HOCHSCHILD COHOMOLOGY OF $GL_2(\overline{\mathbb{F}}_p)$

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Abstract. We compute the Hochschild cohomology algebra of GL_2 over an algebraically closed field of characteristic $p > 2$.

1. Intro.

Hochschild cohomology is a basic invariant which sends a finite dimensional algebra A to a super-commutative algebra $HH(A) = \mathrm{Ext}^*_{A\text{-mod-}A}(A)$. The algebra $HH(A)$ can be thought of as the derived centre of the algebra A, given as it is by the formula $HH(A) = H^* \mathrm{End}_{A\text{-mod-}A}(\tilde{A})$, where $\tilde{A}$ is a projective resolution of A in the category A-mod-A of A-A-bimodules; to see the analogy compare with the formula $Z(A) = \mathrm{End}_{A\text{-mod-}A}(A)$ for the classical centre $Z(A)$ of A. If M is any A-module, then the natural algebra homomorphism $Z(A) \to \mathrm{Hom}_A(M, M)$ extends to a natural algebra homomorphism $HH(A) \to \mathrm{Ext}^*_A(M, M)$.

Like other algebras obtained by taking derived endomorphisms, Hochschild cohomology and its variants can be endowed with additional structures; these have been the source of diverse interest [14]; the most basic such is known as the Gerstenhaber bracket [5]. But even without further decoration, the algebra $HH(A)$ has proved difficult to compute in specific examples, and its behaviour difficult to predict. One delicacy is the issue of finite generation of $HH(A)$ which is not guaranteed for a finite dimensional algebra A, even modulo the ideal of nilpotent elements [15]; yet there are finite dimensional self-injective algebras whose Hochschild cohomology is not merely finitely generated but finite dimensional [2].

The subject of this article is the computation of HH in a basic example arising in the representation theory of algebraic groups. We examine the Hochschild cohomology of Schur algebras $S(2, r)$, which are finite dimensional algebras whose representation theory underlies the rational representation theory of the algebraic group $G = GL_2(F)$, where F is an algebraically closed field of characteristic p. Indeed, we compute the Hochschild cohomology of G for $p > 2$, which we define to be a certain inverse limit over r of Hochschild cohomologies of Schur algebras $S(2, r)$ [6]. The algebras $S(2, r)$ increase in complexity as r increases, but we are nevertheless able to develop sufficiently sharp homological tools to achieve the calculation of their HH algebras. We apply a theory of algebraic operators (2-functors) on certain 2-categories which underlies the representation theory of G [8], [9]. We also use: the theory of quasi-hereditary algebras [3], the theory of Koszul duality [1], the formalism of differential graded algebras and their derived categories [7], a theory of homological duality for algebraic operators, explicit analysis of certain bimodules associated with a well-known quasi-hereditary algebra **c** and its homological duals, and a formalism of algebras with a polytopal basis.

2. The answer.

Suppose $\Gamma = \bigoplus_{i,j,k\in\mathbb{Z}} \Gamma^{ijk}$ is a $\mathbb{Z}$-trigraded algebra. We have a combinatorial operator $\mathfrak{O}_\Gamma$ which acts on the collection of $\mathbb{Z}$-bigraded algebras Σ after the formula

$$\mathfrak{O}_\Gamma(\Sigma)^{ik} = \bigoplus_{j,k_1+k_2=k} \Gamma^{ijk_1} \otimes_F \Sigma^{jk_2},$$

where we take the super tensor product with respect to the k-grading.

Let $p > 2$. In the main body of the paper we define an ijk-graded algebra $\spadesuit$ with an explicit, canonically defined basis $\mathcal{B}_\spadesuit$. A complete description of the algebra $\spadesuit$, its basis, and its product, is given in section 12.

There is a natural algebra homomorphism $F \leftarrow \spadesuit$ which is a splitting of the map sending 1 to the identity in $\spadesuit$. This lifts to a morphism of operators $\mathfrak{O}_F \leftarrow \mathfrak{O}_\spadesuit$. Since $\mathfrak{O}_F^2 = \mathfrak{O}_F$ we have a sequence of operators

$$\mathfrak{O}_F \leftarrow \mathfrak{O}_F\mathfrak{O}_\spadesuit \leftarrow \mathfrak{O}_F\mathfrak{O}_\spadesuit^2 \leftarrow \mathfrak{O}_F\mathfrak{O}_\spadesuit^3 \leftarrow ...$$

We define $hh_l = \mathfrak{O}_F\mathfrak{O}_\spadesuit^l(F[z,z^{-1}])$. We prove that the map $hh_l \to hh_{l-1}$ is surjective for $l \geq 1$, define hh to be the inverse limit of the sequence of algebras hh_l. and establish the following:

Theorem 1. *Every block of the Hochschild cohomology of G is isomorphic to hh.*

Proof. See section 13. □

Remark 2 For every l the algebra hh_l inherits an explicit basis from $\spadesuit$ with an explicit product as described in Corollary 25; these bases are compatible with the surjective maps $hh_l \to hh_{l-1}$.

3. Guidebook.

The proof of Theorem 1 passes through a number of counties of diverse character; here we briefly describe some of these. In Section 4 we review some elements of the theory of Schur algebras, which are finite dimensional algebras commonly used in the study of rational representations of GL_n; we make some comments that are related to the study of the Hochschild cohomology of Schur algebras. The Schur algebras we are interested in are not Koszul algebras; nevertheless, they are closely related to certain Koszul algebras and we make use of some pretty generalities concerning the Hochschild cohomology of Koszul algebras; in Section 5 we give an account of these. In Section 6 we introduce certain algebraic operators and gather together some facts about these that we have established in previous papers. In Section 7 we describe an interaction of these operators with Hochschild cohomology and Koszul duality. In Section 8 we recall from another paper [8] how special examples of our algebraic operators can be used to describe the representation theory of $GL_2(F)$. In Section 9 we show that this description of the representation theory of $GL_2(F)$ via algebraic operators along with the Section 7 analysis of the behaviour of Hochschild cohomology under such algebraic operators can be used to describe the Hochschild cohomology for the algebras relevant to $GL_2(F)$ in terms of an algebraic operator $\mathfrak{O}_{\mathbb{HH}(\clubsuit)}$; here $\clubsuit = \mathbb{H}(\mathbb{T}_\Omega(\underline{\mathbf{t}}^!))$ for a certain Koszul algebra Ω and a certain pair of dg Ω-Ω bimodules $\underline{\mathbf{t}}^!$, and $\mathbb{HH}$ is the operator that sends a graded algebra $x = \bigoplus_i x^i$ to a graded algebra $\bigoplus_i HH(x^0, x^i)$. In Section 10 we give a combinatorial description of the algebra $\clubsuit$ via certain bimodules; to do this we invoke a study of the negative part $\clubsuit^-$ of $\clubsuit$ made in a previous article [9], and Serre duality for Ω. In Section 11 we perform a detailed combinatorial analysis of the Hochschild cohomology of certain bimodules appearing in the algebra $\clubsuit$. A fact emerging here is that a certain quotient Θ of Ω, commonly known as the preprojective algebra of type A, possesses an involution σ such that

$$\Theta^\sigma \cong \Theta^*, \qquad HH(\Omega, \Theta^\sigma) \cong HH(\Omega, \Theta)^*;$$

the first of these formulas asserts the well known self-injectivity of Θ, but the second asserts something similar holds under $HH(\Omega, -)$. In Section 12 we use the analysis of the preceding section to give a combinatorial description of $\spadesuit = \mathbb{HH}(\clubsuit)$, firstly in terms of certain bimodules and maps between them, secondly via a monomial basis. Finally in Section 13 we reach our destination, and give a proof Theorem 1.

4. Schur algebras.

The group $GL_n(F)$ of invertible $n \times n$ matrices is an algebraic group over F. The Schur algebra $S(n) = \bigoplus_{r \geq 0} S(n,r)$ is by definition the graded dual of the bialgebra $F[M_n(F)]$ of polynomial functions on the algebra $M_n(F)$ of $n \times n$ matrices. The category of finite dimensional representations of the finite dimensional algebra $S(n,r)$ is equivalent to the category of finite dimensional polynomial representations of degree r of the group $GL_n(F)$; this category contains for example the symmetric, exterior and tensor powers of the natural representation of degree r. The functor

$$S(n,r)\text{-mod} \to S(n, r+n)\text{-mod}$$

sending a module M to $M \otimes \det$ is a fully faithful embedding. Taking the union $\cup_{l \geq 0} S(n, r+ln)$-mod over the corresponding ascending sequence of such embeddings gives us a category $\mathcal{C}_r$ that is equivalent to the category of rational representations of degree r; this category contains representations of the form $V \otimes \det^{-l}$ where V is a polynomial representation of degree $r + ln$. The category $GL_n(F)$-mod of rational representations of GL_n is a direct sum of categories $\bigoplus_{r \in \mathbb{Z}} \mathcal{C}_r$. There are natural maps between the Hochschild cohomologies of Schur algebras as n and r vary, which we now record.

Hochschild cohomologies of $S(n,r)$ as r varies. We have a surjective algebra homomorphism $S(n,r) \twoheadrightarrow S(n, r-n)$ which is dual to the functor

$$S(n, r-n)\text{-mod} \to S(n,r)\text{-mod}$$

sending a module M to $M \otimes \det$. We thus have functors

$$S(n, r-n) \otimes_{S(n,r)} - \otimes_{S(n,r)} S(n, r-n) :$$

$$D^b(S(n,r)\text{-mod-}\, S(n,r)) \to D^b(S(n, r-n)\text{-mod-}\, S(n, r-n)).$$

Note that $S(n,r)$ is Δ-filtered as an $S(n,r)$-$S(n,r)$-bimodule because it is a quasi-hereditary algebra. It is also the case that $S(n,r-n) \otimes S(n,r-n)$ is Δ-filtered as an $S(n,r)$-$S(n,r)$-bimodule, since $S(n,r-n)$ is Δ-filtered as a left and a right $S(n,r)$-module. For a quasi-hereditary algebra A we have $Tor_A^d(\Delta, \Delta') = 0$ for $d > 0$ [4, Lemma 5]. We thus have isomorphisms in the derived category of $S(n,r)$-$S(n,r)$-bimodules

$$\begin{aligned} &S(n,r-n) \otimes^L_{S(n,r)} S(n,r) \otimes^L_{S(n,r)} S(n,r-n) \\ &\qquad \cong S(n,r-n) \otimes_{S(n,r)} S(n,r) \otimes_{S(n,r)} S(n,r-n) \\ &\qquad \cong S(n,r-n). \end{aligned}$$

The functor

$$S(n,r-n) \otimes^L_{S(n,r)} - \otimes^L_{S(n,r)} S(n,r-n)$$

thus sends $S(n,r)$ to $S(n,r-n)$, giving us natural maps

$$\begin{aligned} &\mathrm{Hom}_{D^b(S(n,r)\text{-mod-}S(n,r))}(S(n,r), S(n,r)[k]) \\ &\qquad \to \mathrm{Hom}_{D^b(S(n,r-n)\text{-mod-}S(n,r-n))}(S(n,r-n), S(n,r-n)[k]) \end{aligned}$$

and therefore a graded algebra homomorphism

$$HH(S(n,r)) \to HH(S(n,r-n)).$$

In case $n = 2$ we prove that this algebra homomorphism is surjective for all r (Theorem 27), and define $HH(GL_2(F))$ to be the inverse limit of these algebras; it thus comes equipped with a surjective algebra homomorphism $HH(GL_2) \twoheadrightarrow HH(S(2,r))$ for every r. We do not know whether something similar is possible for $n \geq 2$.

Hochschild cohomologies of $S(n,r)$ as n varies. Let $\mathbf{n} = \{1, 2, ..., n\}$. Suppose we have an embedding $\iota : \mathbf{m} \to \mathbf{n}$. This gives us an embedding

$$S(m,r) = \xi S(n,r) \xi \to S(n,r)$$

where $\xi = \sum_{l_o \in \mathbf{m}} \xi_{l_1,l_1} \xi_{l_2,l_2} ... \xi_{l_r,l_r}$ is an idempotent [6]. The algebra $S(n,r)$ has standard modules indexed by partitions of r with $\leq n$ parts, while the algebra $S(m,r)$ has standard modules indexed by partitions of r with $\leq m$ parts. The collection of partitions of r with $\leq m$ parts is a coideal in the collection of partitions of r with $\leq m$ parts. Consequently ξ is a good idempotent. Truncating by ξ gives a functor $\mathrm{Hom}(S(n,r)\xi \otimes \xi S(n,r), -)$:

$$D^b(S(n,r)\text{-mod-}S(n,r)) \to D^b(S(m,r)\text{-mod-}S(m,r))$$

which sends $S(n,r)$ to $\xi S(n,r) \xi = S(m,r)$. We therefore have a graded algebra homomorphism

$$\o : HH(S(n,r)) \to HH(S(m,r)).$$

5. Hochschild cohomology of Koszul algebras.

Grading conventions. We will be using multi-graded algebras, and we generally name these gradings by i, j, k-gradings. The k-grading will always be a homological grading, and differentials always have k-degree 1, and (i,j)-degree $(0,0)$. When we speak of a differential (bi-, tri-)graded algebra, we mean (bi-, tri-)graded algebra which is a differential graded algebra with respect to the k-grading.

Generalities. The general conventions we follow concerning Koszul duality are to be found in another paper we have written [10]. If you would like more details you can consult Appendix 2 of that paper, or the article of Beilinson, Ginzburg and Soergel [1]. Throughout this section, A denotes a Koszul algebra with degree 0 part A^0 isomorphic to a direct product of a number of copies of F. The Koszul dual of A is denoted $A^!$, and $C = A \otimes_{A^0} A^!$ denotes the dg A-$A^!$-bimodule with internal differential given by $\alpha \otimes a \mapsto \sum_{\rho \in B^1} \alpha\rho \otimes \rho^* a$, where B^1 is a basis for A^1 with dual basis $\{\rho^* | \rho \in B^1\}$. The differential on C is given by internal multiplication by the unit element $\iota = \sum_{\rho \in B^1} \rho \otimes \rho^* \in A^1 \otimes A^{!1}$ whose square is zero; the formula $\iota^2 = 0$ results from the fact A and $A^!$ are quadratic duals: elements of A^2 are dual to degree 2 defining relations for $A^!$ whilst elements of $A^{!2}$ are dual to degree 2 defining relations for A. The corresponding $A^!$-A-bimodule $A^! \otimes_{A^0} A$ is denoted $C^!$. The Koszul complex K is defined to be $C \otimes_{A^!} A^{!*}$.

The dg algebra D. Let D denote the diagonal subalgebra $\bigoplus_{s,t}(e_s \otimes e_s)(A \otimes A^{!op})(e_t \otimes e_t)$ of $A \otimes A^{!op}$, which equals $\bigoplus_{s,t}(e_s A e_t \otimes_F e_t A^{!op} e_s)$ as a vector space. Multiplication, by the sign conventions for taking tensor products and opposites, is given by $(\alpha \otimes a)(\beta \otimes b) = (-1)^{|a||\beta| + |a||b|} \alpha\beta \otimes ba$. The element ι belongs to D, has degree 1, and squares to zero. We give this algebra a differential d defined by the super-commutator with $-\iota$,

$$d(\alpha \otimes a) = [-\iota, \alpha \otimes a] = \sum_{\rho \in B^1} (-1)^{|\alpha|} \alpha\rho \otimes \rho^* a - (-1)^{(|a|+|\alpha|)|\rho^*|} \rho\alpha \otimes a\rho^*$$

To see this gives D the structure of a dg algebra, note that the super commutator $d_y = [y, \square] = y\square - (-1)^{|\square|}\square y$ gives a graded algebra Y the structure of a dg algebra for any $y \in Y$ of degree 1 such that $y^2 = 0$ thanks to the following formulas:

$$d_y(ab) = yab - (-1)^{|a|+|b|} aby,$$

$$d_y(a)b = (ya - (-1)^{|y||a|} ay)b,$$

$$(-1)^{|a|} a d_y(b) = (-1)^{|a|} a(yb - (-1)^{|b|} by),$$

$$d_y(ab) = d_y(a)b + (-1)^{|a|} a d_y(b).$$

$$d_y(d_y(a)) = d_y(ya - (-1)^{|a|} ay) =$$

$$y^2 a - (-1)^{|a|+1} yay - (-1)^{|a|} yay + (-1)^{|a|+|a|+1} ay^2 = 0.$$

There are two cases we are most interested in: when A is concentrated in k-degree 0 and $A^!$ in nonnegative k-degrees, and when $A^!$ is concentrated in k-degree 0 and A in nonnegative k-degrees; in the first case we write $D = D_A$, in the second we write $D = D'_A$.

