Proposing (ω_1, β)-morasses for $\omega_1 \leq \beta$

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Abstract

Firstly, I propose a notion of (ω_1, β)-morass for the case that $\omega_1 \leq \beta$. Secondly, I define κ-standard morasses such that every $\omega_{1+\beta}$-standard morass is an (ω_1, β)-morass. Thirdly, I justify these notions by proving: If there is a κ-standard morass, then there is an $L_\kappa[X]$ with $Card^{L_\kappa[X]} = Card \cap \kappa$ for which the fine structure theory and condensation hold.

1 Introduction

In set theory, structures are often obtained by first recursively constructing small structures and then taking a direct limit to get a bigger one. Usually a chain of structures of size $< \kappa$ is constructed by induction along a cardinal κ. In this way, a direct limit of size κ can be obtained. Morasses are index sets along which structures of size $< \omega_\alpha$ can be constructed by induction in such a way that the limit has size $\omega_{\alpha+\beta}$. The appropriate morass for such a construction is called an (ω_α, β)-morass.

Morasses were invented by R. Jensen in the early 1970s. He used them to prove the model-theoretic cardinal transfer theorems (see [ChKe]) in Gödel's constructible universe L. If κ, λ are infinite cardinals, a structure $\mathfrak{A}$ is said to have type (κ, λ) if $\mathfrak{A} = \langle A, X^{\mathfrak{A}}, \ldots \rangle$ where $card(A) = \kappa$ and $card(X^{\mathfrak{A}}) = \lambda$. The simplest cardinal transfer theorem states that if $\mathfrak{A}$ is a structure of type (κ^+, κ) then there exists a structure $\mathfrak{B}$ of the same language of type (ω_1, ω) which is elementary equivalent to $\mathfrak{A}$. This is proved by the construction of an elementary chain that has $\mathfrak{B}$ as its direct limit. Using morasses, Jensen obtains in L statements of this type for bigger gaps between κ and λ.

The most general cardinal transfer theorem is shown in a hand-written set of notes [Jen]. Here, he defines the notion of (κ, β)-morasses for $\beta < \kappa$, κ regular. How an $(\omega_1, 1)$-morass is used to prove the gap-2 cardinal transfer theorem may be found in [Dev]. The theory of morasses is very far developed and very well examined. In particular it is known how to construct morasses in L (see [Dev], [Fri], [Jen]) and how to force them ([Sta1], [Sta2]). Moreover, D. Velleman has defined so-called simplified morasses, along which morass constructions can be carried out very easily compared to classical morasses ([Vel1], [Vel2], [Vel4]). They are equivalent to usual morasses ([Don], [Mor]). Besides the cardinal transfer theorems, there are many combinatorial applications of morasses. One is for example the construction of κ^{++}-super-Suslin trees by S. Shelah and L. Stanley [ShSt1]. Other applications need strengthenings of morasses, like simplified morasses with linear limits [Vel4].

For the case $\kappa \leq \beta$, (κ, β)-morasses have never been defined. I want to propose a notion of (ω_1, β)-morass for this case. In addition, I will define κ-standard morasses such that every $\omega_{1+\beta}$-standard morass is an (ω_1, β)-morass. I will prove that if there is a κ-standard morass, then there is an $L_\kappa[X]$ with $Card^{L_\kappa[X]} = Card \cap \kappa$ for which the fine structure theory and condensation hold. In a forthcoming paper [Irr2], I show that if there is an $L_\kappa[X]$ with $Card^{L_\kappa[X]} = Card \cap \kappa$ for which the fine structure theory and condensation hold, then there is a κ-standard morass. On the one hand, this justifies my definitions. On the other hand, it shows that the definition of κ-standard morass is best possible in the sense that it completely captures the combinatorics of an $L_\kappa[X]$ with $Card^{L_\kappa[X]} = Card \cap \kappa$, fine structure theory and condensation. Moreover, I conjecture that the existence of an $\omega_{1+\beta}$-standard morass is actually equivalent to the existence of an (ω_1, β)-morass.

One notion that is related to my definitions of (ω_1, β)-morass for $\omega_1 \leq \beta$ and κ-standard morass is the premorass that was studied by Jensen in the context of his proofs of global square. In [DJS], H.-D. Donder, R. Jensen and L. Stanley derive from the existence of an appropriate premorass that global square and the combinatorial principle squared scales holds. But they derive global square and squared scales directly from the premorass they construct in L without explicitly axiomatizing the notion of premorass. A similiar approach is followed in [BJW] to provide the necessary combinatorics for the proof of Jensen's coding theorem. Squared scales was formulated by Avraham and Shelah for their work on strong covering (see [She], chapter VIII]). A strengthening of squared scales, which is also proved by the same approach [ShSt2], was used in [ShSt3] by S. Shelah and L. Stanley to give a combinatorial proof of Jensen's coding theorem.

Another related notion is that of a smooth category which was introduced by R. Jensen and M. Zeman to prove global square in the core model for measures of order 0 [JeZe]. Similiar systems were studied (again without giving an axiomatic account) in [SchZe1] and [ScheZe2] by E. Schimmerling and M. Zeman to prove that Jensen extender models satisfy the Gap 1 Morass principle and $\square_\kappa$ for all κ that are not subcompact.

It is a natural question to ask in which inner models (ω_1, β)-morasses and κ-standard morasses exist. By the usual argument that ω_1-Erdös cardinals do not exist in L (see e.g. theorem V 1.8 of [Dev]), it is easy to see that an inner model M with an ω_1-Erdös cardinal cannot be of the form $M = L[X]$ such that $L[X]$ satisfies condensation. But that does not mean, that it is impossible that inner models with ω_1-Erdös cardinals (or even larger cardinals) contain κ-standard morasses.

This is a part of my dissertation [Irr1]. I thank Dieter Donder for being my adviser, Hugh Woodin for an invitation to Berkeley, where part of the work was done, and the DFG-Graduiertenkolleg "Sprache, Information, Logik" in Munich for their support.

2 (ω_1, β)-Morasses

Let me briefly recall how an object of size ω_2 is constructed from countable objects in Gödel's constructible universe L. That is, let me briefly describe how a construction along an $(\omega_1, 1)$-morass works. Let an ordinal ν be called ω_2-like, if $L_\nu \models$ there exists exactly one uncountable cardinal. Let $S^0 = \{\alpha \in Lim \mid L_\nu \models (\alpha = \omega_1)$ for some ω_2-like ordinal $\nu\}$. Then there are different kinds of ω_2-like ordinals, namely for every $\alpha \in S^0$ there is the set $S_\alpha = \{\nu \mid \nu$ is ω-like and $L_\nu \models \alpha = \omega_1\}$ of those which believe that $\alpha = \omega_1$. Now, a morass construction proceeds as follows: On the one hand, one constructs for every $\alpha \in S^0 \cap \omega_1$ by induction over $\nu \in S_\alpha$ a countable chain $\langle \mathfrak{A}_\nu \mid \nu \in S_\alpha \rangle$ of countable structures $\mathfrak{A}_\nu$. On the other hand, one constructs by induction over $\alpha \in S^0$ a system of embeddings between these chains. As direct limit of this system of embeddings, one obtains a chain $\langle \mathfrak{A}_\nu \mid \nu \in S_{\omega_1} \rangle$ of length ω_2 of structures $\mathfrak{A}_\nu$ of size $\leq \omega_1$. Finally, the structure $\mathfrak{A}$ of size ω_2 that one wants to construct is obtained as the direct limit of this chain.

The approach is generalized by Jensen to all $\beta < \omega_1$. Let an ordinal ν be $\omega_{1+\beta}$-like, if the set $\{\alpha \mid L_\nu \models \alpha$ is an uncountable cardinal$\} \cup \{\nu\}$ has order-type $\beta + 1$. The basic construction is first carried out for countable structures $\mathfrak{A}_\nu$ and all $\omega_{1+\beta}$-like ordinals ν with $\nu < \omega_1$, and then directed systems of embeddings are used to blow it up to $\omega_{1+\beta}$. This motivates his definition of (ω_α, β)-morasses. They describe axiomatically the properties of the $\omega_{1+\beta}$-like ordinals which enable such constructions. This short introduction to morasses explains already why Jensen never introduced (ω_α, β)-morasses for the case $\omega_1 \leq \beta$, namely because then there are no $\omega_{1+\beta}$-like ordinals below ω_1.

To explain how I circumvent this problem, let me first introduce the notation $f : \bar{\nu} \Rightarrow \nu$ from Jensen's approach. As explained, he considers the sets $S_\alpha = \{\nu \mid L_\nu \models \alpha$ is the largest cardinal$\}$. Let α_ν be the largest cardinal of L_ν. Then he constructs, on the one hand, by induction over the ordinals in the sets S_α chains $\langle \mathfrak{A}_\nu \mid \nu \in S_\alpha \rangle$ of structures $\mathfrak{A}_\nu$. On the other hand, he considers maps f which map under certain conditions $S_{\alpha_{\bar{\nu}}} \cap \bar{\nu}$ into $S_{\alpha_\nu} \cap \nu$ in such a way, that f can be extended to an embedding from $\langle \mathfrak{A}_\tau \mid \tau \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu} \rangle$ into $\langle \mathfrak{A}_\tau \mid \tau \in S_{\alpha_\nu} \cap \nu \rangle$. For such a map he uses the notation $f : \bar{\nu} \Rightarrow \nu$. The possibility to extend the maps to uniform constructions is guaranteed by the so-called logical preservation axiom (see axiom LP1 below). If $f : \bar{\nu} \Rightarrow \nu$, then in Jensen's case $\bar{\nu}$ and ν are of the same type, that is, if $\bar{\nu}$ is ω_η-like, then ν is also ω_η-like. In my case, they can have different types, i.e. if $\bar{\nu}$ is ω_η-like, then ν can be ω_γ-like for some $\gamma \geq \eta$. This is done in such a way that in the limit a construction along the $\omega_{1+\beta}$-like cardinals takes place.

As consequence, also the chains $\langle \mathfrak{A}_\tau \mid \tau \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu} \rangle$ and $\langle \mathfrak{A}_\tau \mid \tau \in S_{\alpha_\nu} \cap \nu \rangle$ of ordinals τ of different types will have to fit together. The idea is to take care of this in the recursive definition of the chains. However, this makes it necessary to introduce a second logical preservation axiom which guarantees that if $\bar{\nu}$ and ν are of different types and $f : \bar{\nu} \Rightarrow \nu$ is cofinal, then f can be extended to an embedding from $\langle \mathfrak{A}_\tau \mid \tau \in S_{\alpha_{\bar{\nu}}} \cap \bar{\nu} \rangle$ into $\langle \mathfrak{A}_\tau \mid \tau \in S_{\alpha_\nu} \cap \nu \rangle$. The second logical preservation axiom (see LP2 below) is inspired by and closely related to the construction of $\Box$-sequences. Unfortunately, I do not have an example of a typical recursive morass construction which can be carried out with my morasses but not with Jensen's morasses or a $\Box$-principle.

Morasses are also closely related to Jensen's fine structure theory in the following way. If for a map f the relation $f : \bar{\nu} \Rightarrow \nu$ holds, this does not only mean that $f : \bar{\nu} \rightarrow \nu$ but also that it can be extended to a map $f : \mu_{\bar{\nu}} \rightarrow \mu_\nu$ where $\mu_{\bar{\mu}} \geq \bar{\nu}$ and $\mu_\nu \geq \nu$ depend on $\bar{\nu}$ and ν. In this sense, it can be interpreted as saying not only that f is a map from $\bar{\nu}$ to ν, but that it is a Σ_1-elementary map from $\langle L_{\bar{\nu}}, \bar{A}\rangle$ into $\langle L_\nu, A\rangle$ where $\bar{A}$ is a predicate coding $L_{\mu_{\bar{\nu}}}$ and A is a predicate coding L_{μ_ν}. To show the above mentioned property that from my morasses an inner model with fine structure can be constructed, I include into my definition of morasses explicitly such a coding property. Moreover, the fine structural coding property for Σ_n-elementary maps is represented by relations $\Rightarrow_n$.

