Some new Inequalities of Hardy-Hilbert Type with general kernel

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Abstract: In this paper, by using hardy inequality, we establish some new integral inequalities of Hardy-Hilbert type with general kernel. As applications, equivalent forms and some particular results are built; the corresponding to the double series inequalities are given.reverse forms are considered also.

Keywords: Hardy-Hilbert's inequality; Hardy's inequalities; Hölder inequality

Mathematics subject classification: 26D15.

1 Introduction

If $p > 1$,$1/p + 1/q = 1$, $f(x)$, $g(x) \geq 0$, $0 < \int_0^\infty f^p(x)dx < \infty$, and

$0 < \int_0^\infty g^q(x)dx < \infty$,then the famous Hardy–Hilbert's inequality (see [1]) and an equivalent form are given by

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy < \frac{\pi}{\sin(\pi/p)}[\int_0^\infty f^p(x)dx]^{1/p}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{1.1}$$

And

$$\int_0^\infty [\int_0^\infty \frac{f(x)}{x+y}dx]^p dy < [\frac{\pi}{\sin(\pi/p)}]^p \int_0^\infty f^p(x)dx, \tag{1.2}$$

Where the constant factor $\frac{\pi}{\sin(\pi/p)}$ and $[\frac{\pi}{\sin(\pi/p)}]^p$ are the best possible.

Hardy et al. [1] gave an inequality and its equivalent form , under the same condition of (1.1), similar to (1.1) as :

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dxdy < pq[\int_0^\infty f^p(x)dx]^{1/p}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{1.3}$$

And

$$\int_0^\infty [\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx]^p dy < [pq]^p \int_0^\infty f^p(x)dx, \tag{1.4}$$

Where the constant factor pq and $[pq]^p$ are the best possible.

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Inequalities (1.1), (1.2) ,(1.3) and (1.4) are important in analysis and its applications(see [2]). In the recent years , many generalization and refinements of these inequalities have been also obtained (see [3-8]).

Recently Das and Sahoo [8] have given a new inequality similar to Hardy–Hilbert inequality (1.1) as follows:

Let $p > 1$, $1/p + 1/q = 1$, $r, s, \lambda > 0$, $r + s = \lambda f(x)$, $g(x) \geq 0$,

$F(x) = \int_0^x f(t)dt, G(x) = \int_0^x g(t)dt$, if $0 < \int_0^\infty f^p(x)dx < \infty$,

$0 < \int_0^\infty g^q(x)dx < \infty$,then the following two integral inequalities holds:

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1} F(x)G(y)}{(x+y)^\lambda} dxdy < pqB(r,s)[\int_0^\infty f^p(x)dx]^{1/p}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{1.5}$$

And

$$\int_0^\infty [\int_0^\infty \frac{x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}} F(x)}{(x+y)^\lambda} dx]^p dy < [qB(r,s)]^p \int_0^\infty f^p(x)dx, \tag{1.6}$$

where the constant factors $pqB(r, s)$ and $[qB(r, s)]^p$ are the best possible.

Sulaiman [7, Theorem 1] derived a new integral inequality similar to (**??**) as follows:

Let $p > 1$,$1/p + 1/q = 1$, $p = \lambda - \alpha - 1 > 1$, $p = \lambda - \beta - 1 > 1$, $\alpha, \beta > -1$,$f(x)$, $g(x) \geq 0$, $F(x) = \int_0^x f(t)dt$,$G(x) = \int_0^x g(t)dt$, if $0 < \int_0^\infty f^p(x)dx < \infty 0 < \int_0^\infty g^q(x)dx < \infty$,then the following two integral inequalities holds:

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{x^{\frac{\beta}{q}} y^{\frac{\alpha}{p}} F(x)G(y)}{\max\{x^\lambda, y^\lambda\}} dxdy \\ &< \frac{p^{1-\frac{1}{p}} q^{1-\frac{1}{q}}}{(\alpha+1)^{\frac{1}{p}} (\beta+1)^{\frac{1}{q}} (p-1)(q-1)} [\int_0^\infty f^p(x)dx]^{1/p} [\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \end{aligned} \tag{1.7}$$

in [7],Sulaiman does not prove whether the constant factor is best possible or not. very recently, Das and Sahoo [9] have given a new generalization of (1.7). the constant factor is the best possible to prove.