The complex $C = A \otimes A^!$ is in a natural way a left dg $A \otimes A^{!op}$-module and we define a right D-module structure on C by defining the action of $\alpha \otimes a \in D$ on $\mu \otimes m \in C$ by $(\mu \otimes m) \circ (\alpha \otimes a) = (-1)^{|a||m| + |\alpha||m|} \mu\alpha \otimes am$. This defines an algebra action which commutes with the natural left action of $A \otimes A^{!op}$. To check that this defines the structure of dg $A \otimes A^{!op}$-D-bimodule we need to check that $d((\mu \otimes m) \circ (\alpha \otimes a)) = d(\mu \otimes m) \circ (\alpha \otimes a) + (-1)^{|\mu|+|m|}(\mu \otimes m) \circ d(\alpha \otimes a)$. In the situation we wish to apply our theory to, A will be concentrated in k-degree zero and $A^!$ will be concentrated in non-negative k-degrees, so for the checking done here, we restrict to this case to avoid an overload of signs. In that case we have

$$\begin{aligned} d((\mu \otimes m) \circ (\alpha \otimes a)) &= (-1)^{|a||m|} d(\mu\alpha \otimes am) \\ &= (-1)^{|a||m|} \sum_{x \in A^1} \mu\alpha x \otimes x^* am \end{aligned}$$

as well as

$$\begin{aligned} &d(\mu \otimes m) \circ (\alpha \otimes a) + (-1)^{|m|} (\mu \otimes m) \circ d(\alpha \otimes a) \\ &\quad = \sum_{x \in A^1} (\mu x \otimes x^* m) \circ (\alpha \otimes a) \\ &\quad + (-1)^{|m|} (\mu \otimes m) \circ (\sum_{x \in A^1} (\alpha x \otimes x^* a - (-1)^{|a|} x\alpha \otimes a x^*)) \\ &\quad = (-1)^{|a||m| + |a|} \sum_{x \in A^1} (\mu x \alpha \otimes a x^* m) \\ &\quad + (-1)^{|a||m|} \sum_{x \in A^1} (\mu \alpha x \otimes x^* am - (-1)^{|a|} \mu x \alpha \otimes a x^* m) \\ &\quad = (-1)^{|a||m|} \sum_{x \in A^1} \mu \alpha x \otimes x^* am \end{aligned}$$

which proves the claim.

Resolutions. Consider the complex $B = C \otimes_{A^!} A^{!*} \otimes_{A^!} C^!$ which is isomorphic, as a vector space, to $A \otimes_{A^0} A^{!*} \otimes_{A^0} A$. We have a natural map from this complex to the complex A, sending $\alpha \otimes a \otimes \alpha'$ to $\alpha a \alpha'$ for $a \in A^{0*} \cong A^0$ and zero otherwise. If we identify B with $KC^!$, this map represents the counit of the adjunction $(K, C^!)$, and therefore our map defines a quasi-isomorphism of complexes of A-A-bimodules $B \to A$. Since B is a projective A-A-bimodule, we thus have a resolution $B \to A$ of A as an A-A-bimodule.

Lemma 3. *We have isomorphisms of vector spaces,*

$$\begin{aligned} &\mathrm{Hom}_{A \otimes A^{op}}(B, A) \cong \mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^! \otimes_{A^0} A) \\ &\cong \bigoplus_{s,t} e_s A^! e_t \otimes_F e_t A e_s \cong \bigoplus_{s,t} (e_s \otimes e_s)(A^! \otimes_F A^{op})(e_t \otimes e_t) \\ &\cong \bigoplus_{s,t} e_t A e_s \otimes_F e_s A^! e_t \cong \bigoplus_{s,t} (e_t \otimes e_t)(A \otimes_F A^{!op})(e_s \otimes e_s). \end{aligned}$$

Proof. The second, third, fourth, and fifth isomorphisms hold by definition. The first holds by a sequence of adjunctions:

$$\begin{aligned}
\mathrm{Hom}_{A\otimes A^{op}}(A\otimes_{A^0} A^{!*}\otimes_{A^0} A, A) &\cong \mathrm{Hom}_A(A\otimes_{A^0} A^{!*}\otimes_{A^0} A, A)^{A^{op}}\\
&\cong \mathrm{Hom}_{A^0}(A^{!*}\otimes_{A^0} A, \mathrm{Hom}_A(A, A))^{A^{op}}\\
&\cong \mathrm{Hom}_{A^{op}}(A^{!*}\otimes_{A^0} A, A)^{A^0}\\
&\cong \mathrm{Hom}_{A^{0op}}(A^{!*}, \mathrm{Hom}_{A^{op}}(A, A))^{A^0}\\
&\cong \mathrm{Hom}_{A^0\otimes A^{0op}}(A^{!*}, A)\\
&\cong \mathrm{Hom}_{A^0\otimes A^{0op}}(A^0\otimes_{A^0} A^{!*}, A)\\
&\cong \mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, \mathrm{Hom}_{A^0}(A^{!*}, A))\\
&\cong \mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, A^{!}\otimes_{A^0} A).
\end{aligned}$$

□

Theorem 4. *(cf.* [13, Theorem 1.1]*) We have natural dg homomorphisms $D_A \to \mathrm{End}(B)$ and $D'_{A^!} \to \mathrm{End}(B)$ which restrict to isomorphisms $H(D_A) \cong HH(A)$ and $H(D'_{A^!}) \cong HH(A)$.*

Proof. The algebra D_A acts on the right of the tensor factor C of B, naturally commuting with the left action of $A \otimes A^{op}$, which gives the first homomorphism; similarly, the algebra $D_{A^!}$ acts on the right of $C^!$ giving the second homomorphism. The first homomorphism composes with the natural map $\mathrm{Hom}(B, B) \to \mathrm{Hom}(B, A)$ and a series of adjunctions like those of Lemma 3 to give a sequence of quasi-isomorphisms

$$D_A \to \mathrm{Hom}(B, B) \to \mathrm{Hom}(B, A) \to D_A$$

whose composite is the identity. The dg homomorphism $D_A \to \mathrm{End}(B)$ is consequently a quasi-isomorphism and therefore $H(D_A) \cong H(\mathrm{End}(B)) \cong HH(A)$ as required. Likewise $H(D'_{A^!}) \cong HH(A)$. □

We deduce the following classical result:

Corollary 5. *$HH(A)$ is super-commutative.*

Proof. Since $D_A \cong {D'_{A^!}}^{op}$ under the map $a \otimes \alpha \mapsto (-1)^{|a||\alpha|}\alpha \otimes a$, we have

$$HH(A) \cong H(D_A) \cong H(D'_{A^!})^{op} \cong HH(A)^{op},$$

which is to say $HH(A)$ is super-commutative. □

Lemma 6. *We have an isomorphism $HH(A, A^*) \cong A^0$.*

Proof. We have derived isomorphisms

$$\begin{aligned}\mathrm{Hom}_{A\otimes A^{op}}(A, A^*) &\cong \mathrm{Hom}_{A^!\otimes A^{op}}(C^! \otimes_A A, C^! \otimes_A A^*)\\ &\cong \mathrm{Hom}_{A^!\otimes A^{op}}(A^! \otimes_{A^0} A, A^0)\\ &\cong \mathrm{Hom}_{A^0\otimes A^{op}}(A, \mathrm{Hom}_{A^!}(A^!, A^0))\\ &\cong \mathrm{Hom}_{A^0\otimes A^{op}}(A, A^0)\\ &\cong \mathrm{Hom}_{A^0\otimes A^{op}}(A^0 \otimes_{A^0} A, A^0)\\ &\cong \mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, \mathrm{Hom}_{A^{op}}(A, A^0))\\ &\cong \mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, A^0)\\ &\cong A^0.\end{aligned}$$

Taking H^* gives the result. □

Bimodules. Now suppose that X is a j-graded complex of A-A-bimodules.

Theorem 7. *We have an isomorphism*

$$HH(A, X) \cong \bigoplus_q H(\bigoplus_{s,t} e_s A^! e_t \otimes e_t H^q(X)e_s)$$

where the differential on $\bigoplus_{s,t} e_s A^! e_t \otimes e_t H^q(X)e_s$ *is given by*

$$a \otimes x \mapsto \sum_{\rho\in B^1} (-1)^{|x|} a\rho \otimes \rho^* x - (-1)^{(|a|+|x|)|\rho^*|} \rho a \otimes x\rho^*$$

Proof. We assume A is concentrated in k-degree 0 and $A^!$ is concentrated in non-negative k-degrees. We can prove $HH(A, X)$ is isomorphic to the homology of a complex

$$\mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, A^! \otimes_{A^0} X)$$

with differential

$$a \otimes x \mapsto \sum_{\rho\in B^1} (-1)^{|x|} a\rho \otimes \rho^* x - (-1)^{(|a|+|x|)|\rho^*|} \rho a \otimes x\rho^*$$

as we proved that $HH(A)$ is isomorphic to the homology of

$$D'_{A^!} = \mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, A^! \otimes_{A^0} A).$$

Previously we used a sequence of natural dg homomorphisms

$$D_{A^!} \to \mathrm{Hom}(B, B) \to \mathrm{Hom}(B, A) \to D_{A^!}$$

whose composite was the identity. In this case we rather use a sequence of natural dg homomorphisms

$$\mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, A^! \otimes_{A^0} X) \to \mathrm{Hom}(B, X) \to \mathrm{Hom}_{A^0\otimes A^{0op}}(A^0, A^! \otimes_{A^0} X)$$

where the first dg homomorphism sends an element $a \otimes x$ of $(A^! \otimes_{A^0} X)^{A^0}$ to the element

$$\alpha \otimes \eta \otimes \alpha' \mapsto (-1)^{(|x|+|\eta|)|\alpha'|)} \alpha\langle \eta, a\rangle x\alpha'$$

of $\mathrm{Hom}(B, X)$, and the second dg homomorphism is obtained by a sequence of adjunctions like those in Lemma 3; the proof continues mutatis mutandis. The complex $\mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^! \otimes_{A^0} X)$ has a filtration inherited from the radical filtration on A, whose sections are isomorphic to

$$\mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^{!p} \otimes_{A^0} X)$$

We consequently have a spectral sequence converging to $HH(A, X)$ whose E_1 page is

$$\bigoplus_{p,q} H^q \,\mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^{!p} \otimes_{A^0} X) \cong \bigoplus_{p,q} \mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^{!p} \otimes_{A^0} H^q(X)).$$

Koszulity of $A^!$ implies $d_l = 0$ for $l \geq 2$, because all differentials in the Koszul complex have degree one. In other words, we have degeneration at page 2 of the spectral sequence, and the E_2 page gives us $HH(A, X)$. Thus $HH(A, X)$ is the homology of the complex $\mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^! \otimes_{A^0} H(X))$ whose differential is obtained by restricting the differential on $\mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^! \otimes_{A^0} X)$. This establishes that

$$HH(A, X) \cong \bigoplus_q H(\mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^! \otimes_{A^0} H^q(X)))$$

We have

$$\mathrm{Hom}_{A^0 \otimes A^{0op}}(A^0, A^! \otimes_{A^0} H^q(X))) \cong \bigoplus_{s,t} e_s A^! e_t \otimes e_t H^q(X) e_s$$

The differential is as stated, and preserves the components $\bigoplus_{s,t} e_s A^! e_t \otimes e_t H^q(X) e_s$, and therefore the theorem holds. □

6. Some old things.

Here we gather an assortment of notions and facts we have established in previous articles. More details can be found in those articles [8], [9], [10].

Bonded bimodules. Let A be a finite dimensional algebra. We say a pair $\underline{M} = (M, M')$ of A-A-bimodules are *bonded* if we have homomorphisms $M \otimes_A M' \to A$, $M' \otimes_A M \to A$, such that the resulting pair of maps

$$M \otimes_A M' \otimes_A M \to M$$

are equal, and the resulting pair of maps

$$M' \otimes_A M \otimes_A M' \to M'$$

are equal. We call the pair and the maps a *bonding*. Given a bonded pair $\underline{M}$ of A-A-bimodules, we have a naturally defined i-graded algebra $\mathbb{T}_A(\underline{M})$, with i-degree 0 part given by A, degree i part $M^{\otimes_A i}$ for $i > 0$, and degree i part $M'^{\otimes_A -i}$ for $i < 0$ [10, Lemma 2]. If M is a differential graded A-A-bimodule which is projective on the left and right as an A-module, then M and $\mathrm{Hom}_A(M, A)$ are a bonded pair of dg bimodules [10, Lemma 3].

Algebraic operators. Let $\mathcal{T}$ denote the collection of $(A, \underline{M})$ where A is a differential k-graded algebra and $\underline{M}$ is a bonded pair of differential k-graded A-A-bimodules.

We define a *Rickard object* of $\mathcal{T}$ to be an object (A, M, M') of $\mathcal{T}$, where A is an algebra (aka a dg algebra concentrated in degree zero with trivial differential), and M, M' are adjoint two-sided tilting complexes [11].

Let $(a,\underline{m}) = (a,m,m')$ be a j-graded object of $\mathcal{T}$. We define

$$\mathbb{O}_{a,\underline{m}} \circlearrowright \mathcal{T}$$

be the 2-functor given by

$$\mathbb{O}_{a,m,m'}(A,M,M') = (a(A,M,M'), m(A,M,M'), m'(A,M,M')),$$

where

$$\alpha(A,M,M') = (\bigoplus_{j<0} \alpha^j \otimes M'^{\otimes_A j}) \oplus (\alpha^0 \otimes A) \oplus (\bigoplus_{j>0} \alpha^j \otimes M^{\otimes_A j})$$

for $\alpha \in \{a,m,m'\}$.

We now recall the definition of the operator $\mathfrak{O}$. Let $\Gamma = \bigoplus \Gamma^{ijk}$ be a differential trigraded algebra. We have an operator

$$\mathfrak{O}_\Gamma \circlearrowright \{\Sigma |\ \Sigma = \bigoplus \Sigma^{jk} \text{ a differential bigraded algebra } \}$$

given by

$$\mathfrak{O}_\Gamma(\Sigma)^{ik} = \bigoplus_{j,k_1+k_2=k} \Gamma^{ijk_1} \otimes \Sigma^{jk_2}. \tag{1}$$

The algebra structure and differential are obtained by restricting the algebra structure and differential from $\Gamma \otimes \Sigma$. If we forget the differential and the k-grading, the operator $\mathfrak{O}_\Gamma$ is identical to the operator $\mathfrak{O}_\Gamma$ defined in the introduction.

The following Lemma is a generalisation to the bonded setting of a result we proved previously [9, Lemma 7]; as in that case, the first two parts follow directly from the definitions, and the third part follows from another previous result [10, Lemma 5]:

Lemma 8. *(i) We have an isomorphism of objects of* $\mathcal{T}$

$$\mathbb{O}_{a,\underline{m}}(A,\underline{M}) \cong (\mathfrak{O}_{\mathbb{T}_a(\underline{m})}(\mathbb{T}_A(\underline{M}))^{0\diamond\bullet}, \mathfrak{O}_{\mathbb{T}_a(\underline{m})}(\mathbb{T}_A(\underline{M}))^{1\diamond\bullet}, \mathfrak{O}_{\mathbb{T}_a(\underline{m})}(\mathbb{T}_A(\underline{M}))^{-1\diamond\bullet}),$$

where the k-grading on the components of $\mathbb{O}_{a,m}(A,\underline{M})$ can be identified with the k-grading on $\mathfrak{O}_{\mathbb{T}_a(\underline{m})}(\mathbb{T}_A(\underline{M}))$.

(ii) We have an isomorphism of differential bigraded algebras

$$\mathfrak{O}_{\mathbb{T}_a(\underline{m})}(\mathbb{T}_A(\underline{M})) \cong \mathbb{T}_a(\underline{m})(A,\underline{M}).$$

(iii) We have an isomorphism of differential bigraded algebras

$$\mathbb{T}_a(\underline{m})(A,\underline{M}) \cong \mathbb{T}_{a(A,\underline{M})}(m(A,\underline{M})).$$

7. Algebraic operators and Hochschild cohomology.

Given a differential ijk-graded algebra $x = \bigoplus_i x^i$, let $\mathbb{HH}(x) = \bigoplus_i HH(x^0, x^i)$ with jk-grading inherited from that on x. We write $a = x^{0\diamond\bullet}$ for the degree 0 part of x in the i-grading.

Assume that x^i is a Rickard tilting complex over x^0 for all i, and the product map $x^h x^i \to x^{h+i}$ is a quasi-isomorphism for all h,i. In this case

$$HH(x^0, x^i) = H^* \operatorname{Hom}_{x^0 \otimes x^{0op}}(x^0, x^i) \cong H^* \operatorname{Hom}_{x^0 \otimes x^{0op}}(x^h, x^{h+i}).$$

Identifying $HH(x^0, x^i)$ with $H^* \operatorname{Hom}_{x^0 \otimes x^{0op}}(x^h, x^{h+i})$ via this isomorphism gives us a product

$$HH(x^0, x^h) \otimes HH(x^0, x^i) \to HH(x^0, x^{h+i})$$

that gives $\mathbb{HH}(x)$ the structure of an ijk-graded associative algebra.

