_\lambda\}$.** It is common to infer the Tate conjecture for a surface from its validity for some cover (cf. [7]). Here we reverse this argument and verify the Tate conjecture for the quintics S through K3 quotients. A similar technique was applied in [11], but the situation here is more complicated since the surfaces in question have different geometric genus.

Proposition 4.8. *The Tate conjecture holds true for any non-degenerate quintic S in the pencil $\{S_\lambda\}$ over any finite field.*

Proof. Let k denote some finite field and assume at first that the quintic S is smooth. Recall from the proof of Corollary 4.4 that the surface S has enough symmetries over k so that all of $H^2_{\text{ét}}(S \otimes \bar{k}, \mathbb{Q}_\ell)$ is governed by copies of the K3 quotient X as in 4.1 via pull-back. As this K3 surface admits an elliptic fibration with section (12), the Tate conjecture holds for X by [2]. Pulling back divisors from X to S via the various quotient maps, we infer that the Tate conjecture holds for S.

The same argument works for the desingularisations of singular non-degenerate members of the pencil. Subsequently it can also be verified for the singular members themselves where we work with the original definition of the zeta function as exponential sum involving numbers of points. □

We can use the Tate conjecture to compute the Picard number of S_a in any characteristic. For instance, from Table 1 it follows that

$$\rho(S_a \otimes \bar{\mathbb{F}}_p) = 45 \quad \text{for} \quad p = 19, 23.$$

For details on the computations see the proof of the following corollary, especially (17). In general, there are two alternatives for the Picard number as we indicate below. Here we only have to rule out that S_a degenerates mod $\mathfrak{p}$. This happens exactly in characteristic 5 for the $\mathbb{F}_5$-rational root of (4). Characteristic 2 also plays a special role, see Example 4.10.

Corollary 4.9. *Let $p \neq 2$ be a prime and $a \in \mathbb{F}_q \subset \bar{\mathbb{F}}_p$ a root of (4) such that $(a, p) \neq (-2, 5)$. Denote by S the desingularisation of $S_a \otimes \bar{\mathbb{F}}_p$ (if necessary). Then*

$$\rho(S \otimes \bar{\mathbb{F}}_p) = \begin{cases} 45, & \text{if } \#S(\mathbb{F}_q) \not\equiv 1 \mod q, \\ 53, & \text{if } \#S(\mathbb{F}_q) \equiv 1 \mod q. \end{cases}$$

Proof. Since the Tate conjecture holds for $S/\mathbb{F}_q$, it suffices to compute the characteristic polynomial $\chi_q(T)$ of Frob_q on $H^2_{\text{ét}}(S \otimes \bar{\mathbb{F}}_p, \mathbb{Q}_\ell)$. Presently we have

$$\chi_q(T) = (T - q)^{40}(T \mp q)\chi'_q(T)^4$$

where the first two factors come from the lines and the extra generator of $\text{NS}(S_a \otimes \mathbb{C})$ and the last corresponds to $T(S_a)$. That is, the degree 3 polynomial $\chi'_q(T)$ comes from the motive V of the K3 surface X_a. Thus it takes the shape

$$\chi'_q(T) = (T \mp q)(T^2 - a_q T \pm q^2)$$

where the alternative $-q^2$ may only persist if $a_q = 0$. In particular,

$$\rho(X_a \otimes \bar{\mathbb{F}}_p) = \begin{cases} 20, & \text{if } a_q \not\equiv 0 \mod q, \\ 22, & \text{if } a_q \equiv 0 \mod q. \end{cases} \tag{17}$$

By Proposition 4.8 the corresponding statement for S reads

$$\rho(S \otimes \bar{\mathbb{F}}_p) = \begin{cases} 45, & \text{if } a_q \not\equiv 0 \mod q, \\ 53, & \text{if } a_q \equiv 0 \mod q. \end{cases}$$

In order to translate to the number of points, we apply the Lefschetz fixed point formula to find

$$\#S(\mathbb{F}_q) = 1 + 40q \pm q + 4(a_q \pm q) + q^2. \tag{18}$$

Outside characteristic 2, the congruence for a_q is equivalent to that for $\#S(\mathbb{F}_q)$ from the corollary. □

In practice it is often easier to use the condition (17) involving the quotient K3 surface. The following example in characteristic 2 illustrates this very well.

Example 4.10. In characteristic 2 the quintic S_a reduces to S_0. One can prove that S_0 is supersingular (i.e. $\rho(S_0 \otimes \bar{\mathbb{F}}_2) = 53$) by counting points on X_0 and using (17). Indeed for the singular double sextic model compactified over $\mathbb{P}_2$ we find $\#X_0(\mathbb{F}_2) = 11$. Since each exceptional divisor from the resolution of singularities adds $q = 2$ points, this yields supersingularity.

Alternatively we can pursue an explicit approach on S_0. Namely in characteristic 2, the quintic S_a contains 60 additional lines. Following [16] these can be given as $\mathfrak{S}_5$-orbits of

$$x_0 + x_1 = x_2 + \omega x_3 = 0$$

and

$$x_0 + x_1 - \omega^2 x_4 = x_0 + x_2 - \omega x_3 = 0$$

where ω denotes a primitive third root of unity. With a machine it is easily verified that the Gram matrix of the 135 lines in total has rank 53. Thus S_a is supersingular in characteristic 2, and in fact $\rho(S_a \otimes \mathbb{F}_{16}) = 53$.

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