On the convergence of Le Page series in Skohorod space

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Abstract

We consider the problem of the convergence of the so-called Le Page series in the Skohorod space $\mathbb{D}^d = \mathbb{D}([0,1], \mathbb{R}^d)$ and provide a simple criterion based on the moments of the increments of the random process involved in the series. This provides a simple sufficient condition for the existence of an α-stable distribution on $\mathbb{D}^d$ with given spectral measure.

Key words: stable distribution, Le Page series, Skohorod space.
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1 Introduction

We are interested in the convergence in the Skohorod space $\mathbb{D}^d = \mathbb{D}([0,1], \mathbb{R}^d)$ endowed with the J_1-topology of random series of the form

$$X(t) = \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \varepsilon_i Y_i(t), \quad t \in [0,1], \tag{1}$$

where $\alpha \in (0,2)$ and

- $(\Gamma_i)_{i\geq 1}$ is the increasing enumeration of the points of a Poisson point process on $[0,+\infty)$ with Lebesgue intensity;
- $(\varepsilon_i)_{i\geq 1}$ is an i.i.d. sequence of real random variables;

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- $(Y_i)_{i\geq 1}$ is an i.i.d. sequence of $\mathbb{D}^d$-valued random variables;
- the sequences (Γ_i), (ε_i) and (Y_i) are independent.

Note that a more constructive definition for the sequence $(\Gamma_i)_{i\geq 1}$ is given by

$$\Gamma_i = \sum_{j=1}^{i} \gamma_j, \quad i \geq 1,$$

where $(\gamma_i)_{i\geq 1}$ is an i.i.d. sequence of random variables with exponential distribution of parameter 1, and independent of (ε_i) and (Y_i).

Series of the form (1) are known as Le Page series. For fixed $t \in [0,1]$, the convergence in $\mathbb{R}^d$ of the series (1) is ensured as soon as one of the following conditions is satisfied:

- $0 < \alpha < 1$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$,
- $1 \leq \alpha < 2$, $\mathbb{E}\varepsilon_1 = 0$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}|Y_1(t)|^\alpha < \infty$.

Here $|.|$ denotes the usual Euclidean norm on $\mathbb{R}$ or on $\mathbb{R}^d$. The random variable $X(t)$ has then an α-stable distribution on $\mathbb{R}^d$. Conversely, it is well known that any α-stable distributions on $\mathbb{R}^d$ admits a representation in terms of Le Page series (see for example Samorodnitsky and Taqqu [9] section 3.9).

There is a vast literature on symmetric α-stable distributions on separable Banach spaces (see e.g. Ledoux and Talagrand [7] or Araujo and Giné [1]). In particular, any symmetric α-stable distribution on a separable Banach space can be represented as an almost surely convergent Le Page series (see Corollary 5.5 in [7]). The existence of a symmetric α-stable distribution with a given spectral measure is not automatic and is linked with the notion of stable type of a Banach space; see Theorem 9.27 in [7] for a precise statement. In [3], Davydov, Molchanov and Zuyev consider α-stable distributions in the more general framework of abstract convex cones.

The space $\mathbb{D}^d$ equipped with the norm

$$\|x\| = \sup\{|x_i(t)|, \quad t \in [0,1],\ i = 1, \cdots, d\}, \quad x = (x_1, \cdots, x_d) \in \mathbb{D}^d,$$

is a Banach space but is not separable. The uniform topology associated with this norm is finer than the J_1-topology. On the other hand, the space $\mathbb{D}^d$ with the J_1-topology is Polish, i.e. there exists a metric on $\mathbb{D}^d$ compatible with the J_1-topology that makes $\mathbb{D}^d$ a complete and separable metric space. However, such a metric can not be compatible with the vector space structure since the addition is not continuous in the J_1-topology. These properties explains why the general theory of stable distributions on separable Banach space can not be applied to the space $\mathbb{D}^d$.