Let $\omega_1 \leq \beta$, $S = Lim \cap \omega_{1+\beta}$ and $\kappa := \omega_{1+\beta}$.

We write $Card$ for the class of cardinals and $RCard$ for the class of regular cardinals.

Let $\lhd$ be a binary relation on S such that:

(a) If $\nu \lhd \tau$, then $\nu < \tau$.
For all $\nu \in S - RCard$, $\{\tau \mid \nu \lhd \tau\}$ is closed.
For $\nu \in S - RCard$, there is a largest μ such that $\nu \unlhd \mu$.

Let μ_ν be this largest μ with $\nu \unlhd \mu$.

Let

$$\nu \sqsubseteq \tau :\Leftrightarrow \nu \in Lim(\{\delta \mid \delta \lhd \tau\}) \cup \{\delta \mid \delta \unlhd \tau\}.$$

(b) $\sqsubseteq$ is a (many-rooted) tree.

Hence, if $\nu \notin RCard$ is a successor in $\sqsubset$, then μ_ν is the largest μ such that $\nu \sqsubseteq \mu$. To see this, let μ_ν^* be the largest μ such that $\nu \sqsubseteq \mu$. It is clear that $\mu_\nu \leq \mu_\nu^*$, since $\nu \unlhd \mu$ implies $\nu \sqsubseteq \mu$. So assume that $\mu_\nu < \mu_\nu^*$. Then $\nu \ntrianglelefteq \mu_\nu^*$ by the definition of μ_ν. Hence $\nu \in Lim(\{\delta \mid \delta \lhd \mu_\nu^*\})$ and $\nu \in Lim(\{\delta \mid \delta \sqsubseteq \mu_\nu^*\})$. Therefore, $\nu \in Lim(\sqsubseteq)$ since $\sqsubseteq$ is a tree. That contradicts our assumption that ν is a successor in $\sqsubset$.

The properties of $\omega\nu \lhd \omega\tau$ are an axiomatic description of the relation "$\omega\nu$ is regular in L_τ". If $\omega\nu \lhd \omega\tau$ really is this relation, then $\omega\nu \sqsubset \omega\tau$ implies that $\omega\nu$ is a cardinal in L_τ, while the converse implication is not true in general. This is a crucial difference to Jensen's morasses, where $\omega\nu \sqsubset \omega\tau$ is an axiomatic description of "$\omega\nu$ is a cardinal in L_τ", and it is the reason why $\lhd$ is introduced. However, if there exists a maximal cardinal in L_ν and $\nu < \tau$, then the two interpretations of $\omega\nu \sqsubset \omega\tau$ coincide.

For $\alpha \in S$, let $|\alpha|$ be the rank of α in this tree. Let

$S^+ := \{\nu \in S \mid \nu \text{ is a successor in } \sqsubset\}$

$S^0 := \{\alpha \in S \mid |\alpha| = 0\}$

$\widehat{S^+} := \{\mu_\tau \mid \tau \in S^+ - RCard\}$

$\widehat{S} := \{\mu_\tau \mid \tau \in S - RCard\}$.

Let $S_\alpha := \{\nu \in S \mid \nu \text{ is a direct successor of } \alpha \text{ in } \sqsubset\}$. For $\nu \in S^+$, let α_ν be the direct predecessor of ν in $\sqsubset$. For $\nu \in S^0$, let $\alpha_\nu := 0$. For $\nu \notin S^+ \cup S^0$, let $\alpha_\nu := \nu$.

(c) For $\nu, \tau \in (S^+ \cup S^0) - RCard$ such that $\alpha_\nu = \alpha_\tau$, suppose:

$$\nu < \tau \quad \Rightarrow \quad \mu_\nu < \tau.$$

For all $\alpha \in S$, suppose:

(d) S_α is closed

(e) $card(S_\alpha) \leq \alpha^+$

$card(S_\alpha) \leq card(\alpha)$ if $card(\alpha) < \alpha$

(f) $\omega_1 = max(S^0) = sup(S^0 \cap \omega_1)$

$\omega_{1+i+1} = max(S_{\omega_{1+i}}) = sup(S_{\omega_{1+i}} \cap \omega_{1+i+1})$ for all $i < \beta$.

Let $D = \langle D_\nu \mid \nu \in \widehat{S} \rangle$ be a sequence such that $D_\nu \subseteq J^D_\nu$. To simplify matters, my definition of J^D_ν is such that $J^D_\nu \cap On = \nu$ (see section 3 or [SchZe]).

Let an $\langle S, \lhd, D \rangle$-maplet f be a triple $\langle \bar{\nu}, |f|, \nu \rangle$ such that $\bar{\nu}, \nu \in S - RCard$ and $|f| : J^D_{\mu_{\bar{\nu}}} \to J^D_{\mu_\nu}$.

Let $f = \langle \bar{\nu}, |f|, \nu \rangle$ be an $\langle S, \lhd, D \rangle$-maplet. Then we define $d(f)$ and $r(f)$ by $d(f) = \bar{\nu}$ and $r(f) = \nu$. Set $f(x) := |f|(x)$ for $x \in J^D_{\mu_{\bar{\nu}}}$ and $f(\mu_{\bar{\nu}}) := \mu_\nu$. But $dom(f)$, $rng(f)$, $f \upharpoonright X$, etc. keep their usual set-theoretical meaning, i.e. $dom(f) = dom(|f|)$, $rng(f) = rng(|f|)$, $f \upharpoonright X = |f| \upharpoonright X$, etc.

For $\bar{\tau} \leq \mu_{\bar{\nu}}$, let $f^{(\bar{\tau})} = \langle \bar{\tau}, |f| \upharpoonright J^D_{\mu_{\bar{\tau}}}, \tau \rangle$ where $\tau = f(\bar{\tau})$. Of course, $f^{(\bar{\tau})}$ needs not to be a maplet. The same is true for the following definitions. Let $f^{-1} = \langle \nu, |f|^{-1}, \bar{\nu} \rangle$. For $g = \langle \nu, |g|, \nu' \rangle$ and $f = \langle \bar{\nu}, |f|, \nu \rangle$, let $g \circ f = \langle \bar{\nu}, |g| \circ |f|, \nu' \rangle$. If $g = \langle \nu', |g|, \nu \rangle$ and $f = \langle \bar{\nu}, |f|, \nu \rangle$ such that $rng(f) \subseteq rng(g)$, then set $g^{-1}f = \langle \bar{\nu}, |g|^{-1} \mid f \mid, \nu' \rangle$. Finally set $id_\nu = \langle \nu, id \upharpoonright J^D_{\mu_\nu}, \nu \rangle$.

Let $\mathfrak{F}$ be a set of $\langle S, \lhd, D \rangle$-maplets $f = \langle \bar{\nu}, |f|, \nu \rangle$ such that the following holds:

(0) $f(\bar{\nu}) = \nu$, $f(\alpha_{\bar{\nu}}) = \alpha_\nu$ and $|f|$ is order-preserving.

(1) For $f \neq id_{\bar{\nu}}$, there is some $\beta \sqsubseteq \alpha_{\bar{\nu}}$ such that $f \upharpoonright \beta = id \upharpoonright \beta$ and $f(\beta) > \beta$.

(2) If $\bar{\tau} \in S^+$ and $\bar{\nu} \sqsubset \bar{\tau} \sqsubseteq \mu_{\bar{\nu}}$, then $f^{(\bar{\tau})} \in \mathfrak{F}$.

(3) If $f, g \in \mathfrak{F}$ and $d(g) = r(f)$, then $g \circ f \in \mathfrak{F}$.

(4) If $f, g \in \mathfrak{F}$, $r(g) = r(f)$ and $rng(f) \subseteq rng(g)$, then $g^{-1} \circ f \in \mathfrak{F}$.

We write $f : \bar{\nu} \Rightarrow \nu$ if $f = \langle \bar{\nu}, |f|, \nu \rangle \in \mathfrak{F}$. If $f \in \mathfrak{F}$ and $r(f) = \nu$, then we write $f \Rightarrow \nu$. The uniquely determined β in (1) shall be denoted by $\beta(f)$.

Say $f \in \mathfrak{F}$ is minimal for a property $P(f)$ if $P(f)$ holds and $P(g)$ implies $g^{-1}f \in \mathfrak{F}$.

Let

$$f_{(u,x,\nu)} = \text{the unique minimal } f \in \mathfrak{F} \text{ for } f \Rightarrow \nu \text{ and } u \cup \{x\} \subseteq rng(f),$$

if such an f exists. The axioms of the morass will guarantee that $f_{(u,x,\nu)}$ always exists if $\nu \in S - RCard^{L_\kappa[D]}$. Therefore, we will always assume and explicitly mention that $\nu \in S - RCard^{L_\kappa[D]}$ when $f_{(u,x,\nu)}$ is mentioned.

Say $\nu \in S - RCard^{L_\kappa[D]}$ is independent if $d(f_{(\beta,0,\nu)}) < \alpha_\nu$ holds for all $\beta < \alpha_\nu$.

For $\tau \sqsubseteq \nu \in S - RCard^{L_\kappa[D]}$, say ν is ξ-dependent on τ if $f_{(\alpha_\tau,\xi,\nu)} = id_\nu$.

For $f \in \mathfrak{F}$, let $\lambda(f) := sup(f[d(f)])$.

For $\nu \in S - RCard^{L_\kappa[D]}$, let

$$C_\nu = \{\lambda(f) < \nu \mid f \Rightarrow \nu\}$$

$$\Lambda(x, \nu) = \{\lambda(f_{(\beta,x,\nu)}) < \nu \mid \beta < \nu\}.$$

It will be shown that C_ν and $\Lambda(x,\nu)$ are closed in ν.

Recursively define a function $q_\nu : k_\nu + 1 \to On$, where $k_\nu \in \omega$:

$q_\nu(0) = 0$

$q_\nu(k+1) = max(\Lambda(q_\nu \restriction (k+1), \nu))$

if $max(\Lambda(q_\nu \restriction (k+1), \nu))$ exists. The axioms will guarantee that this recursion breaks off (see lemma 4 below), i.e. there is some k_ν such that either

$\Lambda(q_\nu \restriction (k_\nu + 1), \nu) = \emptyset$

or

$\Lambda(q_\nu \restriction (k_\nu + 1), \nu)$ is unbounded in ν.

Define by recursion on $1 \leq n \in \omega$, simultaneouly for all $\nu \in S - RCard^{L_\kappa[D]}$, $\beta \in \nu$ and $x \in J^D_{\mu_\nu}$ the following notions. Here definitions are to be understood in Kleene's sense, i.e., that the left side is defined iff the right side is, and in that case, both are equal.