Theorem 9. *Let $(a, \underline{m})$ be a jk-graded object in $\mathcal{T}$ with a Koszul, and let $(A, \underline{M})$ be a Rickard object in $\mathcal{T}$. Then we have*

$$\mathbb{HH}\mathfrak{O}_{\mathbb{T}_a(\underline{m})}(\mathbb{T}_A(\underline{M})) \cong \mathfrak{O}_{\mathbb{HH}(\mathbb{T}_a(\underline{m}))}(\mathbb{HH}(\mathbb{T}_A(\underline{M})))$$

as ijk-graded algebras.

Proof. For notational convenience, write $y = \mathbb{T}_A(\underline{M})$ and for any j-graded a-a-bimodule n write $n(y)$ for $n(A, \underline{M})$. We assume that a is Koszul. Then $(a \otimes_{a^0} a^{!*} \otimes_{a^0} a) \twoheadrightarrow a$ is a projective a-a-bimodule resolution of a. Let $\tilde{y} \twoheadrightarrow y$ be a projective y-y-bimodule resolution of y (which has a natural j-grading inherited from the tensor grading on y) and as for y write $n(\tilde{y}) = \bigoplus_j n^j \otimes \tilde{y}^j$ for any j-graded a-a-bimodule n. Then we claim that $a(y) \otimes_{a^0(y)} a^{!*}(\tilde{y}) \otimes_{a^0(y)} a(y) \twoheadrightarrow a(y)$ is a projective $a(y)$-$a(y)$ bimodule resolution. Indeed, as $a^{!*}$ is projective over $a^0 \otimes a^{0op}$, y^j is projective over A on both sides and $(\tilde{y})^j$ is projective over $A \otimes A^{op}$ for every j, we have that $a^{!*}(\tilde{y})$ is projective over $a^0 \otimes A \otimes (a^0 \otimes A)^{op} = a^0(y) \otimes a^0(y)^{op}$. Furthermore, $a(y)$ is projective over $a^0(y) = a^0 \otimes A$ on both sides so $a(y) \otimes a(y)^{op}$ is projective in $a^0(y) \otimes a^0(y)^{op}$-mod, so we the induced module $a(y) \otimes_{a^0(y)} a^{!*}(\tilde{y}) \otimes_{a^0(y)} a(y)$ is projective in $a(y) \otimes a(y)^{op}$-mod. It is obviously quasi-isomorphic to $(a \otimes_{a^0} a^{!*} \otimes_{a^0} a)(y)$ and hence to $a(y)$.

Now we have isomorphisms

$$\begin{aligned} HH(a(y), m^i(y)) &\cong H \operatorname{Hom}_{a(y) \otimes a(y)^{op}}(a(y) \otimes_{a^0(y)} a^{!*}(\tilde{y}) \otimes_{a^0(y)} a(y), m^i(\tilde{y})) \\ &\cong H \operatorname{Hom}_{a^0(y) \otimes a^0(y)^{op}}(a^{!*}(\tilde{y}), m^i(\tilde{y})) \\ &= H \operatorname{Hom}_{a^0 \otimes A \otimes (a^0 \otimes A)^{op}}(a^{!*}(\tilde{y}), m^i(\tilde{y})) \end{aligned}$$

by a quasi-isomorphism $m^i(\tilde{y}) \to m^i(y)$, projectivity of $a(y) \otimes_{a^0(y)} a^{!*}(\tilde{y}) \otimes_{a^0(y)} a(y)$ and adjunctions. Similarly to $a^{!*}(\tilde{y})$ being projective in $a^0(y) \otimes a^0(y)^{op}$-mod we have that

$$a^0(\tilde{y}) \otimes_{a^0(y)} a^{!*}(\tilde{y}) \cong (a^0 \otimes \tilde{A}) \otimes_{a^0 \otimes A} a^{!*}(\tilde{y}) \cong \bigoplus_j a^{!*j} \otimes \tilde{A} \otimes_A \tilde{y}^j$$

is projective over $a^0 \otimes A \otimes (a^0 \otimes A)^{op}$. It is quasi-isomorphic to $a^{!*}(\tilde{y})$, so we have

$$\begin{aligned} &H \operatorname{Hom}_{a^0 \otimes A \otimes (a^0 \otimes A)^{op}}(a^{!*}(\tilde{y}), m^i(\tilde{y})) \\ &\qquad \cong H \operatorname{Hom}_{a^0 \otimes A \otimes (a^0 \otimes A)^{op}}((a^0 \otimes \tilde{A}) \otimes_{a^0 \otimes A} a^{!*}(\tilde{y}), m^i(\tilde{y})). \end{aligned}$$

Now using adjunction again, we obtain

$$\begin{aligned} &H \operatorname{Hom}_{a^0 \otimes A \otimes (a^0 \otimes A)^{op}}((a^0 \otimes \tilde{A}) \otimes_{a^0 \otimes A} a^{!*}(\tilde{y}), m^i(\tilde{y})) \\ &\qquad \cong H \operatorname{Hom}_{a^0 \otimes A \otimes (a^0 \otimes A)^{op}}(a^0 \otimes \tilde{A}, \operatorname{Hom}_{a^0 \otimes A}(a^{!*}(\tilde{y}), m^i(\tilde{y}))). \end{aligned}$$

Next we claim that $\mathrm{Hom}_{a^0\otimes A}(a^{!*}(\tilde{y}), m^i(\tilde{y}))$ is quasi-isomorphic to $\mathrm{Hom}_{a^0\otimes A}(a^{!*}, m^i)(y)$. Indeed, denoting quasi-isomorphisms by $\to^{qim}$, we have

$$\begin{aligned}
\mathrm{Hom}_{a^0\otimes A}(a^{!*}(\tilde{y}), m^i(\tilde{y})) &\cong \bigoplus_{j_1\in\mathbb{Z}, j_2\geq 0} \mathrm{Hom}_{a^0\otimes A}(a^{!*j_2}\otimes \tilde{y}^{j_2}, (m^i)^{j_1+j_2}\otimes \tilde{y}^{j_1+j_2}) \\
&\cong \bigoplus_{j_1\in\mathbb{Z}, j_2\geq 0} \mathrm{Hom}_{a^0}(a^{!*j_2}, (m^i)^{j_1+j_2})\otimes \mathrm{Hom}_A(\tilde{y}^{j_2}, \tilde{y}^{j_1+j_2}) \\
&\to^{qim} \bigoplus_{j_1\in\mathbb{Z}, j_2\geq 0} \mathrm{Hom}_{a^0}(a^{!*j_2}, (m^i)^{j_1+j_2})\otimes \mathrm{Hom}_A(y^{j_2}, y^{j_1+j_2}) \\
&\text{by quasi-isos } \tilde{y}^j \to y^j \in A\text{-proj} \\
&\to^{qim} \bigoplus_{j_1\in\mathbb{Z}, j_2\geq 0} \mathrm{Hom}_{a^0}(a^{!*j_2}, (m^i)^{j_1+j_2})\otimes y^{j_1} \\
&\cong \bigoplus_{j_1\in\mathbb{Z}} \mathrm{Hom}_{a^0}(a^{!*}, m^i)^{j_1}\otimes y^{j_1} \\
&\cong \mathrm{Hom}_{a^0}(a^{!*}, m^i)(y).
\end{aligned}$$

As $a^0\otimes\tilde{A}$ is projective over $a^0\otimes A\otimes(a^0\otimes A)^{op}$, we have

$$\begin{aligned}
&H\,\mathrm{Hom}_{a^0\otimes A\otimes(a^0\otimes A)^{op}}(a^0\otimes\tilde{A}, \mathrm{Hom}_{a^0\otimes A}(a^{!*}(\tilde{y}), m^i(\tilde{y}))) \\
&\qquad\cong H\,\mathrm{Hom}_{a^0\otimes A\otimes(a^0\otimes A)^{op}}(a^0\otimes\tilde{A}, \mathrm{Hom}_{a^0}(a^{!*}, m^i)(y)), \\
&\qquad\cong H\,\mathrm{Hom}_{a^0\otimes A\otimes(a^0\otimes A)^{op}}(a^0\otimes\tilde{A}, (a^!\otimes_{a^0} m^i)(y))
\end{aligned}$$

Putting everything together we obtain

$$\begin{aligned}
HH(a(y), m^i(y)) &\cong H\,\mathrm{Hom}_{a(y)\otimes a(y)^{op}}(a(y)\otimes_{a^0(y)} a^{!*}(\tilde{y})\otimes_{a^0(y)} a(y), m^i(\tilde{y})) \\
&\cong H\,\mathrm{Hom}_{a^0(y)\otimes a^0(y)^{op}}(a^{!*}(\tilde{y}), m^i(\tilde{y})) \\
&\cong H\,\mathrm{Hom}_{a^0\otimes A\otimes(a^0\otimes A)^{op}}(a^0(\tilde{y})\otimes_{a^0(y)} a^{!*}(\tilde{y}), m^i(\tilde{y})) \\
&\cong H\,\mathrm{Hom}_{a^0\otimes A\otimes(a^0\otimes A)^{op} a^0\otimes A\otimes(a^0\otimes A)^{op}}(a^0(\tilde{y}), (a^!\otimes m^i)(y)) \\
&= H\,\mathrm{Hom}_{a^0\otimes A\otimes(a^0\otimes A)^{op}}(a^0\otimes\tilde{A}, (a^!\otimes m^i)(y)) \\
&= \bigoplus_j H\,\mathrm{Hom}_{a^0\otimes A\otimes(a^0\otimes A)^{op}}(a^0\otimes\tilde{A}, (a^!\otimes m^i)^j\otimes y^j) \\
&= \bigoplus_j H\,\mathrm{Hom}_{a^0\otimes a^{0op}}(a^0, a^!\otimes m^i)^j\otimes H\,\mathrm{Hom}_{A\otimes A^{op}}(\tilde{A}, y^j) \\
&\cong HH(a, m^i)(\mathbb{HH}(y))
\end{aligned}$$

completing the isomorphism as vector spaces. The fact that this is an isomorphism of ijk-graded algebras follows from the fact that on both sides multiplication is obtained from multiplication on $\mathbb{T}_a(\underline{m})\otimes\mathbb{T}_A(\underline{M})$. □

Lemma 10. *Suppose $X = \bigoplus X^i$ is a differential ijk-graded algebra with $A = X^0$ Koszul and X^i a Rickard tilting complex over A. Then $\mathbb{HH}(X)$ is isomorphic to $H(DX^{op})$, where DX is the dg algebra $\bigoplus_{s,t}(e_s\otimes e_s)(A^!\otimes X^{op})(e_t\otimes e_t)$ with differential*

$$a\otimes x \mapsto \sum_{\rho\in B^1} (-1)^{|x|} a\rho\otimes\rho^* x - (-1)^{(|a|+|x|)|\rho^*|}\rho a\otimes x\rho^*$$

Proof. Consider the space

$$\bigoplus_i H\operatorname{Hom}_{A\otimes A^{op}}(B, X^i).$$

This space admits a natural right action by $\mathbb{HH}(X)$, via the identification of $HH(X^0, X^i)$ with $H\operatorname{Hom}_{X^0\otimes X^{0op}}(X^h, X^{h+i})$, and a natural right action by $H(DX)$ via the dg homomorphism

$$\operatorname{Hom}_{A^0\otimes A^{0op}}(A^0, A^! \otimes_{A^0} X^i) \to \operatorname{Hom}_{A\otimes A^{op}}(B, X)$$

of Lemma 7. These actions commute, and the corresponding map $H(DX^{op}) \to \bigoplus_i \operatorname{End}_{\mathbb{HH}(X)}(B, X^i) \cong \mathbb{HH}(X)$ is an isomorphism, also by Lemma 7. This implies the statement of the lemma. □

For convenience, we record the following.

Lemma 11. *Let $(a, \underline{m})$ be a jk-graded object in $\mathcal{T}$ with a Koszul. Then we have $\mathbb{HH}(\mathbb{T}_a(\underline{m})) = \mathbb{HH}(\mathbb{H}(\mathbb{T}_a(\underline{m})))$.*

Proof. We have

$$\begin{aligned}\mathbb{HH}(\mathbb{T}_a(\underline{m})) &= \bigoplus_i HH(a, m^i)\\ &\text{by definition}\\ &\cong \bigoplus_i \bigoplus_q HH(a, H^q(m^i))\\ &\text{by Theorem 7}\\ &= \mathbb{HH}(\mathbb{H}(\mathbb{T}_a(\underline{m})))\\ &\text{by definition.}\end{aligned}$$

□

8. Representations of $GL_2(F)$.

Let $G = GL_2(F)$. The representation theory of G can be studied explicitly. We have proved [8, Corollary 27] that every block of $GL_2(F)$ is equivalent to a union of categories

$$\cup_l \mathbb{H}\mathbb{O}_{F,0}\mathbb{O}^l_{\mathbf{c},\underline{\mathbf{t}}}(F, (F, F))\text{-mod}\,.$$

Here $\mathbf{c}$ is the finite dimensional algebra generated by

$$\begin{tikzcd} \overset{1}{\bullet} \arrow[r, bend left, "\eta_1"] & \overset{2}{\bullet} \arrow[l, bend left, "\xi_1"] \arrow[r, bend left, "\eta_2"] & \overset{3}{\bullet} \arrow[l, bend left, "\xi_2"] & \cdots & \overset{p-1}{\bullet} \arrow[r, bend left, "\eta_{p-1}"] & \overset{p}{\bullet}, \arrow[l, bend left, "\xi_{p-1}"] \end{tikzcd}$$

modulo the ideal

$$I = (\eta_{p-1}\xi_{p-1}, \xi_{m+1}\xi_m, \eta_m\eta_{m+1}, \xi_m\eta_m + \eta_{m+1}\xi_{m+1}| \mid 1 \le m \le p-2),$$

and $\underline{\mathbf{t}} = (\mathbf{t}, \mathbf{t}^{-1})$, where $\mathbf{t}$ is a Rickard tilting complex representing the canonical tilting bimodule for $\mathbf{c}$. Note that the original statement did not use bonded bimodules as, in this entirely positively graded setting the adjoint never contributes, and worked with the tilting bimodule rather than a Rickard tilting complex and therefore did not need to take homology in the end. That working with $\mathbf{t}$ as defined above results in a quasi-isomorphic algebra was established in [8, Lemma 22].

Since quasi-isomorphic algebras share the same Hochschild cohomology, we define hh_l to be the Hochschild cohomology of the algebra $\mathbb{O}_{F,0}\mathbb{O}^l_{\mathbf{c},\mathbf{t}}(F,(F,F))$. In order to compute the Hochschild cohomology of G we compute hh_l.

9. Reduction.

The following Proposition demonstrates how our formalism of algebraic operators and homological duality reduce the computation of the algebra hh_l to the computation of the algebra $\mathbb{HH}(\mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!))$, where $\underline{\mathbf{t}}^! = (\mathbf{t}^!, \mathbf{t}^{!-1})$ it the image of $\underline{\mathbf{t}}$ under Koszul duality.

Proposition 12. *We have* $hh_l \cong \mathfrak{O}_F\mathfrak{O}^l_{\mathbb{HH}(\mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!)))}(F[z,z^{-1}])$.

