ASYMPTOTIC ZERO DISTRIBUTION OF A CLASS OF HYPERGEOMETRIC POLYNOMIALS

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ABSTRACT. We prove that the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ asymptotically approach the section of the lemniscate $\left\{z : \left|z(1-z)^2\right| = \frac{4}{27}; \mathrm{Re}(z) > \frac{1}{3}\right\}$ as $n \to \infty$. In recent papers (cf. [9], [11]), Martínez-Finkelshtein and Kuijlaars and their co-authors have used Riemann-Hilbert methods to derive the asymptotic zero distribution of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$ when the limits $A = \lim\limits_{n\to\infty} \frac{\alpha_n}{n}$ and $B = \lim\limits_{n\to\infty} \frac{\beta_n}{n}$ exist and lie in the interior of certain specified regions in the AB-plane. Our result corresponds to one of the transitional or boundary cases for Jacobi polynomials in the Kuijlaars Martínez-Finkelshtein classification.

Mathematics Subject Classication: 33C05, 30C15.

Key words: Asymptotic zero distribution, Hypergeometric polynomials, Jacobi polynomials.

1. Introduction

The general hypergeometric function is defined by

$$ {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_p)_k}{(b_1)_k \ldots (b_q)_k} \frac{z^k}{k!} \quad , \quad |z| < 1 $$

where

$$ (\alpha)_k = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+k-1) & , \quad k \geq 1, \\ 1 & , \quad k = 0,\ \alpha \neq 0 \end{cases} $$

is Pochhammer's symbol. When one of the numerator parameters is equal to a negative integer, say $a_1 = -n,\ n \in \mathbb{N}$, the series terminates and the function reduces to a polynomial of degree n in z.

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The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ can be defined in terms of a ${}_2F_1$ hypergeometric polynomial (cf. [12], p. 254), viz,

$$P_n^{(\alpha,\beta)}(z) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-z}{2}\right).$$

The study of the zeros of Jacobi polynomials therefore gives direct information about the zeros of the corresponding ${}_2F_1$ hypergeometric polynomials and vice versa.

In [11], Martínez-Finkelshtein, Martínez-González and Orive consider the asymptotic zero distribution of the Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$ where the limits

$$A = \lim_{n\to\infty} \frac{\alpha_n}{n} \quad \text{and} \quad B = \lim_{n\to\infty} \frac{\beta_n}{n} \tag{1.1}$$

exist. They distinguish five cases depending on the values of A and B and prove results for the "general" cases where A and B lie in the interior of certain regions in the plane.

The boundary lines are "non-general" cases, some of which have been studied (cf. [2], [3], [5], [6], [8]), always in the context of the corresponding hypergeometric function.

We find the asymptotic zero distribution of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ as $n \to \infty$, which corresponds to a non-general case not previously studied. Our results, taken in conjunction with those in [2], [5], [6] and [8] suggest that there may be a general result for the asymptotic zero distribution of the class of Jacobi polynomials $P_n^{(\alpha_n,\beta_n)}$ where the limits in (1.1) are

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = k \text{ and } \lim_{n\to\infty} \frac{\beta_n}{n} = -1.$$

We shall prove the following theorem.

Theorem 1. *The zeros of the hypergeometric polynomial*

$${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$$

approach the section of the lemniscate

$$\left\{z : \left|z(1-z)^2\right| = \frac{4}{27}; Re(z) > \frac{1}{3}\right\},$$

as $n \to \infty$.

Our method, which follows the same approach as that used in [8], involves the asymptotic analysis of an integral of the form

$$A_n \int_0^1 [f_z(t)]^n \, dt \tag{1.2}$$

where A_n is a constant involving n and $f_z(t)$ is a polynomial in the complex variable t and analytic in z.

Expressing ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the form given in (1.2) can be done either by substituting t^2 for t in the Euler integral formula for ${}_2F_1$ functions (cf. [12], p. 47), given by

$${}_2F_1(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \, dt,$$

or, more directly, using an integral representation for ${}_3F_2$ hypergeometric functions, namely, (cf. [7], Cor 2.2)

$${}_3F_2\left(-n, \frac{b}{2}, \frac{b+1}{2}; \frac{c}{2}, \frac{c+1}{2}; z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt^2)^n dt,$$

for $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$.