$f^1_{(\beta,x,\nu)} = f_{(\beta,x,\nu)}$

$\tau(n,\nu) =$ the least $\tau \in S^0 \cup S^+ \cup \widehat{S}$ such that for some $x \in J^D_{\mu_\nu}$

$$f^n_{(\alpha_\tau,x,\nu)} = id_\nu$$

$x(n,\nu) =$ the least $x \in J^D_{\mu_\nu}$ such that $f^n_{(\alpha_{\tau(n,\nu)},x,\nu)} = id_\nu$

$K^n_\nu = \{d(f^n_{(\beta,x(n,\nu),\nu)}) < \alpha_{\tau(n,\nu)} \mid \beta < \nu\}$

$f \Rightarrow_n \nu$ iff $f \Rightarrow \nu$ and for all $1 \leq m < n$

$$rng(f) \cap J^D_{\alpha_{\tau(m,\nu)}} \prec_1 \langle J^D_{\alpha_{\tau(m,\nu)}}, D \restriction \alpha_{\tau(m,\nu)}, K^m_\nu \rangle$$

$$x(m,\nu) \in rng(f)$$

$f^n_{(u,\nu)} =$ the minimal $f \Rightarrow_n \nu$ such that $u \subseteq rng(f)$

$f^n_{(\beta,x,\nu)} = f^n_{(\beta\cup\{x\},\nu)}$

$f : \bar{\nu} \Rightarrow_n \nu :\Leftrightarrow f \Rightarrow_n \nu \; and \; f : \bar{\nu} \Rightarrow \nu.$

Let

$n_\nu =$ the least n such that $f^n_{(\gamma,x,\mu_\nu)}$ is confinal in ν for some $x \in J^D_{\mu_\nu}$, $\gamma \sqsubset \nu$

$x_\nu =$ the least x such that $f^{n_\nu}_{(\alpha_\nu,x,\mu_\nu)} = id_{\mu_\nu}$.

Let

$\alpha^*_\nu = \alpha_\nu$ if $\nu \in S^+$

$\alpha^*_\nu = sup\{\alpha < \nu \mid \beta(f^{n_\nu}_{(\alpha,x_\nu,\mu_\nu)}) = \alpha\}$ if $\nu \notin S^+$.

Let $P_\nu := \{x_\tau \mid \nu \sqsubset \tau \sqsubseteq \mu_\nu, \tau \in S^+\} \cup \{x_\nu\}$.

We say that $\mathfrak{M} = \langle S, \lhd, \mathfrak{F}, D\rangle$ is an (ω_1, β)-morass if the following axioms hold:

(MP – minimum principle)

If $\nu \in S - RCard^{L_\kappa[D]}$ and $x \in J^D_{\mu_\nu}$, then $f_{(0,x,\nu)}$ exists.

(LP1 – first logical preservation axiom)

If $f : \bar{\nu} \Rightarrow \nu$, then $|f| : \langle J^D_{\mu_{\bar{\nu}}}, D \restriction \mu_{\bar{\nu}}\rangle \to \langle J^D_{\mu_\nu}, D \restriction \mu_\nu\rangle$ is Σ_1-elementary.

(LP2 – second logical preservation axiom)

Let $f : \bar{\nu} \Rightarrow \nu$ and $f(\bar{x}) = x$. Then

$$(f \upharpoonright J^D_{\bar{\nu}}) : \langle J^D_{\bar{\nu}}, D \upharpoonright \bar{\nu}, \Lambda(\bar{x}, \bar{\nu})\rangle \to \langle J^D_\nu, D \upharpoonright \nu, \Lambda(x, \nu)\rangle$$

is Σ_0-elementary.

(CP1 – first continuity principle)

For $i \leq j < \lambda$, let $f_i : \nu_i \Rightarrow \nu$ and $g_{ij} : \nu_i \Rightarrow \nu_j$ such that $g_{ij} = f_j^{-1} f_i$. Let $\langle g_i \mid i < \lambda\rangle$ be the transitive, direct limit of the directed system $\langle g_{ij} \mid i \leq j < \lambda\rangle$ and $hg_i = f_i$ for all $i < \lambda$. Then $g_i, h \in \mathfrak{F}$.

(CP2 – second continuity principle)

Let $f : \bar{\nu} \Rightarrow \nu$ and $\lambda = sup(f[\bar{\nu}])$. If, for some $\bar{\lambda}$, $h : \langle J^{\bar{D}}_{\bar{\lambda}}, \bar{D}\rangle \to \langle J^D_\lambda, D \upharpoonright \lambda\rangle$ is Σ_1-elementary and $rng(f \upharpoonright J^D_{\bar{\nu}}) \subseteq rng(h)$, then there is some $g : \bar{\lambda} \Rightarrow \lambda$ such that $g \upharpoonright J^{\bar{D}}_{\bar{\lambda}} = h$.

(CP3 – third continuity principle)

If $C_\nu = \{\lambda(f) < \nu \mid f \Rightarrow \nu\}$ is unbounded in $\nu \in S - RCard^{L_\kappa[D]}$, then the following holds for all $x \in J^D_{\mu_\nu}$:

$$rng(f_{(0,x,\nu)}) = \bigcup\{rng(f_{(0,x,\lambda)}) \mid \lambda \in C_\nu\}.$$

(DP1 – first dependency axiom)

If $\mu_\nu < \mu_{\alpha_\nu}$, then $\nu \in S - RCard^{L_\kappa[D]}$ is independent.

(DP2 – second dependency axiom)

If $\nu \in S - RCard^{L_\kappa[D]}$ is η-dependent on $\tau \sqsubseteq \nu$, $\tau \in S^+$, $f : \bar{\nu} \Rightarrow \nu$, $f(\bar{\tau}) = \tau$ and $\eta \in rng(f)$, then $f^{(\bar{\tau})} : \bar{\tau} \Rightarrow \tau$.

(DP3 – third dependency axiom)

If $\nu \in \widehat{S} - RCard^{L_\kappa[D]}$ and $1 \leq n \in \omega$, then the following holds:

(a) If $f^n_{(\alpha_\tau, x, \nu)} = id_\nu$, $\tau \in S^+ \cup S^0$ and $\tau \sqsubseteq \nu$, then $\mu_\nu = \mu_\tau$.

(b) If $\beta < \alpha_{\tau(n,\nu)}$, then also $d(f^n_{(\beta, x(n,\nu), \nu)}) < \alpha_{\tau(n,\nu)}$.

(DF – definability axiom)

(a) If $f_{(0,z_0,\nu)} = id_\nu$ for some $\nu \in \widehat{S} - RCard^{L_\kappa[D]}$ and $z_0 \in J^D_{\mu_\nu}$, then

$$\{\langle z, x, f_{(0,z,\nu)}(x)\rangle \mid z \in J^D_{\mu_\nu}, x \in dom(f_{(0,z,\nu)})\}$$

is uniformly definable over $\langle J^D_{\mu_\nu}, D \upharpoonright \mu_\nu, D_{\mu_\nu}\rangle$.

(b) For all $\nu \in S - RCard^{L_\kappa[D]}$, $x \in J^D_{\mu_\nu}$, the following holds:

$$f_{(0,x,\nu)} = f^{n_\nu}_{(0,\langle x,\nu,\alpha^*_\nu,P_\nu\rangle,\mu_\nu)}.$$

This finishes the definition of an (ω_1, β)-morass.

A consequence of the axioms is $(\times)$:

Theorem

$$\{\langle z,\tau,x,f_{(0,z,\tau)}(x)\rangle \mid \tau<\nu,\mu_\tau=\nu,z\in J^D_{\mu_\tau},x\in dom(f_{(0,z,\tau)})\}$$
$$\cup\{\langle z,x,f_{(0,z,\nu)}(x)\rangle \mid \mu_\nu=\nu,z\in J^D_{\mu_\nu},x\in dom(f_{(0,z,\nu)})\}$$
$$\cup(\sqsubset\cap\nu^2)$$

is for all $\nu\in S$ uniformly definable over $\langle J^D_\nu, D\restriction\nu, D_\nu\rangle$.

The proof of the property $(\times)$ streches over the next twelve lemmas unto the end of the section. It is proved by induction over $\mu\in\widehat{S}$, i.e. we prove it for all ν with $\mu_\nu=\mu$ assuming that it holds for all τ with $\mu_\tau<\mu$. More precisely, assume it holds for all τ such that $\mu_\tau<\mu$. Then we show that the various minimal maps $f^n_{(u,\nu)}$ exist for all ν such that $\mu_\nu=\mu$ and all $u\subseteq\mu$ (lemmas 1 and 12). And we show that q_ν exists for all ν such that $\mu_\nu=\mu$ (lemma 4). Finally we prove that $(\times)$ holds for all ν with $\mu_\nu=\mu$.

So assume $(\times)$ for all τ such that $\mu_\tau<\mu_\nu:=\mu$. If $\mu=0$, this holds trivially, because then there are no such τ. For the proof we need the following lemmas which are very important in themselves but proved as part of our big induction on μ_ν.

Lemma 1

Let $\nu\in S-RCard^{L_\kappa[D]}$ and $u\subseteq J^D_{\mu_\nu}$. Then there is a minimal $f\in\mathfrak{F}$ for $f\Rightarrow\nu$ and $u\subseteq rng(f)$.

We write $f_{(u,\nu)}$ for this f.

Proof:

(1) For finite $u=\{\xi_1,\dots,\xi_n\}$, we have $f_{(u,\nu)}=f_{(0,\langle\xi_1,\dots,\xi_n\rangle,\nu)}$.

For, by (LP1), $f_{(u,\nu)}:\langle J^D_{\mu_{\bar\nu_1}},D\restriction\mu_{\bar\nu_1}\rangle\to\langle J^D_{\mu_\nu},D\restriction\nu\rangle$ is Σ_1-elementary. Since $J^D_{\mu_\nu}$ is closed under pairs, $u\subseteq rng(f_{(u,\nu)})$ implies $\langle\xi_1,\dots,\xi_n\rangle\in rng(f_{(u,\nu)})$. For the converse, we note that $f_{(0,\langle\xi_1,\dots,\xi_n\rangle,\nu)}:\langle J^D_{\mu_{\bar\nu_2}},D\restriction\mu_{\bar\nu_2}\rangle\to\langle J^D_{\mu_\nu},D\restriction\mu_\nu\rangle$ is Σ_1-elementary by (LP1). Hence $\langle\xi_1,\dots,\xi_n\rangle\in rng(f_{(0,\langle\xi_1,\dots,\xi_n\rangle,\nu)})$ implies $u\subseteq rng(f_{(0,\langle\xi_1,\dots,\xi_n\rangle,\nu)})$. By (MP), $f_{(0,\{\xi_1,\dots,\xi_n\},\nu)}$ exists, and by its minimality, it is as wished.

(2) Now, let u be infinite. Then $I=\{v\subseteq u\mid v\text{ finite }\}$ is directed with regard to $\subseteq$. Let $g_{vw}=f^{-1}_{(w,\nu)}f_{(v,\nu)}$ for $v\subseteq w\in I$. Then $g_{vw}\in\mathfrak{F}$ by (4) and the definition of minimality. Let $\langle g_v\mid v\in I\rangle$ be the transitive, direct limit of $\langle g_{vw}\mid v\subseteq w\rangle$ and $hg_v=f_{(v,\nu)}$ for all $v\in I$. Then $g_v,h\in\mathfrak{F}$ by (CP1). But obviously $h=f_{(u,\nu)}$. □

Lemma 2

Let $\nu\in S-RCard^{L_\kappa[D]}$. Then:

(a) Let $g:\bar\nu\Rightarrow\nu$, $\bar u\subseteq J^D_{\mu_{\bar\nu}}$ and $u=g[\bar u]$. Then $gf_{(\bar u,\bar\nu)}=f_{(u,\nu)}$.

(b) $id_\nu\in\mathfrak{F}$.

(c) If $f\Rightarrow\nu$ and $f\restriction\alpha_\nu=id\restriction\alpha_\nu$, then $f=id_\nu$.

(d) $J^D_{\mu_\nu}=\bigcup\{rng(f_{(\beta,\xi,\nu)})\mid\beta<\alpha_\nu\}$ for all $\xi\in J^D_{\mu_\nu}$.