In this paper, we obtain a generalization of the inequalities (1.5) and (1.7) with general kernel., the constant factor obtained is the best possible. First we prove the integral version of the inequality and some particular results. Then we give the discrete analogue of the inequality. equivalent forms and reverse forms are considered.

2 Some Lemmas

We need the following some inequalities, which are well-known as Hardy's inequalities (cf. Hardy et al. [1]).

Lemma 2.1 *If* $p > 1$,$f(x) \geq 0$, $F(x) = \int_0^x f(t)dt$, *and* $0 < \int_0^\infty f^p(x)dx < \infty$, *then*

$$\int_0^\infty (\frac{F(x)}{x})^p dx < (\frac{p}{p-1})^p \int_0^\infty f^p(x)dx, \tag{2.1}$$

unless $f(x) \equiv 0$, *The constant is the best possible.*

Lemma 2.2 *If* $0 < p < 1$,$f(x) \geq 0$, $F(x) = \int_x^\infty f(t)dt$, *and* $0 < \int_0^\infty f^p(x)dx < \infty$, *then*

$$\int_0^\infty (\frac{F(x)}{x})^p dx > (\frac{p}{1-p})^p \int_0^\infty f^p(x)dx, \tag{2.2}$$

unless $f(x) \equiv 0$, *The constant is the best possible.*

Lemma 2.3 *If* $p > 1$,$a_n \geq 0$, *and* $A_n = \sum\limits_{i=1}^{n} a_i$, *then*

$$\sum_{n=1}^{\infty} (\frac{A_n}{n})^p < (\frac{p}{p-1})^p \sum_{n=1}^{\infty} a_n^p, \tag{2.3}$$

unless all the $a_n = 0$. *The constant is the best possible.*

Lemma 2.4 *If* $0 < p < 1$,$a_n \geq 0$, *and* $A_n = \sum\limits_{i=1}^{n} a_i$, *then*

$$\sum_{n=1}^{\infty} (\frac{A_n}{n})^p < (\frac{p}{1-p})^p \sum_{n=1}^{\infty} a_n^p, \tag{2.4}$$

unless all the $a_n = 0$. *The constant is the best possible*

If$k_\lambda(x, y)$ is a measurable function, satisfying for λ, u, x, $y > 0$, $k_\lambda(ux, uy) = u^{-\lambda} k_\lambda(x, y)$, then we call $k_\lambda(x, y)$ the homogeneous function of -λ-degree.

Lemma 2.5 *If* $r, s, \lambda > 0$,$r + s = \lambda$,$k_\lambda(x, y) > 0$ *is a homogeneous function of* -λ-*degree, and* $k_\lambda(r) := \int\limits_0^\infty k(u, 1)u^{r-1}du$ *a positive number, define the weight functions* $\omega_\lambda(s, x)$ *and* $\omega_\lambda(r, y)$ *as*

$$\omega_\lambda(s, x) = \int_0^\infty k_\lambda(x, y)x^r y^{s-1}dy, \tag{2.5}$$

$$\omega_\lambda(r, y) = \int_0^\infty k_\lambda(x, y)x^{r-1} y^s dx, \tag{2.6}$$

then we have

(i) $\int\limits_0^\infty k(1, u)u^{s-1}du = k_\lambda(r)$;

(ii) $\omega_\lambda(s, x) = \omega_\lambda(r, y) = k_\lambda(r)$.

Proof. (i) Setting $v = \frac{1}{u}$, by the assumption, we obtain

$$\int\limits_0^\infty k(1, u)u^{s-1}du = \int\limits_0^\infty k(v, 1)v^{r-1}dv = k_\lambda(r).$$

(ii) Setting $u = y/x$ in the integrals $\omega_\lambda(s, x)$, in view of (i), we still find that$\omega_\lambda(s, x) = k_\lambda(r)$. Similarly we have $\omega_\lambda(r, y) = k_\lambda(r)$, The lemma is proved.