Putting $b = n+1$ and $c = n+2$, we obtain

$${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = (n+1) \int_0^1 \left[t(1-zt^2)\right]^n dt = (n+1) \int_0^1 [f_z(t)]^n \, dt \tag{1.3}$$

where $f_z(t) = t(1-zt^2)$ is a polynomial in the complex variable t and analytic in z.

We shall denote the two branches of the square root of z by $\pm\sqrt{z}$ where $\sqrt{z}$ is the branch with $\sqrt{1} = 1$ and the square root $\sqrt{z}$ is holomorphic on the plane cut along the negative semi-axis. Note that the function $f_z(t)$ has zeros at $t = 0$ and $t = \pm\frac{1}{\sqrt{z}}$ while the critical points $f_z'(t) = 0$ occur at $t = \pm\frac{1}{\sqrt{3z}}$.

2. Preliminary results

In order to prove our main result, we will need the following lemmas.

First, we recall a version of the classical Eneström-Kakeya theorem (cf. [10] p.136): If $0 < a_0 < a_1 < \ldots < a_n$, then all zeros of the polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ lie in the unit disk $|z| < 1$.

Lemma 2.1. *The zeros of* ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ *are contained in the disk* $|z| < n+1$.

Proof. Let

$$F_n(z) = {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = c_0 + c_1 z + \ldots + c_n z^n,$$

where

$$c_m = \frac{(-n)_m \left(\frac{n+1}{2}\right)_m}{\left(\frac{n+3}{2}\right)_m m!} \tag{2.4}$$

A straightforward computation shows that

$$\left|\frac{c_n}{c_{n-1}}\right| > \frac{1}{n+1}, \quad \text{for } n > 1,$$

which implies that

$$\frac{-(n+1)c_m}{c_{m-1}} > 1, \qquad m = 1, 2, \ldots, n.$$

It follows immediately that the coefficients of the polynomial

$$\begin{aligned} p(z) = F_n\left(-(n+1)z\right) &= c_0 - c_1(n+1)z + \ldots + (-1)^n(n+1)^n z^n \\ &= a_0 + a_1 z + \ldots + a_n z^n \end{aligned}$$

are positive and increasing: $0 < a_0 < a_1 < \ldots < a_n$. By the Eneström-Kakeya theorem, the zeros of $F_n\left(-(n+1)z\right)$ lie in the unit disk $|z| < 1$ and therefore the zeros of $F_n(z)$ lie in the disk $|z| < n+1$. □

Lemma 2.2. *The polynomial* ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ *has at least one zero outside the unit circle* $|z| = 1$*.*

Proof. We have

$$\begin{aligned} {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) &= c_0 + c_1 z + \ldots + c_n z^n \\ &= c_n(z-k_1)(z-k_2)\ldots(z-k_n) \end{aligned}$$

where k_j is the j^{th} zero of the polynomial, $j = 1, 2, \ldots, n$. Now

$$c_0 = c_n k_1 k_2 \ldots k_n (-1)^n$$

so that

$$|k_1 k_2 \ldots k_n| = \left|\frac{c_0}{c_n}\right|.$$

Also, from (2.4), we see that

$$\left|\frac{c_0}{c_n}\right| = \frac{3n+1}{n+1} > 1,$$

and so the product of the zeros has modulus greater than 1. Therefore at least one zero of the polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ must be outside the unit circle $|z| = 1$. □

Lemma 2.3. *If* $Re(\sqrt{z}) > \frac{1}{\sqrt{3}}$*, the function* $|f_z(t)|$ *given in (1.3) has a unique path of steepest ascent from* $\frac{1}{\sqrt{z}}$ *to 1. If* $0 < Re(\sqrt{z}) < \frac{1}{\sqrt{3}}$*, there is a unique path of steepest ascent from 0 to 1.*