Nevertheless, in the case when the series (1) converges, the distribution of the sum X defines an α-stable distribution on $\mathbb{D}^d$. We can determine the associated spectral measure σ on the unit sphere $\mathbb{S}^d = \{x \in \mathbb{D}^d;\ \|x\| = 1\ \}$. It is given by

$$\sigma(A) = \frac{\mathbb{E}\Big(|\varepsilon_1|^\alpha \|Y_1\|^\alpha \mathbf{1}_{\{\mathrm{sign}(\varepsilon_1)Y_1/\|Y_1\| \in A\}}\Big)}{\mathbb{E}(|\varepsilon_1|^\alpha \|Y_1\|^\alpha)}, \quad A \in \mathcal{B}(\mathbb{S}^d).$$

This is closely related to regular variations theory (see Hult and Lindskog [5] or Davis and Mikosch [2]): for all $r > 0$ and $A \in \mathcal{B}(\mathbb{S}^d)$ such that $\sigma(\partial A) = 0$, it holds that

$$\lim_{n\to\infty} n\mathbb{P}\Big(\frac{X}{\|X\|} \in A \,\Big|\, \|X\| > rb_n\Big) = r^{-\alpha}\sigma(A),$$

with

$$b_n = \inf\{r > 0;\ \mathbb{P}(\|X\| < r) \le n^{-1}\}, \quad n \ge 1.$$

The random variable X is said to be regularly varying in $\mathbb{D}^d$ with index α and spectral measure σ.

In this framework, convergence of the Le Page series (1) in $\mathbb{D}^d$ is known in some particular cases only:

- When $0 < \alpha < 1$, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $\mathbb{E}\|Y_1\|^\alpha < \infty$, the series (1) converges almost surely uniformly in $[0,1]$ (see example 4.2 in Davis and Mikoch [2]);
- When $1 \le \alpha < 2$, the distribution of the ε_i's is symmetric, $\mathbb{E}|\varepsilon_1|^\alpha < \infty$ and $Y_i(t) = \mathbf{1}_{[0,t]}(U)$ with $(U_i)_{i\ge1}$ an i.i.d. sequence of random variables with uniform distribution on $[0,1]$, the series (1) converges almost surely uniformly on $[0,1]$ and the limit process X is a symmetric α-stable Lévy process (see Rosinski [8]).

The purpose of this note is to complete these results and to provide a general criterion for almost sure convergence in $\mathbb{D}^d$ of the random series (1). Our main result is the following:

Theorem 1 *Suppose that* $1 \le \alpha < 2$*,*

$$\mathbb{E}\varepsilon_1 = 0 \quad , \quad \mathbb{E}|\varepsilon_1|^\alpha < \infty \quad \text{and} \quad \mathbb{E}\|Y_1\|^\alpha < \infty.$$

Suppose furthermore that there exist $\beta_1, \beta_2 > \frac{1}{2}$ *and* F_1 *,*F_2 *nondecreasing continuous functions on* $[0,1]$ *such that, for all* $0 \le t_1 \le t \le t_2 \le 1$*,*

$$\mathbb{E}|Y_1(t_2) - Y_1(t_1)|^2 \le |F_1(t_2) - F_1(t_1)|^{\beta_1}, \tag{2}$$

$$\mathbb{E}|Y_1(t_2) - Y_1(t)|^2|Y_1(t) - Y_1(t_1)|^2 \le |F_2(t_2) - F_2(t_1)|^{2\beta_2}. \tag{3}$$

Then, the Le Page series (1) *converges almost surely in* $\mathbb{D}^d$*.*

The proof of this Theorem is detailled in the next section. We provide hereafter a few cases where Theorem 1 can be applied.

Example 1 The example considered by Davis and Mikosh [2] follows easily from Theorem 1: let U be a random variable with uniform distribution on $[0,1]$ and consider $Y_1(t) = \mathbf{1}_{[0,t]}(U)$, $t \in [0,1]$. Then, for $0 \le t_1 \le t \le t_2 \le 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = t_2 - t_1 \quad \text{and} \quad \mathbb{E}(Y_1(t_2) - Y_1(t))^2(Y_1(t) - Y_1(t_1))^2 = 0,$$

so that conditions (2) and (3) are satisfied.

Example 2 Example 1 can be generalized in the following way: let $p \geq 1$, $(U_i)_{1\leq i\leq p}$ independent random variables on $[0,1]$ and $(R_i)_{1\leq i\leq p}$ random variables on $\mathbb{R}^d$. Consider

$$Y_1(t) = \sum_{i=1}^{p} R_i \mathbf{1}_{[0,t]}(U_i).$$

Assume that for each $i \in \{1, \cdots, p\}$, the cumulative distribution function F_i of U_i is continuous on $[0,1]$. Assume furthermore that there is some $M > 0$ such that for all $i \in \{1, \cdots, p\}$

$$\mathbb{E}[R_i^4 \mid \mathcal{F}_U] \leq M \quad \text{almost surely,} \tag{4}$$

where $\mathcal{F}_U = \sigma(U_1, \cdots, U_p)$. This is for example the case when the R_i's are uniformly bounded by $M^{1/4}$ or when the R_i's have finite fourth moment and are independent of the U_i's. Simple computations entails that under condition (4), it holds for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 \leq M^{1/2} p^2 |F(t_2) - F(t_1)|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2 (Y_1(t) - Y_1(t_1))^2 \leq M p^4 |F(t_2) - F(t_1)|^4.$$

with $F(t) = \sum_{i=1}^{p} F_i(t)$. So conditions (2) and (3) are satisfied and Theorem (1) can be applied in this case.