Lemma 2.6 *If* $p > 1$,$1/p + 1/q = 1$, $r, s, \lambda > 0$, $r + s = \lambda$,$k_\lambda(x, y) > 0$ *is a homogeneous function of* -λ-*degree, and* $k_\lambda(r) := \int\limits_0^\infty k(u, 1)u^{r-1}du$ *a positive number, for sufficiently small* $\varepsilon > 0$,*setting*

$$I_1 = \int_1^\infty \int_1^\infty k_\lambda(x, y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}dxdy, \tag{2.7}$$

$$I_2 = \int_1^\infty \int_1^\infty k_\lambda(x, y)x^{r-\frac{1}{q}-1}y^{s-\frac{\varepsilon}{q}-1}dxdy), \tag{2.8}$$

$$I_3 = \int_1^\infty \int_1^\infty k_\lambda(x, y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{1}{p}-1}dxdy), \tag{2.9}$$

$$\int_1^\infty x^{-1-\varepsilon}[\int_0^{1/x} k_\lambda(1, u)u^{s-\frac{\varepsilon}{q}-1}du]dx = O_1(1), \tag{2.10}$$

$$\int_1^\infty x^{-\frac{\varepsilon+1}{q}-1}[\int_0^{1/x} k_\lambda(1, u)u^{s-\frac{\varepsilon}{q}-1}du]dx = O_2(1), \tag{2.11}$$

$$\int_1^\infty x^{-\frac{\varepsilon+1}{p}-1}[\int_0^{1/x} k_\lambda(1, u)u^{r-\frac{\varepsilon}{p}-1}du]dx = O_3(1), \tag{2.12}$$

then for $\varepsilon \to 0^+$ *we have*

$$I_1 = \frac{1}{\varepsilon}(k_\lambda(r) + o_1(1)) - O_1(1), \tag{2.13}$$

$$I_2 = \frac{q}{1+\varepsilon}(k_\lambda(r) + o_2(1)) - O_2(1), \tag{2.14}$$

$$I_3 = \frac{p}{1+\varepsilon}(k_\lambda(r) + o_3(1)) - O_3(1). \tag{2.15}$$

Proof. setting $u = y/x$,we have

$$I_1 = \int_1^\infty \int_1^\infty k_\lambda(x, y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}dxdy = \int_1^\infty x^{-1-\varepsilon}[\int_{1/x}^\infty k_\lambda(1, u)u^{s-\frac{\varepsilon}{q}-1}du]dx$$
$$= \int_1^\infty x^{-1-\varepsilon}[\int_0^\infty k_\lambda(1, u)u^{s-\frac{\varepsilon}{q}-1}du]dx - \int_1^\infty x^{-1-\varepsilon}[\int_0^{1/x} k_\lambda(1, u)u^{s-\frac{\varepsilon}{q}-1}du]dx$$
$$= \frac{1}{\varepsilon}(k_\lambda(r) + o_1(1)) - O_1(1).$$

Similarity we can prove (2.14) and (2.15), The lemma is proved.

3 main results

Theorem 3.1 *Let* $p > 1$,$1/p + 1/q = 1$, $r + s = \lambda$, $f(x)$, $g(x) \geq 0$, $\lambda > 0$

$F(x) = \int_0^x f(t)dt, G(x) = \int_0^x g(t)dt$, *if* $0 < \int_0^\infty f^p(x)dx < \infty$

$0 < \int_0^\infty g^q(x)dx < \infty, k_\lambda(r) := \int\limits_0^\infty k(u, 1)u^{r-1}du$ *is a positive number, then the following two integral inequalities holds:*

$$\int_0^\infty \int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}k_\lambda(x, y)F(x)G(y)dxdy < pqk_\lambda(r)[\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{3.1}$$

And

$$\int_0^\infty [\int_0^\infty x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}} k_\lambda(x,y) F(x)dx]^p dy < [qk_\lambda(r)]^p \int_0^\infty f^p(x)dx, \quad (3.2)$$

where the constant factors $pqk_\lambda(r)$ and $[qk_\lambda(r)]^p$ are the best possible.