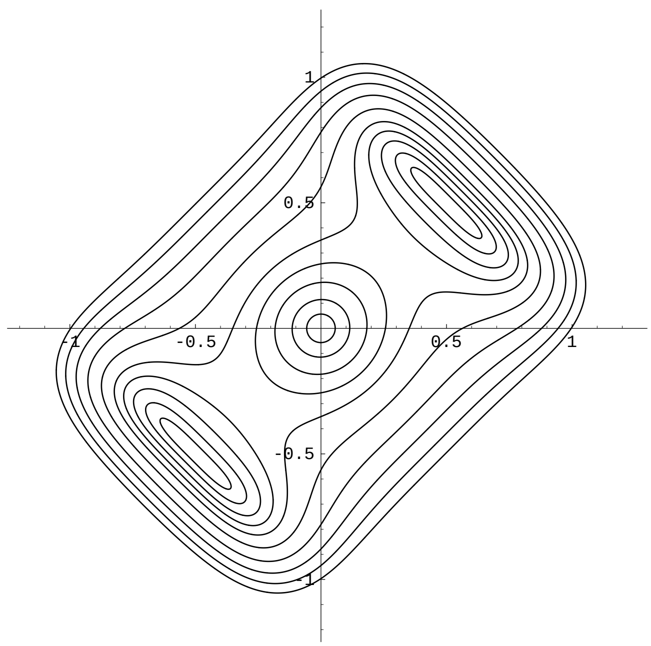

FIGURE 1. The level curves of $f_z(t)$

Proof. First we note that $|f_z(1)| > \left|f_z\left(\frac{1}{\sqrt{3z}}\right)\right| \Leftrightarrow |z(1-z)^2| > \frac{4}{27}$ and that the equivalence for the reverse inequality also holds.

The lines through the saddle-points $t = \pm\frac{1}{\sqrt{3z}}$ perpendicular to the linear segment from $-\frac{1}{\sqrt{z}}$ through 0 to $\frac{1}{\sqrt{z}}$ are "continental divides" that separate the t-plane into three basins containing $-\frac{1}{\sqrt{z}}$, 0 and $\frac{1}{\sqrt{z}}$ respectively.

Any point in the 0-basin is joined to 0 by a unique path of steepest decent, orthogonal to the level curves of $f_z(t)$. Points in the $\frac{1}{\sqrt{z}}$-basin and $-\frac{1}{\sqrt{z}}$-basin can be similarly joined to $\frac{1}{\sqrt{z}}$ and $-\frac{1}{\sqrt{z}}$ respectively.

The point 1 will be located in either the 0-basin or the $\frac{1}{\sqrt{z}}$-basin, but not in the $-\frac{1}{\sqrt{z}}$-basin. Multiplying each point in the t-plane by $\frac{\sqrt{z}}{|z|}$ rotates the figure so that all three zeros of $f_z(t)$ move to the real axis and the lines through the saddle-points $-\frac{1}{\sqrt{3z}}$ and $\frac{1}{\sqrt{3z}}$ are carried to the vertical lines through $-\frac{1}{\sqrt{3}|z|}$ and $\frac{1}{\sqrt{3}|z|}$ respectively.

The point 1 will then be in the $\frac{1}{\sqrt{z}}$-basin if and only if $\mathrm{Re}\left(\frac{\sqrt{z}}{|z|}\right) > \frac{1}{\sqrt{3}|z|}$ which is equivalent to the condition $\mathrm{Re}\left(\sqrt{z}\right) > \frac{1}{\sqrt{3}}$. Similarly, the point 1 will then be in the 0-basin if and only if $\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$ or, more precisely, $0 < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$. $\square$

3. The region $\mathrm{Re}(z) < \frac{1}{3}$

We consider the two possibilities illustrated in Theorem 2.3 separately (refer to Figure 1). The first is the case where $\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$. This section of the z-plane is the area to the left of the parabola with vertex $\frac{1}{3}$ and intercepts $\frac{2}{3}i$ and $-\frac{2}{3}i$. Thus all points z satisfying $\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$ will lie to the left of the vertical line $\mathrm{Re}(z) = \frac{1}{3}$.

We shall prove that no zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ are possible for $\mathrm{Re}(z) < \frac{1}{3}$ and, therefore, no zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ can occur for $\frac{-1}{\sqrt{3}} < \mathrm{Re}\left(\sqrt{z}\right) < \frac{1}{\sqrt{3}}$.