Proof:

(a) On the one hand, we have

$\bar{u} = g^{-1}[u] \subseteq rng(g^{-1}f_{(u,\nu)})$
$\Rightarrow rng(f_{(\bar{u},\bar{\nu})}) \subseteq rng(g^{-1}f_{(u,\nu)})$
$\Rightarrow rng(gf_{(\bar{u},\bar{\nu})}) \subseteq rng(f_{(u,\nu)})$.

On the other hand, we have

$u \subseteq rng(gf_{(\bar{u},\bar{\nu})})$
$\Rightarrow rng(f_{(u,\nu)}) \subseteq rng(gf_{(\bar{u},\bar{\nu})})$.

(b) $id_\nu = f_{(u,\nu)}$ where $u = J^D_{\mu_\nu}$.

(c) Assume $f \neq id_\nu$. Then $\beta(f) \leq \alpha_\nu$ by axiom (1). But $f \upharpoonright \alpha_\nu = id \upharpoonright \alpha_\nu$ by the hypothesis and $f(\alpha_\nu) = \alpha_\nu$ by axiom (0). Contradiction!

(d) If we let $h : \bar{\nu} \Rightarrow \nu$ be the uncollapse of $\bigcup\{rng(f_{(\beta,\xi,\nu)}) \mid \beta < \alpha_\nu\}$, then $h \in \mathfrak{F}$ and $h \upharpoonright \alpha_\nu = id \upharpoonright \alpha_\nu$. So $h = id_\nu$ by (c). □

Lemma 3

Let $\bar{\nu}, \nu \in S$ and let $h : \langle J^{\bar{D}}_{\bar{\nu}}, \bar{D}\rangle \to \langle J^D_\nu, D \upharpoonright \nu\rangle$ be Σ_1-elementary such that there is some $\beta \sqsubseteq \bar{\nu}$ with $h \upharpoonright \beta = id \upharpoonright \beta$. Let $h(\mu_{\bar{\tau}}) = \mu_\tau < \nu$ and $\tau = h(\bar{\tau}) \in S - RCard^{L_\kappa[D]}$. Then $h^{(\bar{\tau})} : \bar{\tau} \Rightarrow \tau$.

Proof: Let $\delta_{\bar{\tau}} \sqsubseteq \bar{\tau}$ and $\delta_{\bar{\nu}} \sqsubseteq \bar{\nu}$ be minimal. If $\delta_{\bar{\tau}} \not\sqsubseteq \bar{\nu}$, then $\mu_{\bar{\tau}} < \delta_{\bar{\nu}}$. To see this, we consider the three cases $\delta_{\bar{\tau}} = \delta_{\bar{\nu}}$, $\delta_{\bar{\tau}} > \delta_{\bar{\nu}}$ and $\delta_{\bar{\tau}} < \delta_{\bar{\nu}}$. The first case is impossible because if $\delta_{\bar{\nu}} = \delta_{\bar{\tau}}$, then $\delta_{\bar{\tau}} \sqsubseteq \bar{\nu}$ by definition of $\delta_{\bar{\nu}}$. The second case is impossible because then by axiom (c) $\mu_{\delta_{\bar{\nu}}} < \delta_{\bar{\tau}}$. But $\delta_{\bar{\tau}} \leq \bar{\tau}$ by definition of $\delta_{\bar{\tau}}$ and $\mu_{\bar{\nu}} \leq \mu_{\delta_{\bar{\nu}}}$ by definition of $\delta_{\bar{\nu}}$ and $\mu_\cdot$. Hence $\mu_{\bar{\nu}} \leq \mu_{\delta_{\bar{\nu}}} < \delta_{\bar{\tau}} \leq \bar{\tau}$ which contradicts the assumption $\mu_{\bar{\tau}} < \bar{\nu}$. Hence $\delta_{\bar{\tau}} < \delta_{\bar{\nu}}$ must hold. But then $\mu_{\delta_{\bar{\tau}}} < \delta_{\bar{\nu}}$ by axiom (c) and therefore $\mu_{\bar{\tau}} \leq \mu_{\delta_{\bar{\tau}}} < \delta_{\bar{\nu}}$ as claimed, by definition of $\delta_{\bar{\tau}}$ and $\mu_\cdot$. So by assumption $h^{(\bar{\tau})} = id_{\bar{\tau}}$ and $id_{\bar{\tau}} \in \mathfrak{F}$ by lemma 2 (b).

Now, let $\bar{\delta} := \delta_{\bar{\tau}} \sqsubseteq \bar{\nu}$ and $f_{(\bar{\delta},x,\bar{\tau})} : \bar{\tau}(x) \Rightarrow \bar{\tau}$. Let $\bar{\delta} \sqsubset \bar{\gamma}(x) \sqsubseteq \bar{\tau}(x)$ where $\alpha_{\bar{\gamma}(x)} = \bar{\delta}$. Then, by (DP2), $f_{(0,x,\bar{\tau}(x))} = f_{(0,x,\bar{\gamma}(x))}$ for all $x \in J^D_{\mu_{\bar{\tau}}}$. And we get $\mu_{\bar{\gamma}(x)} \leq \mu_{\bar{\tau}} < \bar{\nu} \leq \mu_{\bar{\delta}}$. So, by (DP1), $\bar{\gamma}(x)$ is independent. That is, $d(f_{(\beta,0,\bar{\gamma}(x))}) < \alpha_{\bar{\gamma}(x)}$ for all $\beta < \alpha_{\bar{\gamma}(x)}$. Since $J^D_{\mu_{\bar{\gamma}(x)}} = \bigcup\{rng(f_{(\beta,0,\bar{\gamma}(x))}) \mid \beta < \alpha_{\bar{\gamma}(x)}\}$, $x \in rng(f_{(\beta,0,\bar{\gamma}(x))})$ for some $\beta < \alpha_{\bar{\gamma}(x)}$. Hence $d(f_{(0,x,\bar{\gamma}(x))}) < \bar{\delta}$. Altogether, we get

$$d(f_{(0,x,\bar{\tau})}) = d(f_{(\bar{\delta},x,\bar{\tau})} \circ f_{(0,x,\bar{\tau}(x))}) = d(f_{(0,x,\bar{\tau}(x))}) = d(f_{(0,x,\bar{\gamma}(x))}) < \bar{\delta}.$$

By our assuption $h \upharpoonright \bar{\delta} = id \upharpoonright \bar{\delta}$. And by our induction hypothesis, $(\times)$ holds for μ_τ. So by the Σ_1-elementarity of $h : \langle J^D_{\bar{\nu}}, \bar{D}\rangle \to \langle J^D_\nu, D \upharpoonright \nu\rangle$, if $x \in rng(h)$, then even $rng(f_{(0,x,\tau)}) \subseteq rng(h)$. Thus

$$rng(h) \cap J^D_{\mu_\tau} = \bigcup\{rng(f_{(0,x,\tau)}) \mid x \in rng(h) \cap J^D_{\mu_\tau}\}.$$

Therefore,

$$h^{(\bar{\tau})} = f_{(u,\tau)} \in \mathfrak{F} \text{ where } u = rng(h) \cap J^D_{\mu_\tau}.$$ □

Lemma 4

For all $\nu \in S - RCard^{L_\kappa[D]}$, q_ν exists.

Proof: Suppose $q_\nu(k+1) = max(\Lambda(q_\nu \upharpoonright (k+1), \nu))$ exists. Then $q_\nu(k+1) \in \Lambda(q_\nu \upharpoonright (k+1), \nu)$ and there is some β such that $\lambda(f_{(\beta, q_\nu \upharpoonright (k+1), \nu)}) = q_\nu(k+1)$. The set of such β is closed by (CP1). Thus there is a largest such β. Call it β_k. The recursion breaks off if the sequence $\langle \beta_k \mid k\rangle$ is strictly descending, since there is no descending sequence of length ω. But $\beta_k \in rng(f_{(\beta_k, q_\nu \upharpoonright (k+2), \nu)})$ by $(\times)$ and (LP2). Hence $\lambda(f_{(\beta_k, q_\nu \upharpoonright (k+2), \nu)}) = \nu$ by the definition of β_k. Therefore,

$\beta_{k+1} < \beta_k$. $\square$

Lemma 5

Let $f : \bar{\nu} \Rightarrow \nu$, $x \in rng(f)$ and $\lambda = \lambda(f)$. Then $\Lambda(x, \nu) \cap \lambda = \Lambda(x, \lambda)$.

Proof: Let $f(\bar{x}) = x$. Then on the one hand, $(f \restriction J^D_{\bar{\nu}}) : \langle J^D_{\bar{\nu}}, D \restriction \bar{\nu}, \Lambda(\bar{x}, \bar{\nu})\rangle \to \langle J^D_\nu, D \restriction \nu, \Lambda(x, \nu)\rangle$ is Σ_0-elementary by (LP2). But then

$(*)$ $(f \restriction J^D_{\bar{\nu}}) : \langle J^D_{\bar{\nu}}, D \restriction \bar{\nu}, \Lambda(\bar{x}, \bar{\nu})\rangle \to \langle J^D_\lambda, D \restriction \lambda, \Lambda(x, \nu) \cap \lambda\rangle$ is also Σ_0-elementary.

On the other hand, by (CP2) and (LP2),

$(**)$ $(f \restriction J^D_{\bar{\nu}}) : \langle J^D_{\bar{\nu}}, D \restriction \bar{\nu}, \Lambda(\bar{x}, \bar{\nu})\rangle \to \langle J^D_\lambda, D \restriction \lambda, \Lambda(x, \lambda)\rangle$ is also Σ_0-elementary.

Consider the following three cases:

(1) $\Lambda(\bar{x}, \bar{\nu}) = \emptyset$

Then, by $(*)$, $\Lambda(x, \nu) \cap \lambda = \emptyset$ and, by $(**)$, $\Lambda(x, \lambda) = \emptyset$.

(2) $\bar{\eta} := max(\Lambda(\bar{x}, \bar{\nu}))$ exists

Let $f(\bar{\eta}) = \eta$. Then, by $(*)$ and $(**)$,

$\eta = max(\Lambda(x, \nu) \cap \lambda) = max(\Lambda(x, \lambda))$.

And by (CP2), we have

$z \in \Lambda(\bar{x}, \bar{\nu}) \Leftrightarrow z \in \Lambda(\bar{x}, \bar{\eta}) \cup \{\bar{\eta}\}$.

But then, by $(*)$,

$z \in \Lambda(x, \nu) \cap \lambda \Leftrightarrow z \in \Lambda(x, \eta) \cup \{\eta\}$.

and, because of $(**)$,

$z \in \Lambda(x, \lambda) \Leftrightarrow z \in \Lambda(x, \eta) \cup \{\eta\}$.

That's it!

(3) $\Lambda(\bar{x}, \bar{\nu})$ is unbounded in $\bar{\nu}$

Then, by $(*)$, $\Lambda(x, \nu) \cap \lambda$ is unbounded in λ. Hence $\lambda \in \Lambda(x, \nu)$ because $\Lambda(x, \nu)$ is closed. Therefore $\Lambda(x, \lambda) = \Lambda(x, \nu) \cap \lambda$ by (CP2). $\square$

Lemma 6

Let $f : \bar{\nu} \Rightarrow \nu$.

(a) If $q_\nu \restriction k \in rng(f)$, then $f(q_{\bar{\nu}} \restriction k) = q_\nu \restriction k$.

(b) If f is cofinal, then $f(q_{\bar{\nu}}) = q_\nu$.