Proof. By Hölder's inequality with weight (cf. Kuang [9]) and Lemma 2.5, we have

$$\begin{aligned}
&\int_0^\infty \int_0^\infty x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1} k_\lambda(x,y) F(x) G(y) dxdy \\
&= \int_0^\infty \int_0^\infty k_\lambda(x,y) (y^{\frac{s-1}{p}} x^{\frac{r}{p}-1} F(x)) (x^{\frac{r-1}{q}} y^{\frac{s}{q}-1} G(y)) dxdy \\
&\le \{\int_0^\infty \int_0^\infty k_\lambda(x,y)(y^{s-1} x^{r-p} F^p(x)) dxdy\}^{\frac{1}{p}} \\
&\times \int_0^\infty \int_0^\infty k_\lambda(x,y) x^{r-1} y^{s-q} G^q(y)) dxdy \\
&= \{\int_0^\infty \omega_\lambda(s,x) \left(\frac{F(x)}{x}\right)^p dx\}^{\frac{1}{p}} \{\int_0^\infty \omega_\lambda(r,y) \left(\frac{G(y)}{y}\right)^q dy\}^{\frac{1}{q}} \\
&= k_\lambda(r) \{\int_0^\infty \left(\frac{F(x)}{x}\right)^p dx\}^{\frac{1}{p}} \{\int_0^\infty \left(\frac{G(y)}{y}\right)^q dy\}^{\frac{1}{q}},
\end{aligned}$$

Then by Hardy inequality (3.1), (3.1) is valid.

By Hölder's inequality and Lemma 2.5, we get

$$\begin{aligned}
&\int_0^\infty x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}} k_\lambda(x,y) F(x) dx \\
&= \int_0^\infty k_\lambda(x,y) (y^{\frac{s-1}{p}} x^{\frac{r}{p}-1} F(x)) (x^{\frac{r-1}{q}} y^{\frac{s}{q}}) dx \\
&\le \{\int_0^\infty k_\lambda(x,y)(y^{s-1} x^{r-p} F^p(x)) dx\}^{\frac{1}{p}} \\
&\times \{\int_0^\infty k_\lambda(x,y) x^{r-1} y^s dx\}^{\frac{1}{q}} \\
&= k_\lambda^{\frac{1}{q}}(r) \{\int_0^\infty k_\lambda(x,y)(y^{s-1} x^{r-p} F^p(x)) dx\}^{\frac{1}{p}},
\end{aligned}$$

Hence, again applying Lemma 2.5, we obtain

$$\begin{aligned}
&\int_0^\infty [\int_0^\infty x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}} k_\lambda(x,y) F(x) dx]^p dy \\
&\le k_\lambda^{\frac{p}{q}}(r) \int_0^\infty k_\lambda(x,y)(y^{s-1} x^{r-p} F^p(x)) dx \\
&= k_\lambda^p(r) \int_0^\infty (\frac{F(x)}{x})^p dx,
\end{aligned}$$

then by Hardy inequality (2.1), (3.2) is valid.

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}k_\lambda(x,y)F(x)G(y)dxdy\\
&=\int_0^\infty\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}k_\lambda(x,y)F(x)dx(\frac{G(y)}{y})dy\\
&\leq\{\int_0^\infty[\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}k_\lambda(x,y)F(x)dx]^p dy\}^{\frac{1}{p}}\{\int_0^\infty(\frac{G(y)}{y})^q dy\}^{\frac{1}{q}},
\end{aligned}$$

By (2.1) and (3.2), we have (3.1), Hence (3.2) and (3.1) are equivalent. If the constant factor $pqk_\lambda(r)$ is not the best possible, then there exists a positive constant $K$with$K<pqk_\lambda(r)$, thus (**??**) is still valid if we replace $pqk_\lambda(r)$ by K.

For sufficiently small $\varepsilon>0$,Setting $f_\varepsilon(x)$ $g_\varepsilon(x)F_\varepsilon(x)andG_\varepsilon(x)$ as follow

$$f_\varepsilon(x)=\begin{cases}0 & x\in(0,1)\\ x^{-\frac{1}{p}-\frac{\varepsilon}{p}} & x\in[1,\infty)\end{cases}\quad,\quad g_\varepsilon(y)=\begin{cases}0 & y\in(0,1)\\ y^{-\frac{1}{q}-\frac{\varepsilon}{q}} & y\in[1,\infty)\end{cases}$$

$$F_\varepsilon(x)=\begin{cases}0, & x\in(0,1)\\ \frac{q}{1-\varepsilon(q-1)}(x^{\frac{1}{q}-\frac{\varepsilon}{p}}-1), & x\in[1,\infty)\end{cases},$$

$$G_\varepsilon(y)=\begin{cases}0, & y\in(0,1)\\ \frac{p}{1-\varepsilon(p-1)}(y^{\frac{1}{p}-\frac{\varepsilon}{q}}-1), & y\in[1,\infty)\end{cases},$$