Theorem 3.1. *For sufficiently large n, the polynomial ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ has no zeros in the region $Re(z) < \frac{1}{3}$.*

Proof. From (1.3), we know that

$$\begin{aligned} {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) &= (n+1)\int_0^1 [f_z(t)]^n \, dt \\ &= (n+1)\int_0^1 \left[t(1-zt^2)\right]^n dt. \end{aligned}$$

Lemma 2.3 ensures that for $\mathrm{Re}(z) < \frac{1}{3}$, there is a unique path of steepest ascent from 0 to 1. We may thus evaluate the integral involved in (1.3) over this path. In order to find the path of steepest ascent, we use that fact that $f_z(t) = t(1-zt^2)$ will have constant argument along this path (cf. [4]) so that we can parametrise the path by letting

$$f_z(t) = f_z(1)r \qquad 0 \le r \le 1,$$

or equivalently

$$t(1 - zt^2) = r(1 - z). \tag{3.1}$$

Then

$$(1 - 3zt^2)dt = (1 - z)dr$$

and our hypergeometric polynomial can be rewritten as

$${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = (n+1)(1-z)^{n+1} \int_0^1 \frac{r^n}{1 - 3zt^2} dr$$

where $t = t(r)$ is defined implicitly by (3.1), with $t(0) = 0$ and $t(1) = 1$.

Any zeros $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\text{Re}(z) < \frac{1}{3}$ must satisfy

$$(n+1)(1-z)^{n+1} \int_0^1 \frac{r^n}{1 - 3zt^2} dr = 0,$$

or equivalently, using (3.1)

$$n \int_0^1 \frac{(1 - zt^2)tr^{n-1}}{1 - 3zt^2} dr = 0. \tag{3.2}$$

We will prove that the integral in (3.2) is bounded away from 0 and hence deduce that no zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ can lie in the half-plane $\text{Re}(z) < \frac{1}{3}$.

If the zeros are restricted by the inequality $\left|z - \frac{1}{3}\right| \geq \epsilon$ for some $\epsilon > 0$, then the denominator of the integrand in (3.2) satisfies $|1 - 3zt^2| \geq \delta > 0$, where δ is independent of z. Thus for any fixed ρ with $0 < \rho < 1$, we have

$$\begin{aligned} n\left|\int_0^\rho \frac{(1 - zt^2)tr^{n-1}}{1 - 3zt^2} dr\right| &\leq n \int_0^\rho \frac{|(1 - zt^2)t|}{|1 - 3zt^2|}\, r^{n-1} dr \\ &\leq Cn \int_0^\rho |(1 - zt^2)t|\, r^{n-1} dr \end{aligned}$$

since $\frac{1}{|1-3zt^2|} \leq C$ for some constant C.

Lemma 2.1 states that the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ are contained in the disk $|z| < n + 1$. Thus

$$
\begin{aligned}
n\left|\int_0^\rho \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr\right| &\le Cn\int_0^\rho |(1-zt^2)t|\, r^{n-1}dr \\
&\le Cn^2\int_0^\rho r^{n-1}dr \\
&\le Cn\,\rho^n \to 0 \qquad (3.3)
\end{aligned}
$$

as $n\to\infty$ since $0<\rho<1$. On the other hand, for ρ sufficiently close to 1, the integral

$$n\int_\rho^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr \qquad (3.4)$$

is bounded away from zero. To prove this, we first note that since the path $t=t(r)$ must lie on the same side of the "continental divide" as the point 1 (cf. proof of Lemma 2.3), we know that $\mathrm{Re}(zt^2)<\frac{1}{3}$. Our restriction on z further ensures that $\left|zt^2-\frac{1}{3}\right|>\frac{\epsilon}{2}$ for t sufficiently near 1.

Now the linear fractional mapping

$$\omega=\phi(\zeta)=\frac{1-\zeta}{1-3\zeta}$$

sends the region

$$\left\{\zeta:\mathrm{Re}(\zeta)<\frac{1}{3},\left|\zeta-\frac{1}{3}\right|>\frac{\epsilon}{2}\right\}$$

onto a semidisk to the right of the vertical line $\mathrm{Re}(\omega)=\frac{1}{3}$. It follows that

$$\mathrm{Re}\left\{\frac{(1-zt^2)t}{1-3zt^2}\right\}>\frac{1}{6}$$

when t is close enough to 1. This shows that

$$\mathrm{Re}\left\{n\int_\rho^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}\,dr\right\}>\frac{n}{6}\int_\rho^1 r^{n-1}\,dr>\frac{1}{12}$$

for ρ near 1 and all z satisfying $\mathrm{Re}(z)<\frac{1}{3}$. Combining this with (3.3), we see that for sufficiently large n, the polynomial ${}_2F_1\left(-n,\frac{n+1}{2};\frac{n+3}{2};z\right)$ can have no zeros in the region $\mathrm{Re}(z)<\frac{1}{3}$. $\square$

Thus, if any zeros exist in the region $\mathrm{Re}(z)\le\frac{1}{3}$, they must converge uniformly to the point $\frac{1}{3}$ as $n\to\infty$ (since we had the additional restriction of $\left|z-\frac{1}{3}\right|>\epsilon$ for some $\epsilon>0$).