Proof. We have algebra isomorphisms

$$\begin{aligned}
hh_l &\cong \mathbb{HH}\mathbb{O}_{F,0}\mathbb{O}^l_{\mathbf{c},\underline{\mathbf{t}}}(F,(F,F)) \\
&\cong \mathbb{HH}\mathfrak{O}_F\mathfrak{O}^l_{\mathbb{T}_{\mathbf{c}}(\underline{\mathbf{t}})}(F[z,z^{-1}]) && \text{by Lemma 8(i)} \\
&\cong \mathfrak{O}_F\mathbb{HH}\mathfrak{O}^l_{\mathbb{T}_{\mathbf{c}}(\underline{\mathbf{t}})}(F[z,z^{-1}]) \\
&\cong \mathfrak{O}_F\mathfrak{O}^l_{\mathbb{HH}(\mathbb{T}_{\mathbf{c}}(\underline{\mathbf{t}}))}(\mathbb{HH}(F[z,z^{-1}])) && \text{by Lemma 8(ii) and Theorem 9} \\
&\cong \mathfrak{O}_F\mathfrak{O}^l_{\mathbb{HH}(\mathbb{T}_{\mathbf{c}}(\underline{\mathbf{t}}))}(F[z,z^{-1}]).
\end{aligned}$$

Rather than computing $\mathbb{HH}(\mathbb{T}_{\mathbf{c}}(\underline{\mathbf{t}}))$ directly as $\bigoplus_i HH(\mathbf{c},\mathbf{t}^i)$, we pull $\mathbf{c}$ through Koszul duality. We have derived equivalences ([10], Appendix 2)

$$D(\mathbf{c}\text{-bigr}_{\mathrm{jk}}) \cong D(\mathbf{c}^!\text{-bigr}_{\mathrm{jk}}), \qquad D(\mathrm{bigr}_{\mathrm{jk}}\text{-}\mathbf{c}) \cong D(\mathrm{bigr}_{\mathrm{jk}}\text{-}\mathbf{c}^!)$$

$$\mathbf{c} \mapsto \mathbf{c}^{!*} \otimes_{\mathbf{c}^0} \mathbf{c}, \qquad \mathbf{c} \mapsto \mathbf{c} \otimes_{\mathbf{c}^0} \mathbf{c}^!.$$

Here $D(\mathbf{c}\text{-bigr}_{\mathrm{jk}})$ denotes the derived category of differential jk-bigraded left $\mathbf{c}$-modules and $D(\mathrm{bigr}_{\mathrm{jk}}\text{-}\mathbf{c})$ denotes the derived category of differential jk-bigraded right $\mathbf{c}$-modules. Putting these together we have

$$D(\mathbf{c}\text{-bigr}_{\mathrm{jk}}\text{-}\mathbf{c}) \cong D(\mathbf{c}^!\text{-bigr}_{\mathrm{jk}}\text{-}\mathbf{c}^!)$$

$$\mathbf{c} \mapsto \mathbf{c}^{!*} \otimes_{\mathbf{c}^0} \mathbf{c} \otimes_{\mathbf{c}^0} \mathbf{c}^!,$$

and since the equivalences $(- \otimes_{\mathbf{c}^!} \mathbf{c}^{!*} \otimes_{\mathbf{c}^0} \mathbf{c}, - \otimes_{\mathbf{c}} \mathbf{c} \otimes_{\mathbf{c}^0} \mathbf{c}^!)$ are adjoint equivalences (cf. [10], Appendix 2, *Adjunction*) we have an isomorphism in the derived category between $\mathbf{c}^!$ and $\mathbf{c}^{!*} \otimes_{\mathbf{c}^0} \mathbf{c} \otimes_{\mathbf{c}^0} \mathbf{c}^!$, Furthermore, by definition $\mathbf{t}^!$ is the image of $\mathbf{t}$ under the above equivalence. We thus have an isomorphism

$$\begin{aligned}
\mathbb{HH}(\mathbb{T}_{\mathbf{c}}(\underline{\mathbf{t}})) &= \bigoplus_i HH(\mathbf{c}, \mathbf{t}^i) \\
&\cong \bigoplus_i H\,\mathrm{Hom}_{\mathbf{c}\otimes\mathbf{c}^{op}}(\mathbf{c}, \mathbf{t}^i) \\
&\cong \bigoplus_i H\,\mathrm{Hom}_{\mathbf{c}^!\otimes\mathbf{c}^{!op}}(\mathbf{c}^!, \mathbf{c}^{!*} \otimes_{\mathbf{c}^0} \mathbf{t}^i \otimes_{\mathbf{c}^0} \mathbf{c}^!) \\
&\cong \bigoplus_i H\,\mathrm{Hom}_{\mathbf{c}^!\otimes\mathbf{c}^{!op}}(\mathbf{c}^!, (\mathbf{c}^{!*} \otimes_{\mathbf{c}^0} \mathbf{t} \otimes_{\mathbf{c}^0} \mathbf{c}^!)^i) \\
&\cong \bigoplus_i H\,\mathrm{Hom}_{\mathbf{c}^!\otimes\mathbf{c}^{!op}}(\mathbf{c}^!, t^{!i}) \\
&= \mathbb{HH}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!)).
\end{aligned}$$

which implies

$$\mathfrak{O}_F\mathfrak{O}^l_{\mathbb{HH}(\mathbb{T}_{\mathbf{c}}(\underline{\mathbf{t}}))}(F[z,z^{-1}]) \cong \mathfrak{O}_F\mathfrak{O}^l_{\mathbb{HH}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!))}(F[z,z^{-1}]),$$

completing the proof of the Proposition. We note that, as we are ultimately interested in ρ with k-grading given by the homological grading on Hochschild cohomology, we work with the gradings that suit this purpose, i.e. $\mathbf{c}$ is assumed to be concentrated in k-degree zero and $\mathbf{c}^!$ is assumed to be concentrated in positive k-degrees.

We have now established that $hh_l \cong \mathfrak{O}_F\mathfrak{O}^l_{\mathbb{HH}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!))}(F[z,z^{-1}])$. But $\mathbb{HH}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!))$ is isomorphic to $\mathbb{HH}(\mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!)))$ by Lemma 11 which completes the proof of the Proposition. □

The above Proposition leaves us with the problem of computing $\mathbb{HH}(\mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!)))$ in the remaining sections. We compute $\mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!))$ in Section 10, then the Hochschild cohomology of the bimodules appearing in $HH(\mathbf{c}^!, H^q(\mathbf{t}^{!i}))$ for various i in Section 11, and finally infer the multiplication on $\mathbb{HH}(\mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!)))$ from that on $\mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!))$ in Section 12.

10. The algebra $\clubsuit = \mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!))$.

In this section we compute the homology algebra of the dg algebra $\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!)$. The latter algebra has interesting homology, entwining the algebra $\mathbf{c}^!$, its dual, its tilting bimodule, and a preprojective algebra Θ in a subtle way.

We first need some notation. The algebra $\mathbf{c}$ has generators ξ and η, and its Koszul dual $\mathbf{c}^! = \Omega$ has dual generators x and y; Here is the quiver of Ω:

$$\begin{tikzcd}
\overset{1}{\bullet} \arrow[r, bend right, "y_1"'] & \overset{2}{\bullet} \arrow[l, bend right, "x_1"'] \arrow[r, bend right, "y_2"'] & \overset{3}{\bullet} \arrow[l, bend right, "x_2"'] & \cdots & \overset{p-1}{\bullet} \arrow[r, bend right, "y_{p-1}"'] & \overset{p}{\bullet} \arrow[l, bend right, "x_{p-1}"']
\end{tikzcd}$$

The relations for Ω are $x_1y_1 = 0$ and $xy = yx$. For notational convenience we use a different convention for the direction of arrows in Ω than we used in our previous article [9].

We define the algebra Θ to be the quotient $\Omega/\Omega e_p\Omega$, and σ is the involution of Θ which switches e_s and e_{p-s}, and x and y. The algebra Θ is called the *preprojective algebra of type* A_{p-1}. It is a self-injective algebra with Nakayama automorphism σ.

Homology of the bimodules $\mathbf{t}^{!i}$. The complex $\mathbf{t}^{!-1}$ can as a left complex of projectives be presented as a two term complex

$$\Omega e_p^{\oplus p} \to \Omega(\sum_{1\leq l\leq p-1} e_l)$$

because $\mathbf{t}^{!-1}$ sends P_l to $P_p \to P_l$ for $l \neq p$ and $P_p[-1]$ otherwise. The adjoint $\mathbf{t}^!$ of this complex can, again as a complex of left projectives be represented as

$$\Omega(\sum_{1\leq l\leq p-1} e_l) \to \Omega e_p^{\oplus p}.$$

The homology of this complex is zero in degree 0 and $\Omega e_p\Omega$ in degree 1.

The $i+1$-term complex representing $\mathbf{t}^{!-i}$

$$\Omega e_p^{\oplus p} \to ... \to \Omega e_p^{\oplus p} \to \Omega(\sum_{1\leq l\leq p-1} e_l)$$

for $i > 0$ has homology

$$\Omega, \Theta^\sigma, \Theta, \Theta^\sigma, \dots, \Theta^{\sigma^i}$$

generated in degrees $-i, -i+1, \dots, -1, 0$ ([9, Lemma 27]). Taking $\mathrm{Hom}_\Omega(\mathbf{t}^{!-i}, \Omega)$ and using the standard duality Ω-mod $\cong$ mod-Ω, the adjoint $\mathbf{t}^{!i}$ can be represented as

$$\Omega(\sum_{1\leq l\leq p-1} e_l) \to \Omega e_p^{\oplus p} \to ... \to \Omega e_p^{\oplus p},$$

which is generated in degrees $0, 1, \dots, i$, and has homology

$$0, 0, \Theta^{\sigma^{i-1}}, \dots, \Theta, \Theta^\sigma, \Theta, \Omega^*,$$

for $i > 1$.

We now describe the algebra $\clubsuit$ via a collection of bimodules, and bimodule homomorphisms. By the above analysis, its structure as an Ω-Ω-bimodule is given as follows, where the rows represent powers of $\mathbf{t}^!$, and the columns represent homological degrees (where the Ω appearing alone in a row is in position $(0,0)$):

$$\begin{array}{ccccccccc}
 & & & & & & \dots & & \\
 & & & & & & \Theta^\sigma & \Theta & \Omega^* \\
 & & & & & & \Theta & \Omega^* & \\
 & & & & & & \Omega^* & & \\
 & & & & & \Omega e_p\Omega & & & \\
 & & & & \Omega & & & & \\
 & & & \Omega & \Theta^\sigma & & & & \\
 & & \Omega & \Theta^\sigma & \Theta & & & & \\
 & \Omega & \Theta^\sigma & \Theta & \Theta^\sigma & & & & \\
\Omega & \Theta^\sigma & \Theta & \Theta^\sigma & \Theta & & & & \\
 & & \dots & & & & & &
\end{array}$$

Remark 13 The bimodule $\Omega e_p \Omega$ is a tilting bimodule for the Ringel self-dual quasi-hereditary algebra Ω. We can recover $\mathbb{T}_{\mathbf{c}^!}(\underline{\mathbf{t}}^!)$ from its homology, since $\mathbf{t}^!$ is quasi-isomorphic to $\Omega e_p \Omega$.

The algebra Ω is (j,k)-graded with both x and y in (j,k)-degree $(-1,1)$ coming from the (j,k)-grading on $\mathbf{c}$, which has η, ξ in (j,k)-degree $(1,0)$.

In [9, Lemma 27(ii)] we established that as differential bigraded modules $\mathbb{H}(\mathbf{t}^{!-i})$ is isomorphic to

$$\Omega\langle -ip\rangle[i(p-1)] \oplus \Theta^{\sigma}\langle -(i-1)p\rangle[(i-1)(p-1)] \oplus \cdots \oplus \Theta^{\sigma^i}\langle 0\rangle[0]$$

The structure of $\clubsuit^- = \mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\mathbf{t}^{!-1}))$ as a k-graded Ω-Ω-bimodule is therefore given by

$$\begin{array}{ccccc}
 & & & & \Omega[0] \\
 & & & \Omega[p-1] & \Theta^{\sigma}[0] \\
 & & \Omega[2p-2] & \Theta^{\sigma}[p-1] & \Theta[0] \\
 & \Omega[3p-3] & \Theta^{\sigma}[2p-2] & \Theta[p-1] & \Theta^{\sigma}[0] \\
\Omega[4p-4] & \Theta^{\sigma}[3p-3] & \Theta[2p-2] & \Theta^{\sigma}[p-1] & \Theta[0] \\
 & & \cdots & &
\end{array}$$

and the structure of $\clubsuit^-$ as a j-graded Ω-Ω-bimodule is given by

$$\begin{array}{ccccc}
 & & & & \Omega\langle 0\rangle \\
 & & & \Omega\langle -p\rangle & \Theta^{\sigma}\langle 0\rangle \\
 & & \Omega\langle -2p\rangle & \Theta^{\sigma}\langle -p\rangle & \Theta\langle 0\rangle \\
 & \Omega\langle -3p\rangle & \Theta^{\sigma}\langle -2p\rangle & \Theta\langle -p\rangle & \Theta^{\sigma}\langle 0\rangle \\
\Omega\langle -4p\rangle & \Theta^{\sigma}\langle -3p\rangle & \Theta\langle -2p\rangle & \Theta^{\sigma}\langle -p\rangle & \Theta\langle 0\rangle. \\
 & & \cdots & &
\end{array}$$

The summands of the complex representing $\mathbf{t}^!$ viewed as a differential bigraded left Ω-module are given by $\mathbf{t}^! e_l = \Omega e_{p-l} \oplus \Omega e_p\langle l\rangle[1-l]$ and has homology $\Omega e_p \Omega e_l\langle p\rangle[1-p]$, which sums up to give total homology $\Omega e_p \Omega\langle p\rangle[1-p]$. For $\mathbf{t}^{!2}$, we obtain a differential bigraded left Ω-module $\mathbf{t}^{!2} e_l = \Omega e_l \oplus \Omega e_p\langle p-l\rangle[l-p+1] \oplus \Omega e_p\langle p+l\rangle[-l-p+2]$. The homology of this is the injective $I_l = (e_l\Omega)^*$, which has socle in j-degree $p+l-(p-1)-(l-1) = 2$ and k-degree $-l-p+2+(p-1)+(l-1) = 0$, hence $\mathbb{H}(\mathbf{t}^{!2}) \cong \Omega^*\langle 2\rangle[0]$.

We obtain $\clubsuit^+ = \mathbb{H}(\mathbb{T}_{\mathbf{c}^!}(\mathbf{t}^!))$ from $\clubsuit^-$ by noting that $\Omega^* \otimes_\Omega -$ is a Serre functor on $D^b(\Omega)$ and hence in the ungraded setting we have

$$\mathbf{t}^{!-i} = \mathrm{Hom}_\Omega(\mathbf{t}^{!i}, \Omega) \cong \mathrm{Hom}_\Omega(\Omega, \Omega^* \otimes \mathbf{t}^{!i})^* \cong \mathrm{Hom}_\Omega(\Omega, \mathbf{t}^{!i+2})^* = \mathbf{t}^{!i+2*}.$$

Putting in gradings, this gives

$$\mathbf{t}^{!-i} = \mathrm{Hom}_\Omega(\Omega, \Omega^* \otimes \mathbf{t}^{!i})^* \cong \mathrm{Hom}_\Omega(\Omega, \mathbf{t}^{!i+2}\langle -2\rangle[0])^* = \mathbf{t}^{!i+2*}\langle 2\rangle[0],$$

so

$$\begin{aligned}
\mathbb{H}(\mathbf{t}^{!i}) &\cong (\mathbb{H}(\mathbf{t}^{!-(i-2)})\langle -2\rangle)^* \\
&\cong (\Omega\langle -(i-2)p\rangle[(i-2)(p-1)] \oplus \Theta^{\sigma}\langle -(i-3)p\rangle[(i-3)(p-1)] \oplus \cdots \oplus \Theta^{\sigma^{i-2}}\langle 0\rangle[0])^*\langle 2\rangle \\
&\cong (\Omega^*\langle (i-2)p\rangle[(i-2)(1-p)] \oplus \Theta^{\sigma *}\langle (i-3)p\rangle[(i-3)(1-p)] \oplus \cdots \oplus \Theta^{\sigma^{i-2}*}\langle 0\rangle[0])\langle 2\rangle \\
&\cong \Omega^*\langle 2+(i-2)p\rangle[(i-2)(1-p)] \oplus \Theta^{\sigma *}\langle 2+(i-3)p\rangle[(i-3)(1-p)] \oplus \cdots \oplus \Theta^{\sigma^{i-2}*}\langle 2\rangle[0]
\end{aligned}$$

Using $\Theta^* \cong \Theta^{\sigma}\langle p-2\rangle[2-p]$, we obtain

$$\mathbb{H}(\mathbf{t}^{!i}) \cong \Omega^*\langle 2+(i-2)p\rangle[(i-2)(1-p)] \oplus \Theta\langle (i-2)p\rangle[(i-2)(1-p)+1] \oplus \cdots \oplus \Theta^{\sigma^{i-3}}\langle p\rangle[2-p]$$

Hence the structure of $\clubsuit^+$ as a k-graded Ω-Ω-bimodule is given by

$$\begin{array}{cccccc}
 & & \dots & & & \\
 & & \Theta[2-p] & \Theta^{\sigma}[3-2p] & \Theta[4-3p] & \Omega^*[3-3p] \\
 & & \Theta^{\sigma}[2-p] & \Theta[3-2p] & \Omega^*[2-2p] & \\
 & & \Theta[2-p] & \Omega^*[1-p] & & \\
 & & \Omega^*[0] & & & \\
 & \Omega e_p \Omega[1-p] & & & & \\
\Omega & & & & &
\end{array}$$

while the structure of $\clubsuit^+$ as a j-graded Ω-Ω-bimodule is given by

$$\begin{array}{cccccc}
 & & \dots & & & \\
 & & \Theta\langle p\rangle & \Theta^{\sigma}\langle 2p\rangle & \Theta\langle 3p\rangle & \Omega^*\langle 2+3p\rangle \\
 & & \Theta^{\sigma}\langle p\rangle & \Theta\langle 2p\rangle & \Omega^*\langle 2+2p\rangle & \\
 & & \Theta\langle p\rangle & \Omega^*\langle 2+p\rangle & & \\
 & & \Omega^*\langle 2\rangle & & & \\
 & \Omega e_p \Omega\langle p\rangle & & & & \\
\Omega & & & & &
\end{array}$$

The symmetric algebra $\clubsuit$. Under favourable circumstances the algebra $\mathbb{H}(\mathbb{T}_A(M, M^{-1}))$ is a symmetric algebra:

Lemma 14. *Suppose A is a finite dimensional dg algebra and M a finite dimensional dg bimodule that is projective on the left and right as an A-module such that $M^l \otimes_A -$ is a Serre functor for the derived category of A, for some l. Then $\mathbb{H}(\mathbb{T}_A(M, M^{-1}))$ is a symmetric algebra via the symmetric bilinear form*

$$\mathbb{H}(\mathbb{T}_A(M, M^{-1})) \otimes \mathbb{H}(\mathbb{T}_A(M, M^{-1})) \to F$$

that is a sum of forms

$$HM^{-i} \otimes HM^{i+l} \to F.$$

Proof. Since tensoring with $M^l = M^{\otimes_A l}$ is a Serre functor for the derived category we have HA-HA-bimodule isomorphisms

$$\begin{aligned} HM^{-i} = H\operatorname{Hom}_A(M^i, A) &\cong H\operatorname{Hom}_A(A, M^l \otimes M^i)^* \\ &\cong H\operatorname{Hom}_A(A, M^{i+l})^* \cong HM^{i+l*}. \end{aligned}$$

This defines our forms. We can place such isomorphisms in a diagram of natural maps

$$\begin{tikzcd}
H\operatorname{Hom}_A(M^i, A) \otimes_F M \arrow[rr, "\sim"] \arrow[d] & & HM^{i+l*} \otimes_F M \arrow[d] \\
H\operatorname{Hom}_A(M^i, M) \arrow[r] & H\operatorname{Hom}_F(M \otimes_F M^{i-1}, M) \arrow[r, "\sim"] & HM^{i+l-1*} \otimes_F M^* \otimes_F M \\
& H\operatorname{Hom}_A(M \otimes_F M^{i-1}, M) \arrow[u] & \\
& H\operatorname{Hom}_A(M \otimes_A M^{i-1}, M) \arrow[u] & \\
H\operatorname{Hom}_A(M^{i-1}, A) \arrow[uuu, "\wr"] \arrow[ur, "\sim"] \arrow[rr, "\sim"] & & HM^{i-1+l*} \arrow[uuu]
\end{tikzcd}$$

that commutes. Adjunctions and the isomorphism $M \to M^{**}$ give us natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_F(M^{i-1+l*}, M^{i+l-1*} \otimes_F M^* \otimes_F M) &\cong \operatorname{Hom}_F(M^{i-1+l*} \otimes_F M^*, M^{i+l-1*} \otimes M^*) \\ &\cong \operatorname{Hom}_F(M^{i-1+l*} \otimes_F M^* \otimes_F M, M^{i+l-1*}) \end{aligned}$$

which allows us to identify the bottom right arrow in our diagram (a counit) with the identity on HM^{i+l-1*}, or an arrow pointing in the other direction (a unit). The commutativity of the bottom half of the above diagram is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
H\operatorname{Hom}_A(M^i, M) \arrow[r] \arrow[d, dash, "\wr"] & HM^{i+l-1*} \otimes_F M^* \otimes_F M \arrow[d] \\
H\operatorname{Hom}_A(M^{i-1}, A) \arrow[r, dash, "\sim"] & HM^{i-1+l*}
\end{tikzcd}$$

Putting this together with the top half of the severed diagram gives us a commutative diagram

$$\begin{tikzcd}
H\operatorname{Hom}_A(M^i, A) \otimes_F M \arrow[r, dash, "\sim"] \arrow[d] & HM^{i+l*} \otimes_F M \arrow[d] \\
H\operatorname{Hom}_A(M^i, M) \arrow[r] \arrow[d, dash, "\wr"] & HM^{i+l-1*} \otimes_F M^* \otimes_F M \arrow[d] \\
H\operatorname{Hom}_A(M^{i-1}, A) \arrow[r, dash, "\sim"] & HM^{i-1+l*}
\end{tikzcd}$$

The left hand vertical of *this* diagram describes the right action of M on $\mathbb{H}(\mathbb{T}_A(M, M^{-1}))$, and the right hand vertical corresponds to the right action of M on $\mathbb{H}(\mathbb{T}_A(M, M^{-1}))^*$. This implies that the identification of $\mathbb{H}(\mathbb{T}_A(M, M^{-1}))$ with its dual induced by our forms is right HM-equivariant for $i > 0$. Likewise the identification is right HM^i-equivariant for $i > 0$, likewise left HM^i-equivariant for $i > 0$, and likewise HM^i-equivariant for $i < 0$. □

Corollary 15. *$\clubsuit$ is a symmetric algebra.*

Proof. We apply the above Lemma in case $A = \Omega$, $M = \mathbf{t}^!$, and $l = 2$. □

The product on $\clubsuit$. We now investigate the algebra structure on $\clubsuit$. In another place [9, Lemma 27(iii)] we showed that $\clubsuit^-$ is nothing but the tensor algebra $\mathbb{T}_\Omega(\Theta^\sigma) \otimes F(\zeta)$. We define here a number of bimodule homomorphisms, which we then show provide the remaining multiplications in $\clubsuit$.

Lemma 16. *We have natural bimodule homomorphisms,*

$$\beta : \Omega e_p \Omega \cong (\Omega e_p \Omega)^*,$$
$$\alpha : \Omega e_p \Omega \hookrightarrow \Omega, \quad \gamma : \Omega^* \twoheadrightarrow \Omega e_p \Omega,$$
$$\epsilon : \Omega^* \otimes_\Omega \Omega^* \cong \Omega^*,$$
$$\zeta_l : \Omega e_p \Omega \otimes_\Omega \Omega^* \cong \Omega^*, \quad \zeta_r : \Omega^* \otimes_\Omega \Omega e_p \Omega \cong \Omega^*,$$
$$\eta : \Omega e_p \Omega \otimes_\Omega \Omega e_p \Omega \cong \Omega^*,$$
$$\theta_l : \Omega \otimes_\Omega \Omega^* \to \Omega e_p \Omega, \quad \theta_r : \Omega^* \otimes_\Omega \Omega \to \Omega e_p \Omega$$
$$\iota_l : \Omega \otimes_\Omega \Omega^* \to \Omega, \quad \iota_r : \Omega^* \otimes_\Omega \Omega \to \Omega,$$
$$\lambda : \Theta \cong \Theta^{*\sigma},$$
$$\kappa : \Omega \twoheadrightarrow \Theta, \quad \mu : \Theta^\sigma \hookrightarrow \Omega^*,$$
$$\nu_l : \Theta \otimes \Theta^\sigma \to \Omega^*, \quad \nu_r : \Theta^\sigma \otimes \Theta \to \Omega^*,$$

Proof. Let us define a form

$$\Omega e_p \Omega \otimes \Omega e_p \Omega \twoheadrightarrow F$$

sending $e_s x^d y^e e_t \otimes e_{s'} x^{d'} y^{e'} e_{t'}$ to 1 if $s = t'$, $t = s'$, and $d+d' = e+e' = p-1$, and to zero otherwise. This is a symmetric associative nondegenerate bilinear form, giving us the isomorphism β. The natural embedding of $\Omega e_p \Omega$ in Ω is α. Composing α and β and then dualising gives us γ. We have a bimodule homomorphism

$$\epsilon : \Omega^* \otimes_\Omega \Omega^* \to \Omega^*,$$

which can be obtained as the composition

$$\Omega^* \otimes_\Omega \Omega^* \to \Omega^* \otimes \Omega e_p \Omega \to \Omega^*,$$

where the first map is $1_{\Omega^*} \otimes \gamma$ and the second is obtained by restricting the action of Ω on Ω^*. Since the kernel of the map $\Omega^* \to \Omega e_p \Omega$ is sent to zero under left or right multiplication by e_p, the map factors through $\Omega e_p \Omega \otimes_\Omega \Omega e_p \Omega$, giving us ζ_l, ζ_r, and η. Note that η is an isomorphism, because

$$\begin{aligned}\mathrm{Hom}_F(\Omega e_p \Omega \otimes_\Omega \Omega e_p \Omega, F) &\cong \mathrm{Hom}_\Omega(\Omega e_p \Omega, \mathrm{Hom}(\Omega e_p \Omega, F)) \\ &\cong \mathrm{Hom}_\Omega(\Omega e_p \Omega, \Omega e_p \Omega) \cong \Omega\end{aligned}$$

where the last iso comes from the fact that $\Omega e_p\Omega$ is a tilting bimodule; taking $\mathrm{Hom}(-, F)$ gives us $\Omega e_p\Omega \otimes_\Omega \Omega e_p\Omega \cong \Omega^*$. Since η is an isomorphism, ζ_l, ζ_r and ϵ are also isomorphisms.

The morphisms θ_l, θ_r are just given by γ composed with the canonical isomorphisms $\Omega\otimes_\Omega\Omega^* \cong \Omega^* \cong \Omega^*\otimes_\Omega\Omega$ and ι_l, ι_r are the compositions of θ_l, θ_r with α respectively.

The preprojective algebra is self-injective with Nakayama automorphism σ, giving us λ. Indeed, the map

$$\Theta \otimes \Theta \to F$$

that sends $e_s x^d y^e e_t \otimes e_{s'} x^{d'} y^{e'} e_{t'}$ to 1 if $s = t'$, $p-t = s'$, and $d+d'+e+e' = p-1$, and to zero otherwise, is an associative nondegenerate bilinear form, such that $\langle a, a'\rangle = \langle a', a^\sigma\rangle$. By definition we have a canonical surjection $\kappa : \Omega \twoheadrightarrow \Theta$, implying we have a dual bimodule homomorphism $\Theta \to \Omega^*$.

Composing the product map $\Theta\otimes\Theta^\sigma \to \Theta^\sigma$ with μ gives us ν_l; similiarly for ν_r. $\square$

Loosely speaking, the naturality of F-linear duality means we have products given by natural maps on both $\clubsuit^+$ and $\clubsuit^-$ so long as we have products given by natural maps on $\clubsuit^-$; this much naturalness we have established in our paper on the Yoneda extension algebra of G [9]. But we should be more precise.

To describe the product on $\clubsuit$ using our natural bimodule homomorphisms we split the algebra into five parts: Ω_-, which consists of all copies of Ω in $\clubsuit^-$, Θ_-, which consists of all copies of Θ or Θ^σ in $\clubsuit^-$, $\Omega e_p\Omega$, which sits in i-degree 1, Ω^*_+, which consists of all copies of Ω^* appearing in $\clubsuit^+$, and Θ_+ consisting of all copies of Θ or Θ^σ in $\clubsuit^+$.

Proposition 17. *The multiplication between these five parts is given by the following table:*

	Ω_-	Θ_-	$\Omega e_p\Omega$	Θ_+	Ω^*_+
Ω_-	a	a	a	a	ι, θ, a
Θ_-	a	a	0	$0, a, \nu$	0
$\Omega e_p\Omega$	a	0	η	0	ζ
Θ_+	a	$0, a, \nu$	0	0	0
Ω^*_+	ι, θ, a	0	ζ	0	ϵ

Here a is our generic notation for an action map. For the products where we give several options, the choice depends on the component in which the product lands. In the case of products between Ω_- and Ω^*_+ this is determined by

Component in which the product lands:	Ω_-	$\Omega e_p\Omega$	Ω^*_+
Natural map describing the product:	ι	θ	a

and in the case of products between Θ_- and Θ_+, it is given by

Component in which the product lands:	$\clubsuit^-$	$\Omega e_p\Omega$	Θ_+	Ω^*_+
Natural map describing the product:	0	0	a	ν.

Proof. The fact that the product on $♣^-$ is as given in the top left 2×2-corner of our table we have already established in a previous paper ([9] Proposition 21).

We consider zero products in ♣. There are two reasons for which the product of a pair of components c_1 and c_2 of ♣ is necessarily zero in ♣. Firstly it may be the case that the only elements of ♣ with the same degree as c_1c_2 is 0, for some grading on ♣; this argument implies that the product of Θ_+ and Θ_+ is zero, and that the product of components of Θ_- and Θ_+ is zero if the component in which the product lands lies in $♣^-$. Secondly it may be the case that $c_1 \otimes_\Omega c_2 = 0$; this argument implies that the product of Θ_+, Θ_- and $\Omega e_p \Omega$, Ω^* is zero, since ${}_\Omega\Omega^*$ and $\Omega e_p \Omega$ are quotients of $\Omega e_p^{\oplus p}$, we have $\Theta \otimes \Omega e_p = 0$, and $\Theta \otimes -$ is left exact. This provides all remaining zeros in the table.

We consider nonzero products of components c_1 and c_2 both of which lie in $♣^+$. The only situation in which such products occur is the case where both c_1 and c_2 and lie on the diagonal, and are thus either isomorphic to $\Omega e_p \Omega$ or Ω^*. The product in this case can be identified with the maps we have defined thanks to the fact that $\mathbf{t}^{!1} = \Omega e_p \Omega$; indeed making this identification means that $\mathbf{t}^{!i}$ can be naturally represented by $Q^{\otimes_\Omega i}$, where Q is a projective bimodule resolution of $\Omega e_p \Omega$. We have a quasi-isomorphism $Q \twoheadrightarrow \Omega e_p \Omega$; the rightmost component of $\mathbb{H}(\mathbf{t}^{!i})$ is thus identified with $\Omega e_p \Omega^{\otimes_\Omega i}$ which is naturally identified with Ω^* for $i \geq 2$ via the natural maps ζ and η; these identifications provide our products.

We consider nonzero multiplication of components c_1 and c_2 where one of these belongs to $♣^-$ and the other belongs to $♣^+$. Since $♣^- = \mathbb{T}_\Omega(\Theta^\sigma)$, we only need to consider the case where one is a component of $\mathbb{H}(\mathbf{t}^{!-1})$ and the other a component of $\mathbb{H}(\mathbf{t}^{!i})$ for $i > 1$. We now use the fact that ♣ is a symmetric algebra via the Serre functor $\mathbf{t}^{!2} \otimes_\Omega -$: for Y a component of $\mathbf{t}^{!-1}$ the product

$$\mathbf{t}^{!-i+3} \otimes Y \to \mathbf{t}^{!-i+2}$$

has dual

$$\mathbf{t}^{!-i+3*} \otimes Y^* \leftarrow \mathbf{t}^{!-i+2*}.$$

Tensoring with Y and contracting with the counit for Y gives us

$$\mathbf{t}^{!-i+2*} \otimes Y \to \mathbf{t}^{!-i+3*} \otimes Y^* \otimes Y \to \mathbf{t}^{!-i+3*},$$

which by Corollary 15 is a map $\mathbf{t}^{!i} \otimes Y \to \mathbf{t}^{!i-1}$. Consequently, whenever the product $\mathbf{t}^{!-i+3} \otimes Y \to \mathbf{t}^{!-i+2}$ can be identified with an action map, the corresponding map $\mathbf{t}^{!i} \otimes Y \to \mathbf{t}^{!i-1}$ can also be identified with an action map. We can recover the product of $♣^+$ and $♣^-$ from previously established products with this observation. To begin with, this argument implies that products of Ω_- with components of Θ_+, products of Ω_- with Ω_+^* landing in Ω_+^*, and products of Θ_- with Θ_+ landing in Θ_+ are as stated. We are left with the problem of checking four types of maps: the product of Ω_- with Ω_+^* landing in $\Omega e_p \Omega$; the product of Ω_- with Ω_+^* landing in Ω_- the product of Ω_- with $\Omega e_p \Omega$ landing in Ω_-; the product of Θ_- with Θ_+ landing in Ω_+^*. These four types can be dealt with the following way: we factor over an action map, whose dual we know to be an action map by the preceding argument. For example, for the fourth type of product, we know the product of a component Θ_- with a component Θ_+ landing in a component Ω^* of Ω_+^* must in fact land in the subset $\Theta^\sigma \subset \Omega^*$ obtained by dualising the map $\Omega \to \Theta$ and identifying Θ^* with Θ^σ; this product must be dual to an action map of Θ on Θ landing in Θ (possibly twisted by σ); this means it must be given by ν_l or ν_r as required. For the third type of product, it is enough to treat the product of $\Omega e_p \Omega$ in i-degree 1 with Ω in i-degree -1, landing in Ω in i-degree 0; this product is dual to the action of $\Omega e_p \Omega$ in i-degree 1 on Ω^* in i-degree 1, factoring over $\gamma : \Omega^* \to \Omega e_p \Omega$ and landing in Ω^* in i-degree 2; this product is dual to the action of $\Omega e_p \Omega$ on Ω as required. For the first and second type of product, it is enough to treat the product of Ω^* in i-degree 2 with Ω in i-degree -1, landing in $\Omega e_p \Omega$ in i-degree 1; this product factors through the map $\Omega^* \to \Omega e_p \Omega$ and allowing for the factorisation is dual to the action of Ω on Ω^*; the products are thus θ_l, θ_r, ι_l, or ι_r, as required.