Proof:

(a) That is proved by induction on k using (LP2) to show $f(max(\Lambda(\bar{x}, \bar{\nu})) = max(\Lambda(x, \nu))$ whenever $max(\Lambda(x, \nu)) \in rng(f)$.

(b) Like (a). Since f is cofinal, $q_\nu \restriction (k+1)$ lies always in $rng(f)$. $\square$

Lemma 7

$\lambda \in C_\nu$ implies $\lambda \in \Lambda(q_\lambda, \nu)$.

Proof: Since $\lambda \in C_\nu$, $q_\lambda \in rng(f)$ for some $f : \bar{\nu} \Rightarrow \nu$ by lemma 6 (b). So $\Lambda(q_\lambda, \nu) \cap \lambda = \Lambda(q_\lambda, \lambda)$ by lemma 5. Therefore, by the definition of q_λ, $max(\Lambda(q_\lambda, \nu) \cap \lambda)$ does not exist. But if $\Lambda(q_\lambda, \nu) \cap \lambda$ is unbounded in λ, then $\lambda \in \Lambda(q_\lambda, \nu)$ by the closedness of $\Lambda(q_\lambda, \nu)$. So let $\Lambda(q_\lambda, \nu) \cap \lambda = \emptyset$. But then $\lambda = \lambda(f_{(0,q_\lambda,\nu)})$. For $\lambda(f_{(0,q_\lambda,\nu)}) \geq \lambda$, since otherwise $\Lambda(q_\lambda, \nu) \cap \lambda \neq \emptyset$. And

$\lambda(f_{(0,q_\lambda,\nu)}) \leq \lambda$, because $\lambda \in C_\nu$. Thus $q_\lambda \in rng(f)$ for some $f : \bar{\nu} \Rightarrow \nu$ by lemma 6 (b). But then $rng(f_{(0,q_\lambda,\nu)}) \subseteq rng(f)$. $\Box$

Lemma 8

Let $\rho \in C_\nu \cap \lambda$ such that $\rho > q_\lambda$. Then q_λ is an initial segment of q_ρ.

Proof:

$$q_\rho(k) = max(\Lambda(q_\rho \upharpoonright k, \rho)) = max(\Lambda(q_\rho \upharpoonright k, \nu) \cap \rho),$$

as long as these maxima exist, because $\rho \in C_\nu$. Hence $q_\rho \upharpoonright k \in rng(f)$ for some $f : \bar{\nu} \Rightarrow \nu$ by lemma 6 (b). So $\Lambda(q_\rho \upharpoonright k, \nu) \cap \rho = \Lambda(q_\rho \upharpoonright k, \rho)$ by lemma 5. Analogously

$$q_\lambda(k) = max(\Lambda(q_\lambda \upharpoonright k, \lambda)) = max(\Lambda(q_\lambda \upharpoonright k, \nu) \cap \lambda) = max(\Lambda(q_\lambda \upharpoonright k, \nu) \cap \rho),$$

as long as these maxima exist, because $q_\lambda < \rho < \lambda$. The lemma follows from these two equations by induction. $\Box$

Lemma 9

C_ν is closed in ν.

Proof: Let $\lambda \in Lim(C_\nu)$. Consider the sequence $\langle q_\rho \mid \rho \in C_\nu \cap \lambda \rangle$. By lemma 8, there is some $\rho_0 \in C_\nu \cap \lambda$ such that $q_\rho = q_{\rho_0}$ for all $\rho_0 < \rho \in C_\nu \cap \lambda$. Therefore, by lemma 7, $\rho \in \Lambda(q_{\rho_0}, \nu)$ for all $\rho_0 < \rho \in C_\nu \cap \lambda$. But $\Lambda(q_{\rho_0}, \nu)$ is closed. Hence $\lambda \in \Lambda(q_{\rho_0}, \nu) \subseteq C_\nu$. $\Box$

Lemma 10

$\lambda \in C_\nu \Rightarrow C_\lambda = C_\nu \cap \lambda$.

Proof by induction on λ and ν. Suppose the lemma to be proved already for all $\rho < \lambda$ and $\mu \leq \nu$. By lemma 7, $\Lambda(q_\lambda, \lambda) = \Lambda(q_\lambda, \nu) \cap \lambda$. Therefore $\rho \in C_\nu \cap C_\lambda$ for all $\rho \in \Lambda(q_\lambda, \lambda)$. Hence $C_\lambda \cap \rho = C_\nu \cap \rho = C_\rho$ by the induction hypothesis. If $\Lambda(q_\lambda, \lambda)$ is unbounded in λ, we are finished. If $\Lambda(q_\lambda, \lambda) = \emptyset$, then $(C_\nu \cap \lambda) - (q_\lambda(k_\lambda) + 1) = \emptyset$ by lemma 8. To see this, assume $(C_\nu \cap \lambda) - (q_\lambda(k_\lambda) + 1) \neq \emptyset$. Let $\rho = min(C_\nu - (q_\lambda(k_\lambda) + 1))$. Then $q_\rho = q_\lambda$ by lemma 8. Hence $\rho \in \Lambda(q_\lambda, \lambda)$. Contradiction! Therefore $(C_\nu \cap \lambda) - (q_\lambda(k_\lambda) + 1) = C_\lambda - (q_\lambda(k_\lambda) + 1) = \emptyset$. If $q_\lambda(k_\lambda) = 0$, then we are finished. If $q_\lambda(k_\lambda) \neq 0$, then $q_\lambda(k_\lambda) = max(C_\lambda) = max(C_\nu \cap \lambda)$. But $C_\lambda \cap q_\lambda(k_\lambda) = C_\nu \cap q_\lambda(k_\lambda) = C_{q_\lambda(k_\lambda)}$. Hence $C_\lambda = C_\nu \cap \lambda$. $\Box$

Lemma 11

Let $f : \bar{\nu} \Rightarrow \nu$. Then $(f \upharpoonright J^D_{\bar{\nu}}) : \langle J^D_{\bar{\nu}}, D \upharpoonright \bar{\nu}, C_{\bar{\nu}} \rangle \rightarrow \langle J^D_\nu, D \upharpoonright \nu, C_\nu \rangle$ is Σ_0-elementary.

Proof: Show $f(C_{\bar{\nu}} \cap \bar{\eta}) = C_\nu \cap f(\bar{\eta})$ for all $\bar{\eta} < \bar{\nu}$. By (LP1), we have $f(C_{\bar{\nu}} \cap \bar{\lambda}) = f(C_{\bar{\lambda}}) = C_\lambda = C_\nu \cap f(\lambda)$ for all $\bar{\lambda} \in C_{\bar{\nu}}$. Therefore, if $C_{\bar{\nu}}$ is cofinal in $\bar{\nu}$, we are finished. If it is not, then $f(q_{\bar{\nu}}) = q_\nu$. If $q_{\bar{\nu}}(k_{\bar{\nu}}) = 0$, then $\Lambda(0, \bar{\nu}) = \Lambda(0, \nu) = \emptyset$, implying that $C_{\bar{\nu}} = C_\nu = \emptyset$. If $q_{\bar{\nu}}(k_{\bar{\nu}}) \neq 0$, then we use $f(max(C_{\bar{\nu}})) = max(C_\nu)$. But $max(C_{\bar{\nu}}) = q_{\bar{\nu}}(k_{\bar{\nu}})$ and $max(C_\nu) = q_\nu(k_\nu)$. $\Box$

Lemma 12

Set $\alpha_{\tau(0,\nu)} = \mu_\nu$ and $x(0,\nu) = \emptyset$ for all ν. Then the following holds for all $0 \leq n$ and $\nu \in \widehat{S}$:

(i) If $f : \bar{\nu} \Rightarrow_{n+1} \nu$, $\alpha := \alpha_{\tau(n,\nu)}$ and $\bar{\alpha} := f^{-1}[\alpha \cap rng(f)]$, then $\bar{\alpha} = \alpha_{\tau(n,\bar{\nu})}$.

(ii) If $f : \bar{\nu} \Rightarrow_{n+1} \nu$, then $f(x(n, \bar{\nu})) = x(n, \nu)$.
(iii) If $f : \bar{\nu} \Rightarrow_{n+1} \nu$ and $\bar{K} = f^{-1}[K^n_\nu \cap rng(f)]$, then $\bar{K} = K^n_{\bar{\nu}}$.
(iv) If $f, g \Rightarrow_{n+1} \nu$ and $rng(f) \subseteq rng(g)$, then $g^{-1}f \Rightarrow_{n+1} d(g)$.
(v) For all $u \subseteq J^D_{\mu_\nu}$, there is $f^{n+1}_{(u,\nu)}$.
(vi) For all $\beta < \nu$ and $x \in J^D_{\mu_\nu}$, $f^{n+1}_{(\beta,x,\nu)}$ is uniformly definable over $\langle J^D_\nu, D \upharpoonright \nu, D_\nu \rangle$.
Proof by induction on n. For $n = 0$, (i) to (v) hold by the morass axioms.
(vi) By (DF), the $rng(f^1_{(0,x,\nu)})$ are uniformly definable over $\langle J^D_\nu, D \upharpoonright \nu, D_\nu \rangle$. Like in the proof of lemma 1, $rng(f^1_{(\beta,z_0,\nu)}) = \bigcup \{rng(f^1_{(0,z,\nu)}) \mid z \in (\beta \cup \{z_0\})^{<\omega}\}$. And $f^1_{(\beta,z_0,\nu)}(x) = y$ may be defined by: There is some $\bar{\nu}$ and some $\bar{z}_0$ such that, for all $z_1 \in \beta^{<\omega}$,

$$d(f_{(0,\langle z_1,\bar{z}_0\rangle,\bar{\nu})}) = d(f_{(0,\langle z_1,z_0\rangle,\nu)})$$

and, for all $t \in J^D_{\bar{\nu}}$, there is some $z_1 \in \beta^{<\omega}$ such that

$$t \in rng(f_{(0,\langle z_1,\bar{z}_0\rangle,\bar{\nu})})$$

and there is some z and some $z_1 \in \beta^{<\omega}$ such that

$$f_{(0,\langle z_1,\bar{z}_0\rangle,\bar{\nu})}(z) = x \Leftrightarrow f_{(0,\langle z_1,z_0\rangle,\nu)}(z) = y.$$