Let $\varphi(\varepsilon)=\frac{pq}{(1-\varepsilon(q-1))(1-\varepsilon(p-1))}$, then $\varphi(\varepsilon)\to pq$, as $\varepsilon\to 0^+$ and

$$\{\int_0^\infty f_\varepsilon^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty g_\varepsilon^q(x)dx\}^{\frac{1}{q}}=\frac{1}{\varepsilon}, \tag{3.3}$$

$$F_\varepsilon(x)G_\varepsilon(y)>\varphi(\varepsilon)(x^{\frac{1}{q}-\frac{\varepsilon}{p}}y^{\frac{1}{p}-\frac{\varepsilon}{q}}-y^{\frac{1}{p}-\frac{\varepsilon}{q}}-x^{\frac{1}{q}-\frac{\varepsilon}{p}}),$$

Hence

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}k_\lambda(x,y)F_\varepsilon(x)G_\varepsilon(y)dxdy\\
&>\varphi(\varepsilon)\int_0^\infty\int_0^\infty k_\lambda(x,y)[x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}-x^{r-\frac{1}{q}-1}y^{s-\frac{\varepsilon}{q}-1}-x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{1}{p}-1}]dxdy,
\end{aligned}$$

By Lemma2.5,we obtian

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}k_\lambda(x,y)F_\varepsilon(x)G_\varepsilon(y)dxdy\\
&>\varphi(\varepsilon)[\int_0^\infty\int_0^\infty k_\lambda(x,y)(x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{\varepsilon}{q}-1}-x^{r-\frac{1}{q}-1}y^{s-\frac{\varepsilon}{q}-1})dxdy\\
&-\int_0^\infty\int_0^\infty k_\lambda(x,y)x^{r-\frac{\varepsilon}{p}-1}y^{s-\frac{1}{p}-1}dxdy]\\
&=\varphi(\varepsilon)[\frac{1}{\varepsilon}(k_\lambda(r)+o_1(1))-\frac{q}{1+\varepsilon}(k_\lambda(r)+o_2(1))-\frac{p}{1+\varepsilon}(k_\lambda(r)+o_3(1))-O(1)],
\end{aligned} \tag{3.4}$$

If the constant factor$pqk_\lambda(r)$ in (3.1) is not the best possible, then there exists a positive constant K, such that $K < pqk_\lambda(r)$ and (3.1) still remains valid if $pqk_\lambda(r)$ is replaced by K. In particular by (3.2) and (3.3), we have

$$\begin{aligned}
&\varphi(\varepsilon)[k_\lambda(r)+o_1(1)-\frac{\varepsilon q}{1+\varepsilon}(k_\lambda(r)+o_2(1))-\frac{\varepsilon p}{1+\varepsilon}(k_\lambda(r)+o_3(1))-\varepsilon O(1)] \\
&< \varepsilon\int_0^\infty\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}k_\lambda(x,y)F_\varepsilon(x)G_\varepsilon(y)dxdy \\
&< \varepsilon K\{\int_0^\infty f_\varepsilon^p(x)dx\}^{\frac{1}{p}}\{\int_0^\infty g_\varepsilon^q(x)dx\}^{\frac{1}{q}} = K,
\end{aligned}$$

Then $pqk_\lambda(r) \le K$ as $\varepsilon \to 0^+$.This contradiction shows that the constant factor $pqk_\lambda(r)$ in (3.1) is the best possible.

If the constant factor $[qk_\lambda(r)]^p$ in (3.2) is not the best possible, then there exists a positive constant $\tilde{K}$such that $\tilde{K} < [qk_\lambda(r)]^p$and (3.2) still remains valid if $[qk_\lambda(r)]^p$ is replaced by $\tilde{K}^p$. Then by Holder inequality, (3.2) and Hardy inequality (2.1), we obtain

$$\begin{aligned}
&\int_0^\infty\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}k_\lambda(x,y)F(x)G(y)dxdy \\
&= \int_0^\infty\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}k_\lambda(x,y)F(x)\frac{G(y)}{y}dxdy \\
&\le \{\int_0^\infty(\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}k_\lambda(x,y)F(x)dx)^p dy\}^{\frac{1}{p}}\{\int_0^\infty(\frac{G(y)}{y})^q dy\}^{\frac{1}{q}} \\
&< p\tilde{K}[\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(y)dy]^{\frac{1}{q}},
\end{aligned}$$

which gives that the constant factor $pqk_\lambda(r)$ in (3.1) is not the best possible. This contradiction shows that the constant factor $[qk_\lambda(r)]^p$ in (3.2) is the best possible. This proves the theorem.