Example 3 A further natural example is the case when $Y_1(t)$ is a Poisson process with intensity $\lambda > 0$ on $[0,1]$. Then, for all $0 \leq t_1 \leq t \leq t_2 \leq 1$,

$$\mathbb{E}(Y_1(t_2) - Y_1(t_1))^2 = \lambda |t_2 - t_1| + \lambda^2 |t_2 - t_1|^2$$

and

$$\mathbb{E}(Y_1(t_2) - Y_1(t))^2 (Y_1(t) - Y_1(t_1))^2 = (\lambda|t_2 - t| + \lambda^2 |t_2 - t|^2)(\lambda|t - t_1| + \lambda^2|t - t_1|^2)$$

and we easily see that conditions (2) and (3) are satisfied.

2 Proof

For the sake of clarity, we divide the proof of Theorem 1 into five steps.

Step 1. According to Lemma 1.5.1 in [9], it holds almost surely that for k large enough

$$|\Gamma_k^{-1/\alpha} - k^{-1/\alpha}| \leq 2\alpha^{-1} k^{-1/\alpha} \sqrt{\frac{\ln \ln k}{k}}. \tag{5}$$

This implies the a.s. convergence of the series

$$\sum_{i=1}^{\infty} |\Gamma_i^{-1/\alpha} - i^{-1/\alpha}|\, |\varepsilon_i|\, \|Y_i\| < \infty. \tag{6}$$

The series (6) has indeed nonnegative terms, and (5) implies that the following conditionnal expectation is finite,

$$\mathbb{E}\left[\sum_{i=1}^{\infty}|\Gamma_i^{-1/\alpha}-i^{-1/\alpha}|\,|\varepsilon_i|\,\|Y_i\|\;\Big|\;\mathcal{F}_\Gamma\right]=\mathbb{E}|\varepsilon_1|\,\mathbb{E}\|Y_i\|\sum_{i=1}^{\infty}|\Gamma_i^{-1/\alpha}-i^{-1/\alpha}|$$

where $\mathcal{F}_\Gamma=\sigma(\Gamma_i, i\geq 1)$.

This proves that (6) holds true and it is enough to prove the a.s. convergence in $\mathbb{D}^d$ of the series

$$Z(t)=\sum_{i=1}^{\infty}i^{-1/\alpha}\varepsilon_i Y_i(t),\quad t\in[0,1], \tag{7}$$

Step 2. Next, consider

$$\widetilde{Z}(t)=\sum_{i=1}^{\infty}i^{-1/\alpha}\tilde{\varepsilon}_i Y_i(t),\quad t\in[0,1]. \tag{8}$$

with

$$\tilde{\varepsilon}_i=\varepsilon_i\mathbf{1}_{\{|\varepsilon_i|^\alpha\leq i\}},\quad i\geq 1.$$

We prove that the series (7) and (8) differ only by a finite number of terms. We have indeed

$$\sum_{i=1}^{\infty}\mathbb{P}\left(\tilde{\varepsilon}_i\neq\varepsilon_i\right)=\sum_{i=1}^{\infty}\mathbb{P}\left(|\varepsilon_i|^\alpha>i\right)\leq\mathbb{E}|\varepsilon_1|^\alpha<\infty$$

and the Borel-Cantelli Lemma implies that almost surely $\tilde{\varepsilon}_i=\varepsilon_i$ for i large enough. So, both series (7) and (8) have the same nature and it is enough to prove the convergence in $\mathbb{D}^d$ of the series (8).