In the above argument we have established the product on $\clubsuit$ in its entirety, but there is a small gap: we have not shown that the isomorphism between Ω^* and $\Omega e_p \Omega^{\otimes_\Omega i}$ used in defining the product on $\clubsuit^+$ is compatible with the identification of the rightmost component of $\mathbb{H}(\mathbf{t}^{!i})$ for $i \geq 2$ with Ω^* by taking the dual of the leftmost component of $\mathbb{H}(\mathbf{t}^{!i})$ for $i \leq 0$. To see this it remains for us to note that the product of $\mathbf{t}^{!i}$ with a component Y' of $\mathbf{t}^{!1}$ can be obtained from the product of $\mathbf{t}^{!-i+1*}$ with Y' by the same argument that establishes the product of $\mathbf{t}^{!i}$ with a component Y of $\mathbf{t}^{!-1}$ is obtained from the product of $\mathbf{t}^{!-i+3*}$ with Y; consequently the identification of the rightmost component with Ω^* by taking duals mentioned above demands that multiplication by the component $\Omega e_p \Omega$ of is given by an action map, which is compatible with the isomorphism between Ω^* and $\Omega e_p \Omega^{\otimes_\Omega i}$ used in defining the product on $\clubsuit^+$, as required. $\square$

11. Explicit Hochschild cohomology of some bimodules.

Here we describe the components of $HH(\mathbf{c}^!, \clubsuit)$ as $HH(\mathbf{c}^!)$-$HH(\mathbf{c}^!)$-bimodules.

Let us first describe the centres of our algebras $\mathbf{c}$ and $\mathbf{c}^!$.

Lemma 18. *The centre of* $\mathbf{c}$ *is* $Z(\mathbf{c}) = F.1 \oplus \mathbf{c}^2$. *The centre of* Ω *is* $Z(\Omega) = F[z]/z^p$ *where* $z = xy$ *has* k-*degree* 2.

Proposition 19. *Suppose* $p > 2$.

(i) $HH(\Omega)$ *is isomorphic to* $Z(\mathbf{c}) \otimes Z(\Omega) \otimes \bigwedge(\kappa)/(\mathbf{c}^2.z, \mathbf{c}^2\kappa, z^{p-1}\kappa)$, *where* $\mathbf{c}^2$ *has* jk-*degree* $(2,0)$, *the* z *has* jk-*degree* $(-2,2)$ *and* κ *has* jk-*degree* $(0,1)$.

(ii) $HH(\Omega, \Theta)$ *is isomorphic to* $HH(\Omega)/(z^{\frac{p-1}{2}})$ *as an* $HH(\Omega)$-$HH(\Omega)$-*bimodule.*

(iii) $HH(\Omega, \Theta^\sigma)$ *is isomorphic to* $HH(\Omega, \Theta)^*$ *as an* $HH(\Omega)$-$HH(\Omega)$-*bimodule.*

(iv) $HH(\Omega, \Omega^*)$ *is isomorphic to* Ω^0.

(v) $HH(\Omega, \Omega e_p \Omega)$ *is isomorphic to the kernel of the natural surjection*

$$HH(\Omega) \to HH(\Omega)/(z^{\frac{p-1}{2}}).$$

Proof. (i) We know $HH(\Omega) = HH(\mathbf{c}) = H(D_{\mathbf{c}})$, and therefore compute the homology of $D_{\mathbf{c}} = \bigoplus_{s,t} e_s \mathbf{c} e_t \otimes e_t \Omega e_s$. The differential on $D_{\mathbf{c}}$ sends $\alpha \otimes a$ to

$$\alpha\xi \otimes ya + \alpha\eta \otimes xa - (-1)^{|a|}\xi\alpha \otimes ay - (-1)^{|a|}\eta\alpha \otimes ax$$

The complex $D_{\mathbf{c}}$ is $\mathbb{Z}^2$-graded, where we give e_s degree $(0,0)$, we give x and y degree $(0,1)$, and we give ξ and η degree $(-1,0)$ (note that this is induced by our normal j-grading). The differential therefore has degree $(-1,1)$. We have a basis for $e_s\mathbf{c}e_t \otimes e_t\Omega e_s$ given by those monomials $e_s\xi^{m_\xi}\eta^{m_\eta}e_t \otimes e_tx^{m_x}y^{m_y}e_s$ which are not zero in this space. In terms of graded subspaces of $D_\mathbf{c}$, the only nonzero components are $D_\mathbf{c}^{-2,0}$, $D_\mathbf{c}^{0,2l}$, $D_\mathbf{c}^{-1,2l+1}$ and $D_\mathbf{c}^{-2,2l+2}$ for $0 \le l \le p-1$. The first is just $\mathbf{c}^2 \otimes 1_\Omega$ and isomorphic to $\mathbf{c}^2$. Setting $z = xy$ and

$$a_{s,l} = e_s\xi e_{s+1} \otimes e_{s+1}yz^le_s, \quad b_{s,l} = e_s\eta e_{s-1} \otimes e_{s-1}xz^le_s, \quad w_{s,l} = e_s\xi\eta e_s \otimes e_sz^{l+1}e_s$$

the other three have bases given by $\{e_s \otimes e_sz^le_s | s = l+1,\dots p\}$, $\{a_{s,l}, b_{s,l} | s = l+1,\dots p-1\}$ and $\{w_{s,l} | s = l+2,\dots,p-1\}$ respectively. Our complex $D_\mathbf{c}$ is then a sum of the complex

$$0 \to \mathbf{c}^2 \to 0$$

and the sum over l of complexes, for $0 \le l \le p-1$,

$$\begin{aligned}(0 \to &D_\mathbf{c}^{(0,2l)} \to D_\mathbf{c}^{(-1,2l+1)} \to D_\mathbf{c}^{(-2,2l+2)} \to 0)\\ &\cong (0 \to F^{p-l} \to F^{2p-2-2l} \to F^{p-2-l} \to 0)\end{aligned}$$

and the differential acts on the l-component by

$$\begin{aligned}e_s \otimes e_sz^le_s &\mapsto a_{s,l} - b_{s,l} - a_{s-1,l} + b_{s-1,l},\\ a_{s,l} &\mapsto w_{s,l} - w_{s+1,l}\\ b_{s,l} &\mapsto w_{s+1,l} - w_{s,l}\end{aligned}$$

from where we see that in the sequence $D_\mathbf{c}^{0,2l} \to D_\mathbf{c}^{-1,2l+1} \to D_\mathbf{c}^{-2,2l+2}$ the last map is surjective, the first has one-dimensional kernel spanned by $\sum_{s=l+1}^p e_s \otimes e_sz^le_s = 1 \otimes z^l$ (the centre of Ω), and one-dimensional homology in the middle spanned by κz^l where $\kappa := \sum_{s=1}^{p-1} a_{s,0}$. The homology of $H(D_\mathbf{c})$ is therefore

$$c^2 \oplus \bigoplus_{l=0}^{p-2} F.\kappa z^l \oplus \bigoplus_{l=0}^{p-1} F.z^l$$

and the multiplication is obvious from this explicit description. In our gradings, the j-grading sees η, ξ, x, y in degrees $1, 1, -1, -1$ respectively, and the k grading has η, ξ, x, y in degrees $0, 0, 1, 1$, so the factor $\mathbf{c}^2$ has (j,k)-degree $(2,0)$, the element z has (j,k)-degree $(-2,2)$ and the element κ has (j,k)-degree $(0,1)$. This completes the proof of (i).

(ii) We need to compute the homology of $D_{\mathbf{c},\Theta} := \bigoplus_{s,t} e_s\mathbf{c}e_t \otimes e_t\Theta e_s$ with differential

$$\alpha \otimes m \mapsto \alpha\xi \otimes ym + \alpha\eta \otimes xm - (-1)^{|m|}\xi\alpha \otimes my - (-1)^{|m|}\eta\alpha \otimes mx.$$

The only nonzero components are $D_{\mathbf{c},\Theta}^{-2,0}$, $D_{\mathbf{c},\Theta}^{0,2l}$, $D_{\mathbf{c},\Theta}^{-1,2l+1}$ and $D_{\mathbf{c},\Theta}^{-2,2l+2}$ for $0 \le l \le \frac{p-3}{2}$. The first is just $\mathbf{c}^2 \otimes 1_\Omega$ and isomorphic to $\mathbf{c}^2$ and contributes to homology as before. The other three have bases given by $\{e_s \otimes e_sz^le_s | s = l+1,\dots p-l-1\}$, $\{a_{s,l}, b_{s,l} | s = l+1,\dots p-l-2\}$ and $\{w_{s,l} | s = l+2,\dots,p-l-2\}$ respectively. Our Our complex $D_\mathbf{c}$ is then a sum of the complex

$$0 \to \mathbf{c}^2 \to 0$$

and the sum over l of complexes, for $0 \le l \le p-1$,

$$\begin{aligned}(0 \to &D_{\mathbf{c},\Theta}^{(0,2l)} \to D_{\mathbf{c},\Theta}^{(-1,2l+1)} \to D_{\mathbf{c},\Theta}^{(-2,2l+2)} \to 0)\\ &\cong (0 \to F^{p-2l-1} \to F^{2p-4l-4} \to F^{p-2l-3} \to 0)\end{aligned}$$

and the differential acts as before on the basis elements. Again the last map is surjective, the first has kernel $\sum_{s=l+1}^{p-l-1} e_s \otimes e_s z^l e_s = 1 \otimes z^l$, and homology in the middle is spanned by $\kappa z^l = \sum_{s=1}^{p-l-2}(-1)^s(a_{s,l} - b_{s,l})$. The homology $H(D_{\mathbf{c},\Theta})$ is therefore

$$\mathbf{c}^2 \oplus \bigoplus_{l=0}^{\frac{p-3}{2}} F.z^l\kappa \oplus \bigoplus_{l=0}^{\frac{p-3}{2}} F.z^l$$

with multiplication naturally induced by multiplication in c and Ω. This completes the proof of (ii).

(iii) To compute $HH(\Omega, \Theta^\sigma)$ we need to compute the homology of

$$D_{\mathbf{c},\Theta^\sigma} := \bigoplus_{s,t} e_s \mathbf{c} e_t \otimes e_t \Theta^\sigma e_s,$$

which as a vector space is isomorphic to $\bigoplus_{s,t} e_s \mathbf{c} e_t \otimes e_t \Theta e_{p-s}$. This has nonzero components $D_{\mathbf{c},\Theta^\sigma}^{(0,p-2-2l)}$ for $l = 0, \ldots, \frac{p-3}{2}$, as well as $D_{\mathbf{c},\Theta^\sigma}^{(-1,p-1-2l)}$ and $D_{\mathbf{c},\Theta^\sigma}^{(-2,p-2l)}$ for $l = 1, \ldots, \frac{p-1}{2}$, with bases given by

$$\{e_s \otimes e_s x^{p-s-l-1} y^{s-l-1} e_{p-s} | s = l+1, \ldots, p-l-1\}$$

$$\{e_s \xi e_{s+1} \otimes e_{s+1} x^{p-s-l-1} y^{s-l} e_{p-s}, e_{s+1}\eta e_s \otimes e_s x^{p-s-l-1} y^{s-l} e_{p-s-1} | s = l, \ldots, p-l-1\}$$

and

$$\{e_s \xi\eta e_s \otimes e_s x^{p-s-l} y^{s-l} e_{p-s} | s = l, \ldots, p-l\}$$

respectively. As the differential has degree $(-1, 1)$, for $l = 0$ we obtain homology spanned by $\{e_s \otimes e_s x^{p-s-1} y^{s-1} e_{p-s} | s = 1, \ldots, p-1\}$ in degree $(0, p-2)$. This is equal to $1 \otimes (\Theta^\sigma)^{p-2}$. The rest of the complex is a sum over l for $l = 1, \ldots, \frac{p-1}{2}$ of

$$\begin{aligned}(0 \to & D_{\mathbf{c},\Theta^\sigma}^{(0,p-2-2l)} \to D_{\mathbf{c},\Theta^\sigma}^{(-1,p-1-2l)} \to D_{\mathbf{c},\Theta^\sigma}^{(-2,p-2l)} \to 0) \\ &\cong (0 \to F^{p-2l-1} \to F^{2p-4l} \to F^{p-2l+1} \to 0).\end{aligned}$$

Setting

$$\begin{aligned} f_{s,l} &= e_s \xi e_{s+1} \otimes e_{s+1} x^{p-s-l-1} y^{s-l} e_{p-s}, \\ g_{s,l} &= e_{s+1} \eta e_s \otimes e_s x^{p-s-l-1} y^{s-l} e_{p-s-1} \\ v_{s,l} &= e_s \xi\eta e_s \otimes e_s x^{p-s-l} y^{s-l} e_{p-s} \end{aligned}$$

respectively, the differential acts as

$$\begin{aligned} e_s \otimes e_s x^{p-s-l-1} y^{s-l-1} e_{p-s} &\mapsto f_{s,l} + f_{s-1,l} + g_{s,l} + g_{s-1,l} \\ f_{s,l} &\mapsto v_{s,l} + v_{s+1,l} \\ g_{s,l} &\mapsto -v_{s,l} - v_{s+1,l}. \end{aligned}$$

It is easy to see that the first map is injective. However, the image of the last map is spanned by $v_{s,l} + v_{s+1,l}$ for $s = l, \ldots, p-l-1$ and is hence only $p-2l$-dimensional, leaving one-dimensional homology in both the middle (spanned by $\mu_l = (f_{\frac{p-1}{2},l} + g_{\frac{p-1}{2},l})$ say) and the end (spanned by $\nu_l = (v_{\frac{p-1}{2},l} - v_{\frac{p+1}{2},l})$, say). In order to describe the structure as $HH(\mathbf{c})$-$HH(\mathbf{c})$-bimodule, we need to determine the action of the generators of $HH(\mathbf{c})$ on this. It is clear that both ν_l and μ_l are annihilated by $\mathbf{c}^2$. Direct computation shows that $\kappa.\mu_l = \nu_l$, $z.\mu_l = \mu_{l-1}$ and $z.\nu_l = \nu_{l-1}$. By graded dimensions, the only other non-zero product could be

$\mathbf{c}^2.D^{(0,p-2)}_{\mathbf{c},\Theta^\sigma}$, which lies in degree $(-2,p-2)$, where ν_1 also lives. Direct computation shows that with our choice of representatives of homology, we obtain

$$(e_s\xi\eta e_s \otimes e_s)(e_s \otimes e_s x^{p-s-1}y^{s-1}e_{p-s}) = \frac{1}{2}(-1)^{\frac{p-1}{2}-s}\nu_1$$

and all other product with non-matching idempotents are obviously zero. The (j,k)-degrees of the basis elements are $(-p+2,p-2)$ for $e_s \otimes e_s x^{p-s-1}y^{s-1}e_{p-s}$ for $s=1,\dots,p-1$, then $(-p+2+2l,p-2l-1)$ for μ_l and $(-p+2+2l,p-2l)$ for ν_l.

This completes our combinatorial description of $HH(\Omega,\Theta^\sigma)$. To define an isomorphism between $HH(\Omega,\Theta^\sigma)$ and $HH(\Omega,\Theta)^*$ we now define a bilinear form

$$|-,-| : HH(\Omega,\Theta^\sigma) \otimes HH(\Omega,\Theta) \to F$$

such that

$$|h,h'h''| = |hh',h''|, \qquad |h,h''h'| = (-1)^{|h'|_k(|h|_k+|h''|_k)}|h'h,h''|,$$

for $h \in HH(\Omega,\Theta^\sigma)$, $h' \in HH(\Omega)$, $h'' \in HH(\Omega,\Theta)$. Indeed the form $|-,-|$ which pairs $2(-1)^{\frac{p-1}{2}-s}(e_s\otimes e_s x^{p-s-1}y^{s-1}e_{p-s}) \in HH(\Omega,\Theta^\sigma)$ with $e_s\xi\eta\otimes 1 \in HH(\Omega,\Theta)$, which pairs z^l with ν_{l+1}, and which pairs $z^l\kappa$ with μ_{l+1} has the required property; in fact all signs $(-1)^{|h'|_k(|h|_k+|h''|_k)}$ are $+1$ when $|h'h,h''|$ is nonzero for elements h,h',h'' of our canonical bases since the super-commutation relations defining $HH(\Omega)$ are all commutation relations, with z lying in degree 2.

(iv) This follows from Lemma 6.