Theorem 3.2 *Let* $p > 1$,$1/p + 1/q = 1$,$p = \lambda - \alpha - 1 > 1$,$p = \lambda - \beta - 1 > 1$,$\alpha, \beta > -1$, $f(x)$, $g(x) \ge 0$, $F(x) = \int_0^x f(t)dt$, $G(x) = \int_0^x g(t)dt$, *if* $0 < \int_0^\infty f^p(x)dx < \infty$, $0 < \int_0^\infty g^q(x)dx < \infty$, $k_\lambda(\alpha) := \int\limits_0^\infty k(1,u)u^\alpha du$ *and* $k_\lambda(\beta) := \int\limits_0^\infty k(u,1)u^\beta du$ *are positive number then the following two integral inequalities holds:*

$$\int_0^\infty\int_0^\infty k_\lambda(x,y)x^{\frac{\beta}{q}}y^{\frac{\alpha}{p}}F(x)G(y)dxdy < pqk_\lambda^{1/p}(\alpha)k_\lambda^{1/q}(\beta)[\int_0^\infty f^p(x)dx]^{1/p}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{3.5}$$

$$\int_0^\infty[\int_0^\infty k_\lambda(x,y)x^{\frac{\beta}{q}}y^{\frac{\alpha}{p}+1}F(x)dx]^p dy < q^p[k_\lambda(\alpha)]^{p-1}k_\lambda(\beta)[\int_0^\infty f^p(x)dx]^{1/p}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{3.6}$$

The proof of Theorem 3.2 is similar to that of Theorem 3.1, so we omit it.

Theorem 3.3 *Let* $p > 1$,$1/p + 1/q = 1$, $f(x)$, $g(x) \geq 0$, $\lambda > 0$, $F(x) = \int_x^\infty f(t)dt, G(x) = \int_x^\infty g(t)dt$,*if* $0 < \int_0^\infty (xf(x))^p dx < \infty$, $0 < \int_0^\infty (xg(x))^q dx < \infty, k_\lambda(p) := \int\limits_0^\infty k(u,1)u^{\frac{1}{p}-1}du$ *is a positive number, then the following two integral inequalities holds:*

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)F(x)G(y)dxdy < pqk_\lambda(r)[\int_0^\infty (xf(x))^p dx]^{\frac{1}{p}}[\int_0^\infty (xg(x))^q dx]^{\frac{1}{q}}, \quad (3.7)$$

And

$$\int_0^\infty [\int_0^\infty k_\lambda(x,y)F(x)dx]^p dy < [qk_\lambda(r)]^p \int_0^\infty (xf(x))^p dx, \quad (3.8)$$

where the constant factors $pqk_\lambda(r)$ *and* $[qk_\lambda(r)]^p$ *are the best possible.*

The proof of Theorem 3.3 is similar to that of Theorem 3.1, so we omit it.

Theorem 3.4 *Let* $0 < p < 1$,$1/p + 1/q = 1$, $r + s = \lambda$,$f(x)$, $g(x) \geq 0$, $\lambda > 0$, $F(x) = \int_x^\infty f(t)dt, G(x) = \int_x^\infty g(t)dt$,*if* $0 < \int_0^\infty f^p(x)dx < \infty$,

$0 < \int_0^\infty g^q(x)dx < \infty, k_\lambda(r) := \int\limits_0^\infty k(u,1)u^{r-1}du$ *is a positive number, then the following two integral inequalities holds:*

$$\int_0^\infty \int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}k_\lambda(x,y)F(x)G(y)dxdy > (-pqk_\lambda(r))[\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \quad (3.9)$$

And

$$\int_0^\infty [\int_0^\infty x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}k_\lambda(x,y)F(x)dx]^p dy > [-qk_\lambda(r)]^p \int_0^\infty f^p(x)dx, \quad (3.10)$$

where the constant factors $[-pqk_\lambda(r)]$ *and* $[-qk_\lambda(r)]^p$ *are the best possible.*