Step 3. As a preliminary for step 4, we prove several estimates involving the moments of the random variables $(\tilde{\varepsilon}_i)_{i\geq 1}$. First, for all $m>\alpha$,

$$C(\alpha,m):=\sum_{i=1}^{\infty}i^{-m/\alpha}\mathbb{E}(|\tilde{\varepsilon}_i|^m)<\infty. \tag{9}$$

We have indeed

$$\begin{aligned}C(\alpha,m) &= \sum_{i=1}^{\infty}i^{-m/\alpha}\mathbb{E}(|\varepsilon_i|^m\mathbf{1}_{\{|\varepsilon_i|\leq i^{1/\alpha}\}})\\ &= \mathbb{E}\left(|\varepsilon_1|^m\sum_{i=1}^{\infty}i^{-m/\alpha}\mathbf{1}_{\{i\geq|\varepsilon_1|^\alpha\}}\right)\\ &\leq C\mathbb{E}(|\varepsilon_1|^m|\varepsilon_1|^{\alpha-m})=C\mathbb{E}(|\varepsilon_1|^\alpha)<\infty\end{aligned}$$

where the constant $C = \sup_{x>0} x^{m/\alpha-1} \sum_{i \ge x} i^{-m/\alpha}$ is finite since for $m > \alpha$

$$\lim_{x\to\infty} x^{m/\alpha-1} \sum_{i\geq x}^{\infty} i^{-m/\alpha} = \frac{\alpha}{m-\alpha}.$$

Similarly, we also have

$$C(\alpha,1) := \sum_{i=1}^{\infty} i^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_i)| < \infty. \tag{10}$$

Indeed, the assumption $\mathbb{E}\varepsilon_i = 0$ implies $\mathbb{E}(\tilde{\varepsilon}_i) = \mathbb{E}(\varepsilon_i \mathbf{1}_{\{|\varepsilon_i|^\alpha > i\}})$. Hence,

$$\begin{aligned}
\sum_{i=1}^{\infty} i^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_i)| &\leq \sum_{i=1}^{\infty} i^{-1/\alpha} \mathbb{E}(|\varepsilon_1| \mathbf{1}_{\{|\varepsilon_1| > i^{1/\alpha}\}}) \\
&= \mathbb{E}\Big(|\varepsilon_1| \sum_{i=1}^{[|\varepsilon_1|^\alpha]} i^{-1/\alpha} \Big) \\
&\leq \mathbb{E}\Big(|\varepsilon_1| C' (|\varepsilon_1|^\alpha)^{1-1/\alpha} \Big) = C' \mathbb{E}|\varepsilon_1|^\alpha < \infty
\end{aligned}$$

where the constant $C' = \sup_{x>0} x^{1/\alpha-1} \sum_{i=1}^{[x]} i^{-1/\alpha}$ is finite.

Step 4. For $n \geq 1$, consider the partial sum

$$\widetilde{Z}_n(t) = \sum_{i=1}^{n} i^{-1/\alpha} \tilde{\varepsilon}_i Y_i(t), \quad t \in [0,1]. \tag{11}$$

We prove that the sequence of processes $(\widetilde{Z}_n)_{n \geq 1}$ is tight in $\mathbb{D}^d$. Following Theorem 3 in Gikhman and Skohorod [4] chapter 6 section 3, it is enough to show that there exists $\beta > 1/2$ and a non decreasing continuous function F on $[0,1]$ such that

$$\mathbb{E}|\widetilde{Z}_n(t_2) - \widetilde{Z}_n(t)|^2 |\widetilde{Z}_n(t) - \widetilde{Z}_n(t_1)|^2 \leq |F(t_2) - F(t_1)|^{2\beta}, \tag{12}$$

for all $0 \leq t_1 \leq t \leq t_2 \leq 1$. Remark that in Gikhman and Skohorod [4], the result is stated only for $F(t) \equiv t$. However, the case of a general continuous non decreasing function F follows easily from a simple change of variable.

We use the notations $Y(t) = (Y^p(t))_{1 \leq p \leq d}$, $[\![1,n]\!] = \{1, \cdots, n\}$ and