(v) We have an exact sequence of Ω-Ω-bimodules,

$$0 \to \Omega e_p \Omega \to \Omega \to \Theta \to 0.$$

Applying $\mathrm{Hom}(\Omega,-)$ gives us an exact triangle

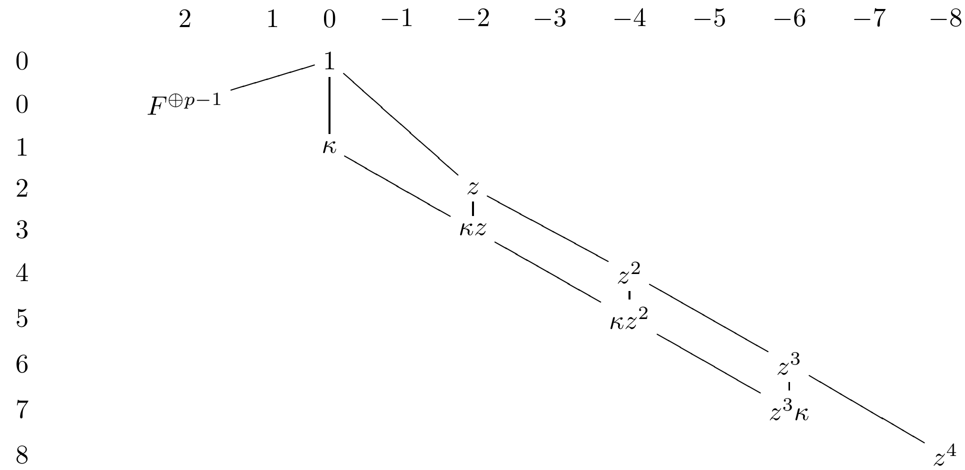
$$\mathrm{Hom}_{\Omega\otimes\Omega^{op}}(\Omega,\Omega e_p\Omega) \to \mathrm{Hom}_{\Omega\otimes\Omega^{op}}(\Omega,\Omega) \to \mathrm{Hom}_{\Omega\otimes\Omega^{op}}(\Omega,\Theta) \rightsquigarrow$$

in the derived category of F-F-bimodules, which corresponds to an exact triangle

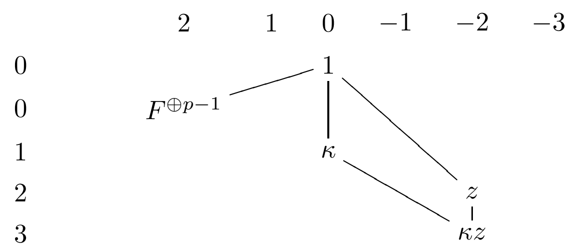
$$HH(\Omega,\Omega e_p\Omega) \to HH(\Omega,\Omega) \to HH(\Omega,\Theta) \rightsquigarrow$$

We know $HH(\Omega,\Omega)$ and $HH(\Omega,\Theta)$, and from our calculations the map between them is visibly the canonical surjection. This completes the proof of (v). □

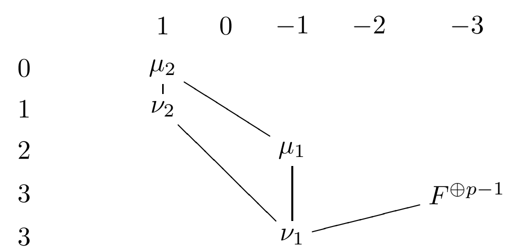
We give some pictures visualising the structure of the bimodules in case $p=5$ (the numbers down the left hand side denote the k-grading and along the top the j-grading Here is $HH(\Omega)$:



Here is $HH(\Omega, \Theta)$:



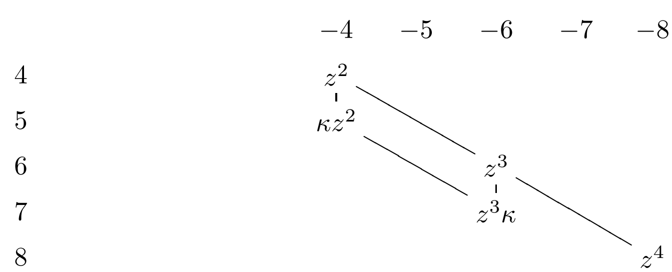
Here is $HH(\Omega, \Theta^\sigma)$:



Here is $HH(\Omega, \Omega^*)$:

$0 \qquad F^{\oplus p}$

Here is $HH(\Omega, \Omega e_p \Omega)$:



Remark 20 The bimodule isomorphism

$$HH(\Omega, \Theta^\sigma) \cong HH(\Omega, \Theta)^*$$

of Proposition 19(iii) is striking, since we also have $\Theta^\sigma \cong \Theta^*$ as bimodules. This duality between Hochschild cohomologies does not follow from basic general principles and therefore deserves further comment. We give a more conceptual explanation of its origin here, and in addition demonstrate that

$$HH(\Omega, \Theta) \cong HH(\Omega, \Theta^\sigma)^* \langle 4-p \rangle [p-2]$$

as jk-graded bimodules. We have a short exact sequence

$$0 \to \Omega e_p \Omega \to \Omega \to \Theta \to 0$$

whose dual is

$$0 \leftarrow \Omega e_p \Omega \langle 2p-2 \rangle [2-2p] \leftarrow \Omega^* \leftarrow \Theta^\sigma \langle p-2 \rangle [2-p] \leftarrow 0.$$

Applying derived $\mathrm{Hom}_{\Omega \otimes \Omega^{op}}(\Omega, -)$ gives us an exact triangle

$$HH(\Omega, \Theta^\sigma)\langle p-2 \rangle [2-p] \to HH(\Omega, \Omega^*) \to HH(\Omega, \Omega e_p \Omega)\langle 2p-2 \rangle [2-2p] \rightsquigarrow$$

We know that $HH(\Omega, \Omega e_p \Omega)$ is the kernel of $HH(\Omega, \Omega \to \Theta)$, an extension of $F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})\langle 1-p \rangle [p-1]$ by $F\langle 2-2p \rangle [2p-2]$; we know that $HH(\Omega, \Omega^*)$ is iso to $F^{\oplus p}\langle 0 \rangle [0]$. Two copies of F cancel in the derived category in our triangle via the map $HH(\gamma)$ (see proof of Lemma 22, the product $\lozenge_l$), leaving us with an exact triangle

$$HH(\Omega, \Theta^\sigma)\langle p-2 \rangle [2-p] \to F^{\oplus p-1} \to F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})\langle 1-p \rangle [p-1]\langle 2p-2 \rangle [2-2p] \rightsquigarrow$$

which we can shift to a triangle

$$F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})\langle p-1 \rangle [1-p]\langle 2-p \rangle [p-2][1] \to HH(\Omega, \Theta^\sigma) \to F^{\oplus p-1}\langle 2-p \rangle [p-2] \rightsquigarrow$$

which is a triangle

$$F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})\langle 1 \rangle [0] \to HH(\Omega, \Theta^\sigma) \to F^{\oplus p-1}\langle 2-p \rangle [p-2] \rightsquigarrow .$$

That is dual to the exact triangle

$$F^{\oplus p-1}\langle 2 \rangle [0] \to HH(\Omega, \Theta) \to F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}}) \rightsquigarrow .$$

Here we use the self-injectivity of $F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})$, which is given by an isomorphism

$$F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}}) \cong F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})^* \langle 3-p \rangle [p-2]$$

of $F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})$-$F[\kappa, z]/(\kappa^2, z^{\frac{p-1}{2}})$-bimodules. We thus have

$$HH(\Omega, \Theta) \cong HH(\Omega, \Theta^\sigma)^* \langle 4-p \rangle [p-2]$$

as jk-graded $HH(\Omega)$-$HH(\Omega)$-bimodules.

Remark 21 The spaces computed in Lemma 19 come with natural bases. Indeed, we have bases for these bimodules, indexed by pairs (d,x) where d denotes a jk-degree and x an idempotent such that $xm_{d,x} = m_{d,x}$:

$$\begin{aligned}
\mathcal{B}_\chi =&\{m_{-2l,2l,1}|0 \le l \le p-1\} \cup \{m_{-2l,2l+1,1}|0 \le l \le p-2\}\\
&\cup \{m_{2,0,e_s}|1 \le s \le p-1\};\\
\mathcal{B}_{\overline{\chi}} =&\{m_{-2l,2l,1}|0 \le l \le \frac{p-3}{2}\} \cup \{m_{-2l,2l+1,1}|0 \le l \le \frac{p-3}{2}\}\\
&\cup \{m_{2,0,e_s}|1 \le s \le p-1\};\\
\mathcal{B}_{\overline{\chi}^*} =&\{m_{2l,-2l,1}|0 \le l \le \frac{p-3}{2}\} \cup \{m_{2l,-2l-1,1}|0 \le l \le \frac{p-3}{2}\}\\
&\cup \{m_{-2,0,e_s}|1 \le s \le p-1\};\\
\mathcal{B}_{\underline{\chi}} =&\mathcal{B}_\chi \backslash \mathcal{B}_{\overline{\chi}};\\
\mathcal{B}_{\Omega^0} =&\{m_{0,0,e_s}|1 \le s \le p\}.
\end{aligned}$$

More precisely we have

$$\begin{aligned}
\mathcal{B}_\chi =&\{1, z^l|0 \le l \le p-1\} \cup \{\kappa z^l|1 \le l \le p-2\} \cup \{e_s\xi\eta \otimes 1|1 \le s \le p-1\};\\
\mathcal{B}_{\overline{\chi}^*} =&\{\nu_{l+1}|0 \le l \le \frac{p-3}{2}\} \cup \{\mu_{l+1}|0 \le l \le \frac{p-3}{2}\}\\
&\cup \{e_s \otimes e_s x^{p-s-1}y^{s-1}e_{p-s}|1 \le s \le p-1\}
\end{aligned}$$

and we identify $\mathcal{B}_{\overline{\chi}}$ and $\mathcal{B}_{\underline{\chi}}$ with subsets of $\mathcal{B}_\chi$ in the natural way. The basis $\mathcal{B}_{\Omega^0}$ is merely the set of idempotents e_s for $1 \le s \le p$.

12. The algebra $\spadesuit = \mathbb{HH}(\clubsuit)$.

Cute as $\clubsuit$ is, to compute the Hochschild cohomology of GL_2 we must diminish it, by taking Hochschild cohomology with respect to Ω. The resulting algebra we call $\spadesuit$. In the remains of the paper we assume $p > 2$.

Description via bimodules. Let $\chi = HH(\Omega)$, let $\overline{\chi} = \chi/z^{\frac{p-1}{2}}$, and let $\underline{\chi}$ denote the kernel of the natural surjection $\chi \to \overline{\chi}$, so we have isomorphisms $HH(\Omega,\Theta) \cong \overline{\chi}$, $HH(\Omega,\Theta^\sigma) \cong \overline{\chi}^*$ and $HH(\Omega,\Omega e_p\Omega) \cong \underline{\chi}$. Then by taking componentwise Hochschild cohomology we see that the structure of $\spadesuit$ as a χ-χ-bimodule is given by

$$
\begin{array}{ccccccccc}
 & & & & & & \cdots & & \\
 & & & & & & \overline{\chi}^* & \overline{\chi} & \Omega^0 \\
 & & & & & & \overline{\chi} & \Omega^0 & \\
 & & & & & & \Omega^0 & & \\
 & & & & & \underline{\chi} & & & \\
 & & & & \chi & & & & \\
 & & & \chi & \overline{\chi}^* & & & & \\
 & & \chi & \overline{\chi}^* & \overline{\chi} & & & & \\
 & \chi & \overline{\chi}^* & \overline{\chi} & \overline{\chi}^* & & & & \\
\chi & \overline{\chi}^* & \overline{\chi} & \overline{\chi}^* & \overline{\chi} & & & & \\
 & & \cdots & & & & & &
\end{array}
$$

From the structure of ♣ as bigraded Ω-Ω-bimodule, we infer the structure of $♠^- = \mathbb{HH}(♣^-)$ as a k-graded χ-χ-bimodule

$$
\begin{array}{ccccc}
 & & & & \chi \\
 & & & \chi[p-1] & \overline{\chi}^*[p-2] \\
 & & \chi[2p-2] & \overline{\chi}^*[2p-3] & \overline{\chi} \\
 & \chi[3p-3] & \overline{\chi}^*[3p-4] & \overline{\chi}[p-1] & \overline{\chi}^*[p-2] \\
\chi[4p-4] & \overline{\chi}^*[4p-5] & \overline{\chi}[2p-2] & \overline{\chi}^*[2p-3] & \overline{\chi} \\
 & & \cdots & & ;
\end{array}
$$

the structure of $♠^-$ as a j-graded χ-χ-bimodule

$$
\begin{array}{ccccc}
 & & & & \chi \\
 & & & \chi\langle -p\rangle & \overline{\chi}^*\langle 4-p\rangle \\
 & & \chi\langle -2p\rangle & \overline{\chi}^*\langle 4-2p\rangle & \overline{\chi} \\
 & \chi\langle -3p\rangle & \overline{\chi}^*\langle 4-3p\rangle & \overline{\chi}\langle -p\rangle & \overline{\chi}^*\langle 4-p\rangle \\
\chi\langle -4p\rangle & \overline{\chi}^*\langle 4-4p\rangle & \overline{\chi}\langle -2p\rangle & \overline{\chi}^*\langle 4-2p\rangle & \overline{\chi} \\
 & & \cdots & & ;
\end{array}
$$

the structure of $\spadesuit^+ = \mathbb{HH}(\clubsuit^+)$ as a k-graded χ-χ-bimodule

$$\begin{array}{cccccc}
 & & \cdots & & & \\
 & & \overline{\chi}[2-p] & \overline{\chi}^*[1-p] & \overline{\chi}[4-3p] & \Omega^0[3-3p] \\
 & & \overline{\chi}^*[0] & \overline{\chi}[3-2p] & \Omega^0[2-2p] & \\
 & & \overline{\chi}[2-p] & \Omega^0[1-p] & & \\
 & & \Omega^0[0] & & & \\
 & \underline{\chi}[1-p] & & & & \\
\chi & & & & & ;
\end{array}$$

and finally the structure of $\spadesuit^+$ as a j-graded χ-χ-bimodule

$$\begin{array}{cccccc}
 & & \cdots & & & \\
 & & \overline{\chi}\langle p\rangle & \overline{\chi}^*\langle 4+p\rangle & \overline{\chi}\langle 3p\rangle & \Omega^0\langle 2+3p\rangle \\
 & & \overline{\chi}^*\langle 4\rangle & \overline{\chi}\langle 2p\rangle & \Omega^0\langle 2+2p\rangle & \\
 & & \overline{\chi}\langle p\rangle & \Omega^0\langle 2+p\rangle & & \\
 & & \Omega^0\langle 2\rangle & & & \\
 & \underline{\chi}\langle p\rangle & & & & \\
\chi\langle 0\rangle & & & & & .
\end{array}$$

In order to give the multiplication on $\spadesuit$, we first define a number of χ-χ-bimodule homomorphisms between the various components of $\spadesuit$.

Lemma 22. *Let $\bigstar$, $\lozenge_l$, $\lozenge_r$, $\blacklozenge$, $\blacklozenge_r$, $\square_l$, $\square_r$ and $\blacktriangle$ be the χ-χ-bimodule homomorphisms obtained by applying $HH(\Omega,-)$ to $a:\Theta^\sigma\otimes\Theta^\sigma\to\Theta$, θ_l, θ_r, ι_l, ι_r, ν_l, ν_r, and η respectively, which we identify with products of components of $H(\mathbf{c}^{op}\otimes\clubsuit)$. Then the products of basis elements in these spaces that are nonzero are given as follows:*

$$\begin{aligned}
\bigstar: \quad & \overline{\chi}^*\otimes_\chi\overline{\chi}^*\to\overline{\chi}\\
& \mu_{\frac{p-1}{2}}\otimes\mu_{\frac{p-1}{2}}\mapsto \xi\eta(e_{\frac{p-1}{2}}-e_{\frac{p+1}{2}})\\
& (e_{\frac{p+1}{2}}\otimes e_{\frac{p+1}{2}}x^{\frac{p-3}{2}}y^{\frac{p-1}{2}}e_{\frac{p-1}{2}})\otimes\mu_{\frac{p-1}{2}}\mapsto\kappa z^{\frac{p-3}{2}}\\
& (e_{\frac{p-1}{2}}\otimes e_{\frac{p-1}{2}}x^{\frac{p-1}{2}}y^{\frac{p-3}{2}}e_{\frac{p+1}{2}})\otimes\mu_{\frac{p-1}{2}}\mapsto\kappa z^{\frac{p-3}{2}}\\
& \mu_{\frac{p-1}{2}}\otimes(e_{\frac{p+1}{2}}\otimes e_{\frac{p+1}{2}}x^{\frac{p-3}{2}}y^{\frac{p-1}{2}}e_{\frac{p-1}{2}})\mapsto\kappa z^{\frac{p-3}{2}}\\
& \mu_{\frac{p-1}{2}}\otimes(e_{\frac{p-1}{2}}\otimes e_{\frac{p-1}{2}}x^{\frac{p-1}{2}}y^{\frac{p-3}{2}}e_{\frac{p+1}{2}})\mapsto\kappa z^{\frac{p-3}{2}}
\end{aligned}$$

$$\begin{array}{llcll}
\lozenge_l: & \chi\otimes_\chi\Omega^0\to\underline{\chi}, & \qquad & \lozenge_r: & \Omega^0\otimes_\chi\chi\to\underline{\chi}\\
& 1\otimes e_p\mapsto z^{p-1} & & & e_p\otimes 1\mapsto z^{p-1}
\end{array}$$

$$\blacklozenge_l : \quad \chi \otimes_\chi \Omega^0 \to \chi, \qquad \blacklozenge_r : \quad \Omega^0 \otimes_\chi \chi \to \chi$$
$$1 \otimes e_p \mapsto z^{p-1} \qquad\qquad e_p \otimes 1 \mapsto z^{p-1}$$

$$\square_l : \quad \overline{\chi} \otimes_\chi \overline{\chi}^* \to \Omega^0,$$
$$1 \otimes (e_s \otimes e_s x^{p-s-1} y^{s-1}) \mapsto e_s, \quad 1 \leq s \leq p-1$$

$$\square_r : \quad \overline{\chi}^* \otimes_\chi \overline{\chi} \to \Omega^0,$$
$$(e_s \otimes e_s x^{p-s-1} y^{s-1}) \otimes 1 \mapsto e_s, \quad 1 \leq s \leq p-1$$

$$\blacktriangle : \quad \underline{\chi} \otimes_\chi \underline{\chi} \to \Omega^0,$$
$$z^{\frac{p-1}{2}} \otimes z^{\frac{p-1}{2}} \mapsto e_p$$