Now, assume that (i) to (vi) are proved already for all $0 \leq m < n$.
(i) Let $B^n(x,\nu) := \{\beta(f^n_{(\gamma,x,\nu)}) < \alpha_{\tau(n,\nu)} \mid \gamma < \nu\} = \{\beta < \alpha_{\tau(n,\nu)} \mid \beta \notin rng(f^n_{(\beta,x,\nu)})\}$. Let $f(\bar{x}) = x(n,\nu)$, $B = B^n(x,\nu)$ and $\bar{B} := f^{-1}[B \cap rng(f)]$. Then $f \circ f^n_{(\bar{u},\bar{\nu})} = f^n_{(u,\nu)}$ for all $\bar{u} \subseteq J^D_{\mu_{\bar{\nu}}}$ and $u = f[\bar{u}]$ by (iv) of the induction hypothesis (cf. lemma 2b). Therefore, if $f(\bar{\beta}) = \beta \in rng(f_{(\beta,x(n,\nu),\nu)})$, then $\bar{\beta} \in rng(f_{(\bar{\beta},\bar{x},\bar{\nu})})$, because $\bar{\beta} = f^{-1}[\beta \cap rng(f)]$. And if $f(\bar{\beta}) = \beta \notin rng(f_{(\beta,x(n,\nu),\nu)})$, then $\bar{\beta} \notin rng(f_{(\bar{\beta},\bar{x},\bar{\nu})})$. So, altogether, $\bar{B} = B^n(\bar{x},\bar{\nu})$. By (DP3)(b) and (iv) of the induction hypothesis, $B^n(x(n,\nu),\nu) = \bigcup\{B^n(x(n,\eta),\eta) \mid \eta \in K^n_\nu\}$. But $B^n(x(n,\nu),\nu)$ is unbounded in α and $rng(f) \cap J^D_\alpha \prec_1 \langle J^D_\alpha, D \upharpoonright \alpha, K^n_\nu \rangle$. Thus $\bar{B} = B^n(\bar{x},\bar{\nu})$ is also unbounded in $\bar{\alpha}$. Assume there were some $z \in J^D_{\mu_{\bar{\nu}}}$ and some $\beta < \bar{\alpha}$ such that $f^n_{(\beta,z,\bar{\nu})} = id_{\bar{\nu}}$. Then there was some $\beta \leq \gamma < \bar{\alpha}$ such that $z \in rng(f^n_{(\gamma,\bar{x},\bar{\nu})})$. For, by (iv) of the induction hypothesis, $f^n_{(\bar{\alpha},\bar{x},\bar{\nu})} = id_{\bar{\nu}}$. So $f^n_{(\gamma,\bar{x},\bar{\nu})} = id_{\bar{\nu}}$. But this contradicts the fact that $B^n(\bar{x},\bar{\nu})$ is unbounded in $\bar{\alpha}$.
(ii) By the proof of (i), $f^n_{(\bar{\alpha},\bar{x},\bar{\nu})} = id_{\bar{\nu}}$ is satisfied for $\bar{\alpha} = \alpha_{\tau(n,\bar{\nu})}$ and $f(\bar{x}) = x(n,\bar{\nu})$. Therefore $x(n,\bar{\nu}) \leq \bar{x}$. Assume $x(n,\bar{\nu}) < \bar{x}$. Then $x(n,\nu) \in rng(f^n_{(\alpha,x,\nu)})$ where $x := f(x(n,\bar{\nu}))$ and $\alpha := \alpha_{\tau(n,\nu)}$. Thus $f^n_{(\alpha,x,\nu)} = id_\nu$ for all $x < x(n,\nu)$. But that contradicts the definition of $x(n,\nu)$.
(iii) Let $f(\bar{\mu}) = \mu$, $K^+ = K^n_\nu - Lim(K^n_\nu)$ and $\bar{K}^+ = K^n_{\bar{\nu}} - Lim(K^n_{\bar{\nu}})$. First, prove $\mu \in K^+ \Rightarrow \bar{\mu} \in \bar{K}^+$. By (i) and (ii), we know that $\bar{B} = f^{-1}[B \cap rng(f)]$ where $B = B^n(x,\nu)$, $\bar{B} = B^n(\bar{x},\bar{\nu})$ and $x = x(n,\nu)$, $\bar{x} = x(n,\bar{\nu})$. Let $\mu = d(f^n_{(\beta,x,\nu)})$. Since $\mu \in K^+ \cap rng(f)$, we may assume $\beta \in B^+ \cap rng(f)$ where $B^+ = B - Lim(B)$. Let δ be the predecessor of β in B. Then $f^n_{(\delta,\langle\delta,x\rangle,\mu)} = id_\mu$. Define $\gamma = \beta$ if $\beta \in S^+ \cup S^0$, and $\gamma = min\{\gamma \sqsubset \beta \mid \delta < \gamma\}$ else. Then $\gamma \in rng(f)$ and $\mu = \mu_\gamma$ by (DP3). Let $f(\bar{\beta}) = \beta$, $f(\bar{\gamma}) = \gamma$. By (iv) of the induction hypothesis, $\bar{\mu} = \mu_{\bar{\gamma}} = d(f^n_{(\bar{\beta},\bar{x},\bar{\nu})}) \in \bar{K}^+$. In the same way, we show

$\bar{\mu} \in \bar{K}^+ \Rightarrow \mu \in K^+$. But $K^n_\nu = \bigcup\{K^n_\eta \mid \eta \in K^+\}$ and $K^n_\nu = \bigcup\{K^n_\eta \mid \eta \in \bar{K}^+\}$. Thus the claim holds.

(iv) follows immediately from (ii), (iii) and the definition of $\Rightarrow_{n+1}$.

(v) First, we notice that $\langle J^D_\alpha, D \restriction \alpha, K^n_\nu\rangle$ where $\alpha := \alpha_{\tau(n,\nu)}$ is rudimentary closed. Then $K^n_\nu \cap \eta = K^n_\eta$ for all $\eta \in K^n_\eta$ by (iv). But, by (vi) of the induction hypothesis, K^n_η is uniformly definable over $\langle J^D_\eta, D \restriction \eta, D_\eta\rangle$. Since $\langle J^D_\alpha, D \restriction \alpha, K^n_\nu\rangle$ is rudimentary closed, by the definition of $\Rightarrow_{n+1}$,

$$f^{n+1}_{(u,\nu)} = f^n_{(w \cup u \cup \{x(n,\nu)\},\nu)}$$

where $w := h[\omega \times (u \cap J^D_\alpha)^{<\omega}]$.

Here, h denotes the canonical Σ_1-Skolem function of $\langle J^D_\alpha, D \restriction \alpha, K^n_\nu\rangle$.

(vi) If $w \prec_1 \langle J^D_{\alpha_{\tau(n,\nu)}}, D \restriction \alpha_{\tau(n,\nu)}, K^n_\nu\rangle$, then there is a uniquely determined $f \Rightarrow_{n+1} \nu$ such that $rng(f) \cap J^D_{\alpha_{\tau(n,\nu)}} = w$.

Existence :

Let $\alpha := \alpha_{\tau(n,\nu)}$ and

$$f_\beta = f^n_{(\beta,x(n,\nu),\nu)}$$

$$\nu(\beta) = d(f_\beta)$$

$$H = \bigcup\{f_\beta[w \cap J^D_{\nu(\beta)}] \mid \beta < \alpha\}.$$

Then $H \cap J^D_\alpha = w$. For $w \subseteq H \cap J^D_\alpha$ is clear, since $f_\beta \restriction J^D_\beta = id \restriction J^D_\beta$. So let $y \in H \cap J^D_\alpha$. Thus $y = f_\beta(x)$ for some $x \in w$ and some $\beta < \alpha$. Let $K^+ = K^n_\nu - Lim(K^n_\nu)$ and $\beta(\eta) = sup\{\beta \mid f^n_{(\beta,x(n,\eta),\eta)} \neq id_\eta\}$. Then

$$\langle J^D_\alpha, D \restriction \alpha, K^n_\nu\rangle \models (\exists y)(\exists \eta \in K^+)(y = f^{m+1}_{(\beta,x(m+1,\eta),\eta)}(x) \in J^D_{\beta(\eta)}).$$

Since $w \prec_1 \langle J^D_\alpha, D \restriction \alpha, K^n_\nu\rangle$, $y = f^n_{(\beta,x(n,\eta),\eta)}(x) \in w$ for all such η and $x \in w$. But since $y = f^n_{(\beta,x(n,\eta),\eta)}(x) \in J^D_{\beta(\eta)}$, we get $f_\beta(x) = f^n_{(\beta,x(n,\eta),\eta)}(x) \in w$.

Let $|f| : J^D_{\bar{\nu}} \to J^D_\nu$ be the uncollapse of H and $f = \langle\bar{\nu}, |f|, \nu\rangle$. Then $f : \bar{\nu} \Rightarrow_{n+1} \nu$. For, for all $\beta < \alpha$ by (DF), $f^{(\bar{\nu}(\beta))} : \bar{\nu}(\beta) \Rightarrow_n \nu(\beta)$ where $f(\bar{\nu}(\beta)) = \nu(\beta)$ if $\nu(\beta) \in rng(f)$. Let $\Gamma = \{\beta < \alpha \mid \nu(\beta) \in rng(f)\}$. For $\beta, \gamma \in \Gamma$, let $g_\beta = f_\beta \circ f^{(\bar{\nu}(\beta))}$ and $g_{\beta\gamma} = g^{-1}_\gamma \circ g_\beta$. Let $\langle h_\beta \mid \beta \in \Gamma\rangle$ be the transitive, direct limit of the directed system $\langle g_{\beta\gamma} \mid \beta \leq \gamma \in \Gamma\rangle$. Then $f \circ h_\beta = g_\beta$ for all $\beta \in \Gamma$. Thus, by (CP1) and (iv) of the induction hypothesis, $f : \bar{\nu} \Rightarrow_n \nu$. But $x(n+1,\nu) \in H = rng(f)$ and $rng(f) \cap J^D_\alpha = w \prec_1 \langle J^D_\alpha, D \restriction \alpha, K^n_\nu\rangle$. Thus $f : \bar{\nu} \Rightarrow_{n+1} \nu$.

Uniqueness:

Let $f : \bar{\nu} \Rightarrow_{n+1} \nu$ such that $rng(f) \cap J^D_{\alpha_{\tau(n,\nu)}} = w$ and $\bar{\alpha} := f^{-1}[\alpha \cap rng(f)]$. Then $\bar{\alpha} = \alpha_{\tau(n,\bar{\nu})}$ by (i). And $f \circ f^{n+1}_{(\bar{\alpha},\bar{\nu})} = f^{n+1}_{(w,\nu)}$ by (iv) (cf. lemma 2a). But $f^{n+1}_{(\bar{\alpha},\bar{\nu})} = id_{\bar{\nu}}$, since $\bar{\alpha} = \alpha_{\tau(n,\bar{\nu})}$. Therefore, $f = f^{n+1}_{(w,\nu)}$ is uniquely determined.

Let $f^n_{(0,\langle x(n,\nu),z_0\rangle,\nu)}(z^*_0) = z_0$. Use $w = h_{(n,\nu)}[\omega \times (\beta^{<\omega} \times \{z_0\})]$ where $h_{(n,\nu)}$ is the canonical Σ_1-Skolem function of $\langle J^D_{\alpha_{\tau(n,\nu)}}, D \restriction \alpha_{\tau(n,\nu)}, K^n_\nu\rangle$. By (vi) of the induction hypothesis, K^n_ν is uniformly definable over $\langle J^D_\nu, D \restriction \nu, D_\nu\rangle$. Therefore, w is uniformly definable over $\langle J^D_\nu, D \restriction \nu, D_\nu\rangle$. Let π be the uncollapse of w.