$\mathbf{i} = (i_1, i_2, i_3, i_4) \in [\![1, n]\!]^4$. We have

$$\begin{aligned}
& \mathbb{E}|\widetilde{Z}_n(t_2) - \widetilde{Z}_n(t)|^2 |\widetilde{Z}_n(t) - \widetilde{Z}_n(t_1)|^2 \\
= \ & \mathbb{E}\Big| \sum_{i=1}^{n} i^{-1/\alpha} \tilde{\varepsilon}_i (Y_i(t) - Y_i(t_1)) \Big|^2 \Big| \sum_{j=1}^{n} j^{-1/\alpha} \tilde{\varepsilon}_j (Y_j(t_2) - Y_j(t)) \Big|^2 \\
= \ & \sum_{1 \le p,q \le d} \sum_{\mathbf{i} \in [\![1,n]\!]^4} (i_1 i_2 i_3 i_4)^{-1/\alpha} \mathbb{E}(\tilde{\varepsilon}_{i_1} \tilde{\varepsilon}_{i_2} \tilde{\varepsilon}_{i_3} \tilde{\varepsilon}_{i_4}) \mathbb{E}[(Y_{i_1}^p(t) - Y_{i_1}^p(t_1)) \quad (13) \\
& \quad (Y_{i_2}^p(t) - Y_{i_2}^p(t_1))(Y_{i_3}^q(t_2) - Y_{i_3}^q(t))(Y_{i_4}^q(t_2) - Y_{i_4}^q(t))] \quad (14) \\
\le \ & d^2 \sum_{\mathbf{i} \in [\![1,n]\!]^4} (i_1 i_2 i_3 i_4)^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_{i_1} \tilde{\varepsilon}_{i_2} \tilde{\varepsilon}_{i_3} \tilde{\varepsilon}_{i_4})| D_{\mathbf{i}}(t, t_1, t_2) \quad (15)
\end{aligned}$$

where

$$D_{\mathbf{i}}(t, t_1, t_2) = \mathbb{E}|Y_{i_1}(t) - Y_{i_1}(t_1)||Y_{i_2}(t) - Y_{i_2}(t_1)||Y_{i_3}(t_2) - Y_{i_3}(t)||Y_{i_4}(t_2) - Y_{i_4}(t)|.$$

Consider $\sim_{\mathbf{i}}$ the equivalence relation on $\{1, \cdots, 4\}$ defined by

$$j \sim_{\mathbf{i}} j' \quad \text{if and only if} \quad i_j = i_{j'}.$$

Let $\mathcal{P}$ be the set of all partitions of $\{1, \cdots, 4\}$ and $\tau(\mathbf{i})$ be the partition of $\{1, 2, 3, 4\}$ given by the equivalence classes of $\sim_{\mathbf{i}}$. We introduce these definitions because, since the Y_i's are i.i.d., the term $D_{\mathbf{i}}(t, t_1, t_2)$ depends on $\mathbf{i}$ only through the associated partition $\tau(\mathbf{i})$. For example, if $\tau(\mathbf{i}) = \{1, 2, 3, 4\}$, i.e. if $i_1 = i_2 = i_3 = i_4$, then

$$D_{\mathbf{i}}(t, t_1, t_2) = \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)|^2.$$

Or if $\tau(\mathbf{i}) = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, i.e. if the indices $i_1, \cdots, i_4$ are pairwise distinct, then

$$D_{\mathbf{i}}(t, t_1, t_2) = (\mathbb{E}|Y_1(t) - Y_1(t_1)| \mathbb{E}|Y_1(t_2) - Y_1(t)|)^2.$$

For $\tau \in \mathcal{P}$, we denote by $D_\tau(t, t_1, t_2)$ the common value of the terms $D_{\mathbf{i}}(t, t_1, t_2)$ corresponding to indices $\mathbf{i}$ such that $\tau(\mathbf{i}) = \tau$. Define also

$$S_{n,\tau} = \sum_{\mathbf{i} \in \{1, \cdots, n\}^4; \tau(\mathbf{i}) = \tau} (i_1 i_2 i_3 i_4)^{-1/\alpha} |\mathbb{E}(\tilde{\varepsilon}_{i_1} \tilde{\varepsilon}_{i_2} \tilde{\varepsilon}_{i_3} \tilde{\varepsilon}_{i_4})|.$$

With these notations, equation (15) can be rewritten as

$$\mathbb{E}|\widetilde{Z}_n(t_2) - \widetilde{Z}_n(t)|^2 |\widetilde{Z}_n(t) - \widetilde{Z}_n(t_1)|^2 \le d^2 \sum_{\tau \in \mathcal{P}} S_{n,\tau} D_\tau(t, t_1, t_2). \quad (16)$$

Under conditions (2) and (3), we will prove that for each $\tau \in \mathcal{P}$, there exist $\beta_\tau > 1/2$, a non decreasing continuous function F_τ on $[0, 1]$ and a constant $S_\tau > 0$ such that

$$D_\tau(t, t_1, t_2) \le |F_\tau(t_1) - F_\tau(t_2)|^{2\beta_\tau}, \quad 0 \le t_1 \le t \le t_2, \quad (17)$$

and

$$S_{n,\tau} \leq S_\tau, \quad n \geq 1. \tag{18}$$

Equations (16),(17) and (18) together imply inequality (12) for some suitable choices of $\beta > 1/2$ and F.