In common with the analysis in [8], we are unable to show that the polynomial never has zeros in this region, although numerical evidence generated by Mathematica suggests that this is the case.

4. The asymptotic zero distribution in the region $\mathrm{Re}(z) > \frac{1}{3}$

We know from section 3 that for n sufficiently large, the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ lie in the region $\mathrm{Re}(z) > \frac{1}{3}$.

From Lemma 2.2, we know that at least one of the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ lies outside the unit circle $|z| = 1$. Since there are no zeros to the left of $\mathrm{Re}(z) = \frac{1}{3}$, we know that this polynomial has at least one zero in the region $\mathrm{Re}(z) > \frac{1}{3}$.

We are now in a position to prove our theorem.

Theorem 4.1. *The zeros of the hypergeometric polynomial*

$$ {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) $$

approach the section of the lemniscate

$$ \left\{ z : \left|z(1-z)^2\right| = \frac{4}{27}; Re(z) > \frac{1}{3} \right\}, $$

as $n \to \infty$.

Proof. From (1.3), we know that

$$ {}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right) = (n+1) \int_0^1 \left[t(1-zt^2)\right]^n dt. $$

Lemma 2.3 guarantees that there is a unique path of steepest ascent from $\frac{1}{\sqrt{z}}$ to 1. We may thus deform the path of integration in (1.3) to write

$$ \int_0^1 [f_z(t)]^n \, dt = \int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt + \int_{\frac{1}{\sqrt{z}}}^1 [f_z(t)]^n \, dt $$

following the linear path from 0 to $\frac{1}{\sqrt{z}}$ and then the unique path of steepest ascent from $\frac{1}{\sqrt{z}}$ to 1. The linear path from 0 to $\frac{1}{\sqrt{z}}$ is orthogonal to the level curves of $f_z(t)$ and is therefore the path of steepest ascent from 0 to the saddle-point $\frac{1}{\sqrt{3z}}$, followed by the path of steepest descent to $\frac{1}{\sqrt{z}}$.

Any zero $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\mathrm{Re}(z) > \frac{1}{3}$ must satisfy

$$\int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt + \int_{\frac{1}{\sqrt{z}}}^1 [f_z(t)]^n \, dt = 0 \tag{4.1}$$

where the integrals are taken over paths of steepest ascent or descent.

Making the substitution $s = zt^2$ for $0 \le s \le 1$, we obtain

$$\begin{aligned}
\int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt &= \int_0^{\frac{1}{\sqrt{z}}} \left[t(1-zt^2)\right]^n dt \\
&= \frac{1}{2(\sqrt{z})^{n+1}} \int_0^1 s^{(n+1)/2}(1-s)^n ds \\
&= \frac{1}{2(\sqrt{z})^{n+1}} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma(n+1)}{\Gamma\left(\frac{3n+3}{2}\right)}.
\end{aligned}$$

Using Stirling's approximation

$$\Gamma(n+1) = e^{-n} n^n \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + O\left(\frac{1}{n^3}\right)\right), \qquad n \to \infty,$$

we then have

$$\int_0^{\frac{1}{\sqrt{z}}} [f_z(t)]^n \, dt = \frac{\sqrt{2\pi}}{3\sqrt{n}(\sqrt{z})^{n+1}} \left(\frac{2}{\sqrt{27}}\right)^n \left[1 + O\left(\frac{1}{n}\right)\right],$$

as $n \to \infty$. The second integral on the right-hand side of (4.1) requires that we find the path of steepest ascent from $\frac{1}{\sqrt{z}}$ to 1. We again use the fact that $f_z(t)$ will have constant argument along this path (cf. [4]) so that we can parametrise this path by letting