Then we can define $\pi(x) = y$ by: There is some $\bar{\nu} \leq \nu$ and some $\bar{z}_0 \leq z_0^*$ such that, for all $i \in \omega$ and $z_1 \in \beta^{<\omega}$,

$$(\exists z \in J^D_{\alpha_{\tau(n,\bar{\nu})}})(z = h_{(n,\bar{\nu})}(i, \langle z_1, \bar{z}_0 \rangle)) \Leftrightarrow (\exists z \in J^X_{\alpha_{\tau(n,\nu)}})(z = h_{(n,\nu)}(i, \langle z_1, z_0^* \rangle))$$

and, for all $z \in J^X_{\alpha_{\tau(n,\bar{\nu})}}$, there is some $i \in \omega$ and some $z_1 \in \beta^{<\omega}$ such that

$$z = h_{(n,\bar{\nu})}(i, \langle z_1, \bar{z}_0 \rangle)$$

and there is some $i \in \omega$ and some $z_1 \in \beta^{<\omega}$ such that

$$h_{(n,\bar{\nu})}(i, \langle z_1, \bar{z}_0 \rangle) = x \Leftrightarrow h_{(n,\nu)}(i, \langle z_1, z_0^* \rangle) = y.$$

By this, $\bar{\nu}$ is uniquely determined. By what was shown above, one can define $f^{n+1}_{(\beta,z_0,\nu)}(x) = f^n_{(w,\nu)}(x) = y$ by: For all $z_0 \in \alpha^{<\omega}_{\tau(n,\bar{\nu})}$,

$$d(f^n_{(0,\langle z_0, x(n,\bar{\nu})\rangle,\bar{\nu})}) = d(f^n_{(0,\langle \pi(z_0), x(n,\nu)\rangle,\nu)})$$

and, for all $t \in J^D_{\bar{\nu}}$, there is some $z_0 \in \alpha^{<\omega}_{\tau(n,\bar{\nu})}$ such that

$$t \in rng(f^n_{(0,\langle z_0, x(n,\bar{\nu})\rangle,\bar{\nu})})$$

and there is some z and some $z_0 \in \alpha^{<\omega}_{\tau(n,\bar{\nu})}$ such that

$$f^n_{(0,\langle z_0, x(n,\bar{\nu})\rangle,\bar{\nu})}(z) = x \Leftrightarrow f^n_{(0,\langle \pi(z_0), x(n,\nu)\rangle,\nu)}(\pi(z)) = y.$$

□

Now, it is an immediate consequence of lemma 12 and (DF) that (×) holds for all ν such that $\mu_\nu = \mu$.

3 The inner model $L[X]$

Of course my definition of (ω_1, β)-morass makes also sense if $\beta < \omega_1$. Hence a natural question is:

Is the existence of an (ω_1, β)-morass in this new sense equivalent to the existence of an (ω_1, β)-morass in Jensen's sense?

In asking this question one has to be careful what an (ω_1, β)-morass in Jensen's sense is, because there are also different definitions. But for the case $\beta = 1$, I expect an equivalence between all existing definitions.

In the following, I will define a strengthening of the notion of a Jensen (ω_1, β)-morass which I also expect to be equivalent to my notion of (ω_1, β)-morass. If we construct a morass in the usual way in L, the properties of this stronger notion hold automatically (see the paper [Irr2] or my dissertation [Irr1]).

A structure $\mathfrak{M} = \langle S, \lhd, \mathfrak{F}, D \rangle$ is called an $\omega_{1+\beta}$-standard morass if it satisfies all axioms of an (ω_1, β)-morass except **(DF)** which is replaced by:

$\nu \lhd \tau \Rightarrow \nu$ is regular in J^D_τ

and there are functions $\sigma_{(x,\nu)}$ for $\nu \in \widehat{S}$ and $x \in J^D_\nu$ such that:

(MP)$^+$

$\sigma_{(x,\nu)}[\omega] = rng(f_{(0,x,\nu)})$

(CP1)$^+$

If $f : \bar{\nu} \Rightarrow \nu$ and $f(\bar{x}) = x$, then $\sigma_{(x,\nu)} = f \circ \sigma_{(\bar{x},\bar{\nu})}$.

(CP3)$^+$

If C_ν is unbounded in ν, then $\sigma_{(x,\nu)} = \bigcup\{\sigma_{(x,\lambda)} \mid \lambda \in C_\nu, x \in J^D_\lambda\}$.

(DF)$^+$

(a) If $f_{(0,x,\nu)} = id_\nu$ for some $x \in J^D_\nu$, then

$$\{\langle i, z, \sigma_{(z,\nu)}(i)\rangle \mid z \in J^D_\nu, i \in dom(\sigma_{(z,\nu)})\}$$

is uniformly definable over $\langle J^D_{\mu_\nu}, D \upharpoonright \mu_\nu, D_{\mu_\nu}\rangle$.

(b) If C_ν is unbounded in ν, then $D_\nu = C_\nu$. If it is bounded, then $D_\nu = \{\langle i, \sigma_{(q_\nu,\nu)}(i)\rangle \mid i \in dom(\sigma_{(q_\nu,\nu)})\}$.

Lemma 13

(DF) and $(\times)$ also hold in a standard morass.

Proof: First, we prove by induction on $\mu \in \widehat{S}$ that the set

$$\{\langle i, x, \sigma_{(x,\mu)}(i)\rangle \mid x \in J^D_\mu, i \in dom(\sigma_{(x,\mu)})\}$$

is uniformly definable over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$ for all $\mu \in \widehat{S}$. Assume that this has been proved already for all $\tau < \mu$, $\tau \in \widehat{S}$.

If C_μ is unbounded in μ, then, by (CP3)$^+$,

$$\sigma_{(x,\nu)} = \bigcup\{\sigma_{(x,\lambda)} \mid \lambda \in C_\nu, x \in J^D_\lambda\}.$$

But, by the induction hypothesis, the $\sigma_{(x,\lambda)}$, $\lambda \in C_\mu$, are uniformly definable over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$. And, by (DF)$^+$(b), $C_\mu = D_\mu$. Thus $\sigma_{(x,\nu)}$ is uniformly definable over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$.

If C_μ is bounded in μ, then $rng(f_{(0,q_\mu,\mu)})$ is unbounded in μ. Therefore, by (CP2),

$$rng(f_{(0,\langle z_0,q_\mu\rangle,\mu)}) = h_\mu[\omega \times (rng(f_{(0,q_\mu,\mu)}) \times \{z_0\})].$$

Here, h_μ is the Σ_1-Skolem function of $\langle J^D_\mu, D \upharpoonright \mu\rangle$. Since $D_\mu = rng(f_{(0,q_\mu,\mu)})$, the $rng(f_{(0,\langle z_0,q_\mu\rangle,\mu)})$ are uniformly definable over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$. Since, by (CP1)$^+$ and lemma 6 (b), for $\bar{\mu} := d(f_{(0,\langle z_0,q_\mu\rangle,\mu)})$,

$$f_{(0,\langle z_0,q_\mu\rangle,\mu)} \circ \sigma_{(q_{\bar{\mu}},\bar{\mu})} = \sigma_{(q_\mu,\mu)}$$

holds, we can define $f_{(0,\langle z_0,q_\mu\rangle,\mu)}(x) = y$ by: There is some $\bar{\mu} \leq \mu$ and some $\bar{z}_0 \leq z_0$ such that, for all $i, j \in \omega$,

$$(\exists z \in J^D_{\bar{\mu}})(z = h_{\bar{\mu}}(i, \langle \sigma_{(q_{\bar{\mu}},\bar{\mu})}(j), \bar{z}_0\rangle)) \Leftrightarrow (\exists z \in J^D_\mu)(z = h_\mu(i, \langle \sigma_{(q_\mu,\mu)}(j), z_0\rangle))$$

and, for all $z \in J^D_{\bar{\mu}}$, there is some $i \in \omega$ and some $j \in \omega$ such that

$$z = h_{\bar{\mu}}(i, \langle \sigma_{(q_{\bar{\mu}},\bar{\mu})}(j), \bar{z}_0\rangle)$$

and there is some $i \in \omega$ and some $j \in \omega$ such that

$$h_{\bar{\mu}}(i, \langle \sigma_{(q_{\bar{\mu}}, \bar{\mu})}(j), \bar{z}_0 \rangle) = x \Leftrightarrow h_\mu(i, \langle \sigma_{(q_\mu, \mu)}(j), z_0 \rangle) = y.$$

If $\alpha_{\tau(1,\mu)} = 0$, then it follows from $(\mathrm{DF})^+$ that $\{\langle i, z_0, \sigma_{(z_0,\mu)}(i)\rangle \mid z_0 \in J^D_\mu, i \in dom(\sigma_{(z_0,\mu)})\}$ is uniformly definable over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$. If $\alpha_{\tau(1,\mu)} > 0$, then, by (DP3)(b), $\bar{\mu} = d(f_{(0,\langle z_0, q_\mu\rangle,\mu)}) < \mu$. But then, by $(\mathrm{CP1})^+$, $\sigma_{(z_0,\mu)} = f_{(0,\langle z_0,q_\mu\rangle,\mu)} \circ \sigma_{(\bar{z}_0,\bar{\mu})}$, where $f_{(0,\langle z_0,q_\mu\rangle,\mu)}(\bar{z}_0) = z_0$, is definable by the induction hypothesis.

From the σs, we calculate $f_{(0,z_0,\mu)}(x) = y$ as follows: There is some $\bar{\mu} \leq \mu$ and some $\bar{z}_0 \leq z_0$ such that, for all $r, s \in \omega$,

$$\sigma_{(\bar{z}_0,\bar{\mu})}(r) \leq \sigma_{(\bar{z}_0,\bar{\mu})}(s) \Leftrightarrow \sigma_{(z_0,\mu)}(r) \leq \sigma_{(z_0,\mu)}(s)$$

and, for all $z \in J^D_{\bar{\mu}}$, there exists some $s \in \omega$ such that

$$z = \sigma_{(\bar{z}_0,\bar{\mu})}(s)$$

and there exists some $s \in \omega$ such that

$$\sigma_{(\bar{z}_0,\bar{\mu})}(s) = x \Leftrightarrow \sigma_{(z_0,\mu)}(s) = y.$$

Since the $f^1_{(0,z_0,\mu)}$ are uniformly definable over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$ and (DF) and $(\times)$ hold by the induction hypothesis for all $\tau \in \widehat{S} \cap \mu$, we can define the $f^n_{(0,z_0,\mu)}$ with $z_0 \in J^D_\mu$ uniformly over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$ like in the proof of lemma 12. Finally,

$$\{\langle z_0, \nu, x, f_{(0,z_0,\nu)}(x)\rangle \mid \nu < \mu, \mu_\nu = \mu, z_0 \in J^D_\mu, x \in dom(f_{(0,z_0,\nu)})\}$$
$$\cup\{\langle z_0, x, f_{(0,z_0,\mu)}(x)\rangle \mid z_0 \in J^D_\mu, x \in dom(f_{(0,z_0,\mu)})\}$$
$$\cup(\sqsubset \cap \mu^2)$$

may be defined over $\langle J^D_\mu, D \upharpoonright \mu, D_\mu\rangle$ using (DF). □

Let $S^X \subseteq Lim$ and $X = \langle X_\nu \mid \nu \in S^X\rangle$ be a sequence.

Let $I_\nu = \langle J^X_\nu, X \upharpoonright \nu\rangle$ for $\nu \in Lim - S^X$ and $I_\nu = \langle J^X_\nu, X \upharpoonright \nu, X_\nu\rangle$ for $\nu \in S^X$ where $X_\nu \subseteq J^X_\nu$ and

$J^X_0 = \emptyset$

$J^X_{\nu+\omega} = rud(I^X_\nu)$

$J^X_\lambda = \bigcup\{J^X_\nu \mid \nu \in \lambda\}$ for $\lambda \in Lim^2 := Lim(Lim)$.

Here, $rud(I^X_\nu)$ is the rudimentary closure of $J^X_\nu \cup \{J^X_\nu\}$ relative to $X \upharpoonright \nu$ if $\nu \in Lim - S^X$ and relative to $X \upharpoonright \nu$ and X_ν if $\nu \in S^X$.

Let $\beta(\nu)$ be the least β such that $J^X_{\beta+\omega} \models \nu$ singular.

Now, let a κ-standard morass be given. I will show that there is an $S^X \subseteq \kappa$ and a sequence X as above that the following holds:

(Amenability) The structures I_ν are amenable.

(Coherence) If $\nu \in S^X$, $H \prec_1 I_\nu$ and $\lambda = sup(H \cap On)$, then $\lambda \in S^X$ and $X_\lambda = X_\nu \cap J^X_\lambda$.