It remains to prove inequalities (17) and (18). If $\tau = \{1, 2, 3, 4\}$,

$$D_\tau(t, t_1, t_2) \leq \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)|^2 \leq |F_2(t_2) - F_2(t_1)|^{2\beta_2}$$

and

$$S_n^\tau = \sum_{i=1}^n i^{-4/\alpha} \mathbb{E}\tilde{\varepsilon}_i^4 \leq C(\alpha, 4).$$

If $\tau = \{1\} \cup \{2\} \cup \{3\} \cup \{4\}$, Cauchy-Schwartz inequality entails

$$D_\tau(t, t_1, t_2) \leq (\mathbb{E}|Y_1(t) - Y_1(t_1)|\mathbb{E}|Y_1(t_2) - Y_1(t)|)^2 \leq |F_1(t_2) - F_1(t_1)|^{2\beta_1}$$

and

$$S_n^\tau \leq \sum_{\mathbf{i} \in \{1, \cdots, n\}^4; \tau(\mathbf{i}) = \tau} (i_1 i_2 i_3 i_4)^{-1/\alpha} |\mathbb{E}\tilde{\varepsilon}_{i_1}||\mathbb{E}\tilde{\varepsilon}_{i_2}||\mathbb{E}\tilde{\varepsilon}_{i_3}||\mathbb{E}\tilde{\varepsilon}_{i_4}| \leq C(\alpha, 1)^4.$$

Similarly, for $\tau = \{1, 2, 3\} \cup \{4\}$,

$$\begin{aligned} D_\tau(t, t_1, t_2) &= \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 |Y_1(t_2) - Y_1(t)| \mathbb{E}|Y_1(t_2) - Y_1(t)| \\ &\leq |F_1(t) - F_1(t_1)|^{\beta_1/2} |F_2(t_2) - F_2(t_1)|^{\beta_2} |F_1(t_2) - F_1(t)|^{\beta_1/2} \\ &\leq |(F_1 + F_2)(t_2) - (F_1 + F_2)(t_1)|^{\beta_1 + \beta_2} \end{aligned}$$

and

$$S_n^\tau \leq \sum_{1 \leq i \neq j \leq n} (i^3 j)^{-1/\alpha} \mathbb{E}|\tilde{\varepsilon}_i|^3 |\mathbb{E}\tilde{\varepsilon}_j| \leq C(\alpha, 3) C(\alpha, 3).$$

or for $\tau = \{1, 2\} \cup \{3\} \cup \{4\}$,

$$\begin{aligned} D_\tau(t, t_1, t_2) &= \mathbb{E}|Y_1(t) - Y_1(t_1)|^2 (\mathbb{E}|Y_1(t_2) - Y_1(t)|)^2 \\ &\leq |F_1(t) - F_1(t_1)|^{\beta_1} |F_1(t_2) - F_1(t)|^{\beta_1} \\ &\leq |F_1(t_2) - F_1(t_1)|^{2\beta_1} \end{aligned}$$

and

$$S_n^\tau \leq \sum_{1 \leq i \neq j \neq k \leq n} (i^2 jk)^{-1/\alpha} \mathbb{E}|\tilde{\varepsilon}_i|^2 |\mathbb{E}\tilde{\varepsilon}_j||\mathbb{E}\tilde{\varepsilon}_k| \leq C(\alpha, 2) C(\alpha, 1)^2.$$

Similar computations can be checked in all remaining cases. The cardinality of $\mathcal{P}$ is equal to 13.

Step 5. We prove Theorem 1. For each fixed $t \in [0, 1]$, Kolmogorov's three-series Theorem implies that $\widetilde{Z}_n(t)$ converge almost surely as $n \to \infty$. So the finite-dimensional distributions of $(\widetilde{Z}_n)_{n \geq 1}$ converge. The tightness in $\mathbb{D}^d$ of the sequence has already been proved in step 4, so $(\widetilde{Z}_n)_{n \geq 1}$ weakly convergence in $\mathbb{D}^d$ as $n \to \infty$. We then apply Theorem 1 in Kallenberg [6] and deduce that $\widetilde{Z}_n$ converges almost surely in $\mathbb{D}^d$. In view of step 1 and step 2, this yields the almost sure convergence of the series (1). □

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