$$f_z(t) = f_z(1)r \qquad 0 \le r \le 1,$$

yielding

$$\int_{\frac{1}{\sqrt{z}}}^1 [f_z(t)]^n \, dt = (1-z)^n \int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2} dr$$

where $t = t(r)$ is defined implicitly by (3.1), with $t(0) = \frac{1}{\sqrt{z}}$ and $t(1) = 1$. Therefore, any zero $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\mathrm{Re}(z) > \frac{1}{3}$ must asymptotically satisfy

$$\left(\frac{2}{\sqrt{27}}\right)^n \frac{\sqrt{2\pi}}{3\sqrt{n}(\sqrt{z})^{n+1}}\left\{1 + O\left(\frac{1}{n}\right)\right\} + (1-z)^n \int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr = 0,$$

as $n \to \infty$, or equivalently

$$(\sqrt{z})^{n+1}(1-z)^n \int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr = -\left(\frac{2}{\sqrt{27}}\right)^n \frac{\sqrt{2\pi}}{3\sqrt{n}}\left\{1 + O\left(\frac{1}{n}\right)\right\}. \tag{4.2}$$

as $n \to \infty$. Taking moduli and n^{th} roots on both sides of (4.2), we obtain

$$|\sqrt{z}|^{\frac{1}{n}} \left|\sqrt{z}(1-z)\right| \left|\int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr\right|^{\frac{1}{n}} = \left|\frac{2}{\sqrt{27}}\right| \left(\frac{\sqrt{2\pi}}{3\sqrt{n}}\right)^{\frac{1}{n}} \left\{1 + O\left(\frac{1}{n}\right)\right\}^{\frac{1}{n}}.$$

It is straightforward to check that, as $n \to \infty$,

$$\left|\int_0^1 \frac{(1-zt^2)tr^{n-1}}{1-3zt^2}dr\right|^{1/n}$$

converges to 1 uniformly in z and the zeros $z = z_{nj}$ of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ in the region $\mathrm{Re}(z) > \frac{1}{3}$ approach the lemniscate

$$\left|\sqrt{z}(1-z)\right| = \frac{2}{\sqrt{27}}$$

or equivalently

$$\left|z(1-z)^2\right| = \frac{4}{27}. \qquad \square$$

In addition, we note that by taking n^{th} roots on both sides of (4.2), for large n, there are n points satisfying (4.2), distinguished by the n choices of $\sqrt[n]{-1}$. All of these points are zeros of the polynomial, spread out near the right-hand branch of the lemniscate.

Figure 2 shows numerical plotting of the zeros for n ranging from 5 to 60.

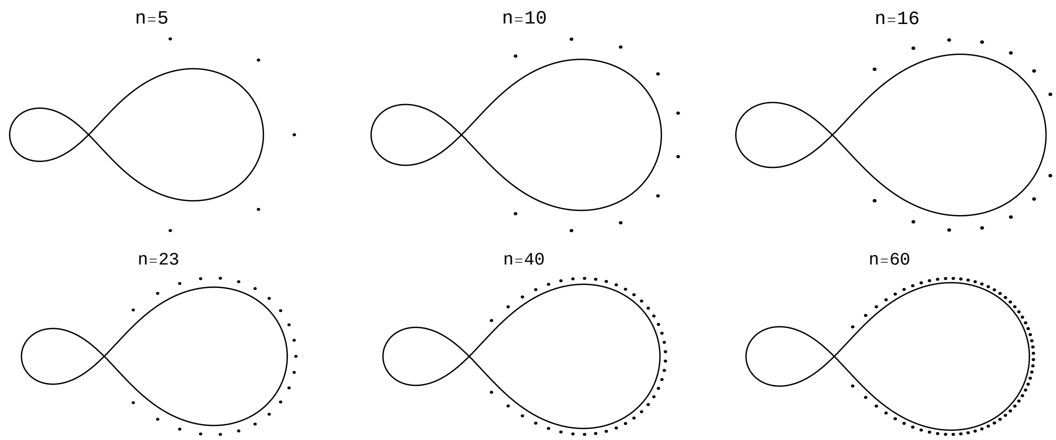

FIGURE 2. The curve $|z(1-z)^2| = \frac{4}{27}$ and the zeros of ${}_2F_1\left(-n, \frac{n+1}{2}; \frac{n+3}{2}; z\right)$ for $n = 5, 10, 16, 23, 40, 60$.

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