(Condensation) If $\nu \in S^X$ and $H \prec_1 I_\nu$, then there is some $\mu \in S^X$ such that $H \cong I_\mu$.
$(*)$ $Card \cap \kappa = Card^{L_\kappa[X]}$.
$(**)$ $S^X = \{\beta(\nu) \mid \nu \text{ singular in } I_\kappa\}$.

These properties are good enough to do fine structure proofs in $L_\kappa[X]$, e.g. to construct a κ-standard morass. This will be shown in a forthcoming paper [Irr2].

To define X, I will use the sets C_ν from (CP3):

If $\nu \in \widehat{S}$ and C_ν is unbounded in ν, then set

$$X_\nu = C_\nu.$$

Let $\nu \in \widehat{S}$ and C_ν be bounded in ν. Then $\Lambda(q,\nu)$ is bounded for all $q \in \nu$. Thus $\Lambda(q_\nu,\nu) = \emptyset$. So $f_{(0,q_\nu,\nu)}$ is cofinal. In this case, set

$$X_\nu = \{\sigma_{(q_\nu,\nu)}[n] \mid n \in \omega\}.$$

Let $S^X = \widehat{S}$.

Lemma 14

If $\nu \in \widehat{S}$, C_ν is unbounded in ν and $f : \langle J^{\bar{D}}_{\bar{\nu}}, \bar{D}, \bar{C}\rangle \to \langle J^D_\nu, D \upharpoonright \nu, C_\nu\rangle$ is Σ_1-elementary, then $\langle \bar{\nu}, f, \nu\rangle \in \mathfrak{F}$.

Proof: Let $z_0 \in rng(f)$, $i \in \omega$ and $y = \sigma_{(z_0,\nu)}(i)$. Then we must prove $y \in rng(f)$. Since C_ν is unbounded in ν, there is some $\lambda \in C_\nu$ such that $y = \sigma_{(z_0,\lambda)}(i)$ by (CP3)$^+$. Since, by lemma 13, the $\sigma_{(z_0,\tau)}$ are definable in $\langle J^D_\nu, D \upharpoonright \nu\rangle$ when $\tau < \nu$, we have $\langle J^D_\nu, D \upharpoonright \nu, C_\nu\rangle \models (\exists y)(\exists \lambda \in C_\nu)(y = \sigma_{(z_0,\lambda)}(i))$. Therefore, also $rng(f) \models (\exists y)(\exists \lambda \in C_\nu)(y = \sigma_{(z_0,\lambda)}(i))$. Thus $y \in rng(f)$. □

Lemma 15

Let $\nu \in \widehat{S}$, $H \prec_1 I_\nu$ and f be the uncollapse of H. Let $f \upharpoonright On : \bar{\nu} \to \nu$. Then $\langle \bar{\nu}, f, \nu\rangle \in \mathfrak{F}$.

Proof: If C_ν is bounded in ν, then $\lambda(f_{(0,q_\nu,\nu)}) = \nu$ and $rng(f_{(0,q_\nu,\nu)}) \subseteq rng(f)$ by the definition of X_ν. In addition, $f \upharpoonright J^D_{\bar{\nu}} : \langle J^D_{\bar{\nu}}, D \upharpoonright \bar{\nu}\rangle \to \langle J^D_\nu, D \upharpoonright \nu\rangle$ is Σ_1-elementary. So the claim follows from (CP2). If C_ν is unbounded in ν, then it follows from lemma 14. □

Lemma 16

(Coherence), (Amenability), (Condensation), $(*)$ and $(**)$ hold for the sequence $X = \langle X_\nu \mid \nu \in S^X\rangle$.

Proof:

(Coherence)

Let $\nu \in S^X$ and $H \prec_1 I_\nu$. If C_ν is unbounded in ν, then $\lambda := sup(H \cap \nu) \in C_\nu$ and $C_\nu \cap \lambda$ is unbounded in λ by lemma 15. But, by lemma 10, $C_\nu \cap \lambda = C_\lambda$. So $X_\lambda = X_\nu \cap \lambda$. But if C_ν is bounded in ν, then $H \cap \nu$ is unbounded in ν by the definition of X_ν. So there is nothing to prove.

(Amenability)

If $\nu \in S^X$ and C_ν is bounded in ν, then $X_\nu \cap J^X_\eta$, $\eta < \nu$, is always finite. Therefore amenability is trivial. If $\nu \in S^X$ and C_ν is unbounded in ν, then $C_\nu \cap \lambda = C_\lambda$ for all $\lambda \in C_\nu$ by lemma 10. Therefore $X_\lambda = X_\nu \cap \lambda$ for all $\lambda \in Lim(C_\nu)$. If $Lim(C_\nu)$ is unbounded in ν, we are finished. If it is not, then let $\lambda := max(Lim(C_\nu))$. Then $X_\nu \cap J^X_\eta = C_\lambda \cup E$ where E is finite for all $\eta > \lambda$.

(Condensation)

If $\nu \in S^X$, $H \prec_1 I_\nu$ and C_ν is unbounded in ν, then condensation holds by lemmas 11 und 15. If $\nu \in S^X$ and C_ν is bounded in ν, then $H \prec_1 I_\nu$ is unbounded in ν by the definition of X_ν. Let π be the uncollapse of H and $\pi \upharpoonright On : \bar{\nu} \to \nu$. By lemmas 6 (b) and 15, $\pi(q_{\bar{\nu}}) = q_\nu$. By the properties of σ_ν and $\sigma_{\bar{\nu}}$, we have condensation.

$(*)$

Let $\omega < \kappa$ be a cardinal. Then all $\nu \in S_\kappa$ are independent by (DP1). Therefore $\bar{\nu} < \alpha_\nu = \kappa$ for all $f_{(\beta,0,\nu)} : \bar{\nu} \Rightarrow \nu$ where $\beta < \alpha_\nu = \kappa$. Thus $rng(F) = \bigcup\{rng(f_{(\beta,0,\nu)}) \cap \nu \mid \beta < \alpha_\nu\} = \nu$ for $F : \{\langle\beta, x\rangle \mid x < d(f_{(\beta,0,\nu)})\} \to \nu$ where $F(\beta, x) = f_{(\beta,0,\nu)}(x)$. By lemma 13, $F \in L_\kappa[X]$. So there is a map from a subset of $\kappa \times \kappa$ onto ν in $L_\kappa[X]$. By axioms (c) and (e), S_κ is unbounded in κ^+. Thus $(\kappa^+)^{L_\kappa[X]} = \kappa^+$. Since $\omega < \kappa$ was arbitrary, we get $Card^{L[X]} - \omega_1 = Card - \omega_1$. It remains to prove $\omega_1^{L_\kappa[X]} = \omega_1$. Let $\nu \in S_{\omega_1}$ and $\eta < \omega_1$. By axiom (1), $\eta \subseteq rng(f_{(0,\eta,\nu)})$. By the definition of X, there exists a map from ω onto $\eta \subseteq rng(f_{(0,\eta,\nu)})$ in $L_\kappa[X]$. If $n_\nu = 1$, then $\sigma_{(\langle\eta,\alpha^*_\nu,P_\nu\rangle,\mu_\nu)}$ is a map as needed by (DF). If $n_\nu > 1$,

$$h(i) := h_{\alpha_{\tau(n_\nu-1,\mu_\nu)},K^{n_\nu-1}_{\mu_\nu}}(i, \langle\eta, \nu^*, \alpha^{**}_\nu, P^*_\nu\rangle)$$

is as needed, by lemma 12 (vi) and (DF), where

$f^{n_\nu-1}_{(\beta,\langle x(n_\nu-1,\mu_\nu),\alpha^*_\nu\rangle,\mu_\nu)}(\alpha^{**}_\nu) = \alpha^*_\nu$

$f^{n_\nu-1}_{(\beta,\langle x(n_\nu-1,\mu_\nu),P_\nu\rangle,\mu_\nu)}(P^*_\nu) = P_\nu$

$\nu^* = \nu$ if $\nu < \alpha_{\tau(n_\nu-1,\mu_\nu)}$ and $\nu^* = 0$ else.

Since $\eta < \omega_1$ was arbitrary, $\omega_1^{L_\kappa[X]} = \omega_1$.

$(**)$

On the one hand, by definition of n_ν in (DF), there exists some $z_0 \in J^D_{\mu_\nu}$ and some $\gamma \sqsubset \nu$ such that $f^{n_\nu}_{(\gamma,z_0,\mu_\nu)}$ is cofinal in ν. If $n_\nu = 1$, then $F : \gamma \times \omega \to \mu_\nu$ where

$$\langle\eta, i\rangle \mapsto \sigma_{(\langle\eta,z_0\rangle,\mu_\nu)}(i)$$

is cofinal in ν. If $n_\nu > 1$, then $F : \gamma \times \omega \to \alpha_{\tau(n_\nu-1,\mu_\nu)}$ where

$$\langle\eta, i\rangle \mapsto h_{\alpha_{\tau(n_\nu-1,\mu_\nu)},K^{n_\nu-1}_{\mu_\nu}}(i, \langle\eta, z^*_0\rangle)$$

is cofinal in ν by the proof of Lemma 12 (vi), where

$f^{n_\nu-1}_{(\beta,\langle x(n_\nu-1,\mu_\nu),z_0\rangle,\mu_\nu)}(z^*_0) = z_0$.

But F is definable over I_{μ_ν} by lemma 13. On the other hand, in a standard morass,

$\nu \lhd \tau \Rightarrow \nu$ regular in J_τ^D.

So ν is regular in I_{μ_ν}. $\Box$

Remark

Let $L[X]$ satisfy (Amenability), (Coherence) and (Condensation). Then we can do fine structure arguments, especially we have the Σ_n-Skolem functions h_ν^n of I_ν. As a result, we get: If $S^X = \{\beta(\nu) \mid \nu$ singular in $I_\kappa\}$, then $S^X = \{\nu \mid \nu$ singular in $I_{\nu+\omega}\}$. Because $\{\nu \mid \nu$ singular in $I_{\nu+\omega}\} \subseteq \{\beta(\nu) \mid \nu$ singular in $I_\kappa\}$ by definition. For $\{\beta(\nu) \mid \nu$ singular in $I_\kappa\} \subseteq \{\nu \mid \nu$ singular in $I_{\nu+\omega}\}$, let n be least such that ν becomes singular over I_{μ_ν}. Let p be minimal such that ν becomes singular over I_{μ_ν} in the parameter p. Let p^* be minimal such that $h^n_{\mu_\nu}(i, p^*) = p$ for some $i \in \omega$. Let $\pi : I_{\bar{\mu}} \to I_{\mu_\nu}$ be the uncollapse of $h^n_{\mu_\nu}[\omega \times (J_\nu^X \times \{p^*\})]$. Let $\pi(\bar{p}) = p^*$. Then ν becomes singular over $I_{\bar{\mu}}$ and $h^n_{\bar{\mu}}[\omega \times (J_\nu^X \times \{\bar{p}\})] = J^X_{\bar{\mu}}$. By the minimality of μ_ν, we get $\bar{\mu} = \mu_\nu$ and that $\mu_\nu \in \{\nu \mid \nu$ is singular in $I_{\nu+\omega}\}$.

Conversely, if $S^X = \{\nu \mid \nu$ singular in $I_{\nu+\omega}\}$, then $S^X = \{\beta(\nu) \mid \nu$ singular in $I_\kappa\}$. We prove $\{\beta(\nu) \mid \nu$ singular in $I_\kappa\} \subseteq \{\nu \mid \nu$ singular in $I_{\nu+\omega}\}$ as above. And $\{\nu \mid \nu$ singular in $I_{\nu+\omega}\} \subseteq \{\beta(\nu) \mid \nu$ singular in $I_\kappa\}$ holds again by definition.

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