Proof. The product $\bigstar$. Let us consider the element $\kappa z^{\frac{p-3}{2}}$ of $HH(\Omega, \Theta)$. From the proof of Lemma 19(i) we find it is equal to $\sum_{s=1}^{p-1} a_{s,0} z^{\frac{p-3}{2}}$. We know that $a_{s,0} z^{\frac{p-3}{2}}$ is zero unless $s = \frac{p-1}{2}$; consequently
$$\kappa z^{\frac{p-3}{2}} = e_{\frac{p-1}{2}} \xi e_{\frac{p+1}{2}} \otimes e_{\frac{p+1}{2}} x^{\frac{p-3}{2}} y^{\frac{p-1}{2}} e_{\frac{p-1}{2}}.$$
The image of $e_{\frac{p-1}{2}} \otimes z^{\frac{p-3}{2}}$ under the differential is
$$e_{\frac{p-1}{2}} \xi e_{\frac{p+1}{2}} \otimes e_{\frac{p+1}{2}} x^{\frac{p-3}{2}} y^{\frac{p-1}{2}} e_{\frac{p-1}{2}} - e_{\frac{p+1}{2}} \eta e_{\frac{p-1}{2}} \otimes e_{\frac{p-1}{2}} x^{\frac{p-1}{2}} y^{\frac{p-3}{2}} e_{\frac{p+1}{2}},$$
and therefore in homology we have
$$\kappa z^{\frac{p-3}{2}} = e_{\frac{p-1}{2}} \xi e_{\frac{p+1}{2}} \otimes e_{\frac{p+1}{2}} x^{\frac{p-3}{2}} y^{\frac{p-1}{2}} e_{\frac{p-1}{2}} = e_{\frac{p+1}{2}} \eta e_{\frac{p-1}{2}} \otimes e_{\frac{p-1}{2}} x^{\frac{p-1}{2}} y^{\frac{p-3}{2}} e_{\frac{p+1}{2}}.$$
We have $\mu_{\frac{p-1}{2}} = e_{\frac{p-1}{2}} \xi e_{\frac{p+1}{2}} \otimes e_{\frac{p+1}{2}} + e_{\frac{p+1}{2}} \eta e_{\frac{p-1}{2}} \otimes e_{\frac{p-1}{2}}$. Multiplying in $\mathbf{c}^{op} \otimes \clubsuit$ gives us $\bigstar$.

The product $\lozenge_l$. Consider the product $\theta_l : \Omega \otimes \Omega^* \to \Omega e_p \Omega$. This factors over the action map $\Omega \otimes \Omega e_p \Omega \to \Omega e_p \Omega$, and consequently $\lozenge_l$ factors over the action map $\chi \otimes \overline{\chi} \to \underline{\chi}$. If we want to know $\lozenge_l$ it therefore suffices to know $HH(\gamma) : \Omega^0 \to \underline{\chi}$. However, since in our computation of $HH(\Omega, \Omega^*)$ the space Ω^0 is identified with the socle of Ω^* in the tensor product $\mathbf{c}^{op} \otimes \Omega^*$, the socle of Ω^* contains $z^{p-1} \in \Omega e_p \Omega$ in the identification of $\Omega e_p \Omega$ with a quotient of Ω^*, and has product zero with e_s for $s \neq p$, we observe the map $HH(\gamma)$ is nothing but the map that sends e_p to z^{p-1}, which fits with the stated structure of $\lozenge_l$.

The product $\blacklozenge_l$. The product ι_l is merely the composition of θ_l and the embedding of $\Omega e_p \Omega$ in Ω. Therefore $\blacklozenge_l$ is the composition of $\lozenge_l$ and the natural embedding of $\underline{\chi}$ in χ.

The product $\square_l$. Consider the product $\nu_l : \Theta \otimes \Theta^\sigma \to \Omega^*$. This is the composite of the action of Θ on Θ^σ and the embedding μ of Θ^σ in Ω^* in which the socle of Θ^σ is identified with the socle of Ω^*. To know $HH(\nu_l)$ it therefore suffices to know

$HH(\mu)$. Since in our computation of $HH(\Omega,\Omega^*)$ the space Ω^0 is identified with the socle of Ω^* in the tensor product $\mathbf{c}^{op}\otimes\Omega^*$, and μ identifies $e_s\otimes e_s x^{p-s-1}y^{s-1}$ with the element of the socle of Ω^* corresponding to $e_s\in\Omega^0$, the product $\Box_l$ is as stated.

The products $\lozenge_r$, $\blacklozenge_r$, and $\Box_r$ are established similarly to $\lozenge_l$, $\blacklozenge_l$, and $\Box_l$.

The product $\blacktriangle$. We know that under $\blacktriangle$ the radical of $\underline{\chi}$ must have product zero with all elements since Ω^0 is semisimple. This leaves us with the problem of finding the square of the element $z^{\frac{p-1}{2}}$ of $\underline{\chi}$ in Ω^0. The element $z^{\frac{p-1}{2}}$ of $\Omega e_p\Omega$ squares to z^{p-1}, and the socle of Ω^* contains $z^{p-1}\in\Omega e_p\Omega$ in the identification of $\Omega e_p\Omega$ with a quotient of Ω^*. The same old socle argument now implies that $\blacktriangle$ is as stated.

$\Box$

We use these maps to describe the product in $\spadesuit$, where we again gather together components which are isomorphic (up to shift), according to whether they lie in $\spadesuit^+$ or $\spadesuit^-$, in a similar way as in Proposition 17.

Theorem 23. *Products between the various components in $\spadesuit$ are given by the following table*

	χ_-	$\overline{\chi}_-$	$\overline{\chi}^*_-$	$\underline{\chi}$	$\overline{\chi}_+$	$\overline{\chi}^*_+$	Ω^0_+
χ_-	a	a	a	a	a	a	$\blacklozenge,\lozenge,a$
$\overline{\chi}_-$	a	a	a	0	$0,a$	$0,a,\Box$	0
$\overline{\chi}^*_-$	a	a	$\bigstar$	0	$0,a,\Box$	0	0
$\underline{\chi}$	a	0	0	$\blacktriangle$	0	0	0
$\overline{\chi}_+$	a	$0,a$	$0,a,\Box$	0	0	0	0
$\overline{\chi}^*_+$	a	$0,a,\Box$	0	0	0	0	0
Ω^0_+	$\blacklozenge,\lozenge,a$	0	0	0	0	0	0

Possible ambiguities are covered by further tables. For the product of Ω^0_+ and χ_-:

Component in which the product lands:	χ	$\underline{\chi}$	Ω^0_+
Natural map describing the product:	$\blacklozenge$	$\lozenge$	a

For the product of $\overline{\chi}_+$ and $\overline{\chi}_-$:

Component in which the product lands:	$\overline{\chi}_+$	$\spadesuit^-$
Natural map describing the product:	a	0

For the product of $\overline{\chi}^*{}_-$ and $\overline{\chi}_+$:

Component in which the product lands:	$\overline{\chi}^*{}_+$	$\underline{\chi}_+$	Ω^0_+	$\spadesuit^-$
Natural map describing the product:	a	0	$\Box$	0

For the product of $\overline{\chi}^*{}_+$ *and* $\overline{\chi}_-$*:*

Component in which the product lands:	$\overline{\chi}^*{}_+$	$\underline{\chi}_+$	Ω^0_+	$\spadesuit^-$
Natural map describing the product:	a	0	$\square$	0

Proof. Why these products? All the action products are inherited from action products in $\clubsuit$; all the zero products are either inherited from zero products in $\clubsuit$ via Lemma 22, or determined by the fact that the products lie in degrees in which there are no nonzero elements with respect to the various gradings; for example $HH(\epsilon) = HH(\zeta) = 0$ by this reasoning. $\square$

A basis. We describe a basis for $\spadesuit$ indexed by elements of a polytope. Roughly, we label basis elements $m_{d,x}$ for $\spadesuit$ by a pair (d, x) where $d \in \mathbb{Z}^3$ denotes a ijk-degree, and x denotes an element of Ω^0, either 1 or an idempotent.

More precisely, here is our basis for $\spadesuit$:

$$\begin{aligned}
\mathcal{B}_\spadesuit =& \mathcal{B}_{\chi_-} \cup \mathcal{B}_{\overline{\chi}_-} \cup \mathcal{B}_{\overline{\chi}^*{}_-} \cup \mathcal{B}_{\underline{\chi}} \cup \mathcal{B}_{\overline{\chi}_+} \cup \mathcal{B}_{\overline{\chi}^*{}_+} \cup \mathcal{B}_{\Omega^0}\\
=&\{m_{a,b,i,j+ap,k+a(1-p),x} | m_{j,k,x} \in \mathcal{B}_\chi, a \leq 0, b = 0, i = a+b\}\\
&\cup\{m_{a,b,i,j+ap,k+a(1-p),x} | m_{j,k,x} \in \mathcal{B}_{\overline{\chi}_-}, a \leq 0, b \leq -2, b \text{ even }, i = a+b\}\\
&\cup\{m_{a,b,i,j+(4-p)+ap,k+(p-2)+a(1-p),x} | m_{j,k,x} \in \mathcal{B}_{\overline{\chi}^*{}_-}, a \leq 0, b \leq -1, b \text{ odd }, i = a+b\}\\
&\cup\{m_{1,0,1,j,k+p,x} | m_{j,k,x} \in \mathcal{B}_{\underline{\chi}}\}\\
&\cup\{m_{a,b,i,j+p+(a-2)p,k+(2-p)+(a-2)(1-p),x} | m_{j,k,x} \in \mathcal{B}_{\overline{\chi}_+}, a \geq 2, b \geq 1, b \text{ odd }, i = a+b\}\\
&\cup\{m_{a,b,i,j+4+(a-2)p,k+(a-2)(1-p),x} | m_{j,k,x} \in \mathcal{B}_{\overline{\chi}^*{}_+}, a \geq 2, b \geq 2, b \text{ even }, i = a+b\}\\
&\cup\{m_{a,b,i,j+2+(a-2)p,k+(a-2)(1-p),x} | m_{j,k,x} \in \mathcal{B}_{\Omega^0}, a \geq 2, b = 0, i = a+b\}
\end{aligned}$$

We describe the ab grading as follows: in our pictures of $\spadesuit$ a shift by a corresponds to a move to the northeast by a and a shift by b corresponds to a move to the north by b. The product of a pair of basis elements in $\spadesuit$ is either another basis element, or the sum of a basis element and the negative of another basis element, or $\pm\frac{1}{2}$ a basis element, or zero; when a product of $m_{a,b,i,j,k,x}.m_{a',b',i',j',k',x'}$ is nonzero, the basis elements in the product take the form $m_{a+a',b+b',i+i',j+j',k+k',y}$. precise formulas for the product are given bt the formulas in the statement of Lemma 22 and the table in the statement of Theorem 23.

13. The algebra $HH(GL_2)$.

The category G-mod has countably many blocks, all of which are equivalent. Correspondingly, the Hochschild cohomologies of all blocks of G are isomorphic. It is therefore sufficient to compute the Hochschild cohomology of the principal block b of G.

Theorem 24. *We have isomorphisms of k-graded algebras*

$$hh_l \cong \mathfrak{O}_F \mathfrak{O}^l_\spadesuit(F[z, z^{-1}]).$$

Proof. This is a restatement of Proposition 12. $\square$

Corollary 25. *The algebra hh_l inherits an explicit basis from $\spadesuit$.*

Proof. We explicitly write down such a basis as follows: let $\mathcal{B}_\spadesuit$ denote our basis for $\spadesuit$. We have a basis for the algebra $\spadesuit^{\otimes_F l} \otimes_F F[z, z^{-1}]$ given by $\mathcal{B}_\spadesuit^{\times l} \times \{z^d | d \in \mathbb{Z}\}$; the product of basis elements is the super $\times$ product. we define the *weight* of a monomial $m_{w^1} \otimes ... \otimes m_{w^q} \otimes z^\alpha$ in $\mathcal{B}_\spadesuit^{\times l} \times \{z^d | d \in \mathbb{Z}\}$ to be

$$(w_i^2 - w_j^1, w_i^3 - w_j^2, ..., w_i^l - w_j^{l-1}, \alpha - w_j^l) \in \mathbb{Z}^{l+1},$$

where (w_i, w_j) denotes the ij-degree of m_w. We have a basis for the algebra $\mathfrak{O}_F \mathfrak{O}_\spadesuit^l (F[z, z^{-1}])$ given by weight zero elements in $\mathcal{B}_\spadesuit^{\times l} \times \{z^d | d \in \mathbb{Z}\}$; the product is the restriction of the product on $\mathcal{B}_\spadesuit^{\times l} \times \{z^d | d \in \mathbb{Z}\}$. □

Corollary 26. *The map $hh_l \to hh_{l-1}$ is surjective for $l \geq 1$.*

Proof. The map $\spadesuit \to F$ is surjective, implying

$$\mathfrak{O}_\spadesuit(a) \to \mathfrak{O}_F(a)$$

is surjective for any a, implying

$$\mathfrak{O}_F \mathfrak{O}_\spadesuit(a) \to \mathfrak{O}_F^2(a) = \mathfrak{O}_F(a)$$

is surjective for any a, implying

$$\mathfrak{O}_F \mathfrak{O}_\spadesuit^l (F[z, z^{-1}]) \to \mathfrak{O}_F \mathfrak{O}_\spadesuit^{l-1} (F[z, z^{-1}])$$

is surjective, implying $hh_l \to hh_{l-1}$ is surjective. □

Corollary 27. *The map $HH(S(2, r)) \to HH(S(2, r-2))$ corresponding to the tensor product with the determinant representation is surjective for $r \geq 2$.*

Definition 28. *We define $HH(GL_2)$ to be the inverse limit over r of the algebras $HH(S(2, r))$.*

By Corollary 27, the natural map $HH(GL_2) \to HH(S(2, r))$ is surjective for every r; furthermore every block of $HH(GL_2)$ is isomorphic to $\lim_l hh_l$. This completes the proof of Theorem 1.

References

[1] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, J. Amer. Math. Soc. 9 (1996), no. 2, 473-527.
[2] R. Buchweitz, E. Green, D. Madsen, and . Solberg, *Finite Hochschild cohomology without finite global dimension*, Math. Res. Lett. 12 (2005), 805-816
[3] E. Cline, B. Parshall and L. Scott, *Finite-dimensional algebras and highest weight categories.* J. Reine Angew. Math. 391 (1988), 85-99.
[4] V. Dlab, C.M. Ringel, *A construction for quasi-hereditary algebras*, Compositio Math., 70 (1989) no.2, 155-175.
[5] V. Ginzburg, *Lectures on Noncommutative geometry*, arXiv:math/0506603v1 .
[6] J. A. Green, *Polynomial representations of* GL_n. Lecture Notes in Mathematics, 830. Springer, Berlin, 1980.
[7] B. Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.
[8] V. Miemietz, W. Turner, *Homotopy, Homology and* GL_2, Proc. London Math. Soc. (3) 100 (2010), no.2, 585-606.
[9] V. Miemietz, W. Turner, *The Yoneda extension algebra of* $GL_2(\bar{\mathbb{F}}_p)$,preprint, arXiv:1106.5411.
[10] V. Miemietz, W. Turner, *The Weyl extension algebra of* $GL_2(\bar{\mathbb{F}}_p)$, preprint, arXiv:1106.5665.

[11] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) 43 (1991), no. 1, 37-48.
[12] R. Rouquier, *Derived equivalences and finite dimensional algebras*, Proceedings of the International Congress of Mathematicians (Madrid, 2006), vol II, pp. 191-221, EMS Publishing House, 2006.
[13] F. Salfelder, *Hochschild cohomology of category $\mathcal{O}$*, felix.salfelder.org/misc/HH.ps
[14] nLab, http://ncatlab.org/nlab/show/Hochschild+cohomology
[15] F. Xu, *Hochschild and ordinary cohomology rings of small categories*, Adv. Math. 219 (2008), 1872-1893.

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