Theorem 3.5 *Let* $0 < p < 1$,$1/p + 1/q = 1$, $f(x)$, $g(x) \geq 0$, $\lambda > 0$, $F(x) = \int_x^\infty f(t)dt, G(x) = \int_x^\infty g(t)dt$,*if* $0 < \int_0^\infty (xf(x))^p dx < \infty$, $0 < \int_0^\infty (xg(x))^q dx < \infty, k_\lambda(p) := \int\limits_0^\infty k(u,1)u^{\frac{1}{p}-1}du$ *is a positive number, then the following two integral inequalities holds:*

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)F(x)G(y)dxdy > [-pqk_\lambda(p)][\int_0^\infty (xf(x))^p dx]^{\frac{1}{p}}[\int_0^\infty (xg(x))^q dx]^{\frac{1}{q}}, \quad (3.11)$$

And

$$\int_0^\infty [\int_0^\infty k_\lambda(x,y)F(x)dx]^p dy > [-qk_\lambda(p)]^p \int_0^\infty (xf(x))^p dx, \quad (3.12)$$

where the constant factors $[-pqk_\lambda(r)]$ *and* $[-qk_\lambda(r)]^p$ *are the best possible.*

4 Discrete analogous

Theorem 4.1 *Let* $p > 1$,$1/p + 1/q = 1$, $r + s = \lambda$,a_n, $b_n \geq 0$, $A_n = \sum\limits_{k=1}^n a_k$, $B_n = \sum\limits_{k=1}^n b_k$, *if* $k(u,1)u^{r-1}$ *and* $k(1,u)u^{s-1}$ *are decreasing in* $(0,\infty)$ *and strictly decreasing in a subinterval of*

$(0,\infty)$, $0<\sum\limits_{n=1}^{\infty} a_n^p<\infty$, $0<\sum\limits_{n=1}^{\infty} b_n^q<\infty$, *then the following two inequalities holds:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1} k_\lambda(m,n)A_mB_n < pqk_\lambda(r)[\sum_{n=1}^{\infty} a_n^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty} b_n^q]^{\frac{1}{q}}, \tag{4.1}$$

And

$$\sum_{n=1}^{\infty}[\sum_{m=1}^{\infty} m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}} k_\lambda(m,n)A_m]^p < [qk_\lambda(r)]^p\sum_{n=1}^{\infty} a_n^p, \tag{4.2}$$

where the constant factors $pqk_\lambda(r)$ and $[qk_\lambda(r)]^p$ are the best possible.

The proof of Theorem 4.1 is similar to that of Theorem 3.1, so we omit it.

Theorem 4.2 *Let $p>1$,$1/p+1/q=1$, $p=\lambda-\alpha-1>1$,$p=\lambda-\beta-1>1$, $\alpha,\beta>-1$, $a_n,b_n\geq 0$, $A_n=\sum\limits_{k=1}^{n} a_k$, $B_n=\sum\limits_{k=1}^{n} b_k$,if $k(u,1)u^\alpha$ and $k(1,u)u^\beta$ are decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$, $0<\sum\limits_{n=1}^{\infty} a_n^p<\infty$, $0<\sum\limits_{n=1}^{\infty} b_n^q<\infty$, then the following two inequalities holds:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{\frac{\beta}{q}} n^{\frac{\alpha}{p}} k_\lambda(m,n)A_mB_n < pqk_\lambda(\alpha)[\sum_{n=1}^{\infty} a_n^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty} b_n^q]^{\frac{1}{q}}, \tag{4.3}$$

And

$$\sum_{n=1}^{\infty}[\sum_{m=1}^{\infty} m^{\frac{\beta}{q}} n^{\frac{\alpha}{p}+1} k_\lambda(m,n)A_m]^p < q^p[k_\lambda(\alpha)]^{p-1}k_\lambda(\beta)\sum_{n=1}^{\infty} a_n^p, \tag{4.4}$$

Theorem 4.3 *Let $p>1$,$1/p+1/q=1$, $\lambda>0$, a_n, $b_n\geq 0$, $A_n=\sum\limits_{k=1}^{n} a_k$, $B_n=\sum\limits_{k=1}^{n} b_k$, $k_\lambda(p):=\int\limits_0^\infty k(u,1)u^{\frac{1}{p}-1}du$,if $k(u,1)u^{\frac{1}{p}-1}$ and $k(1,u)u^{\frac{1}{q}-1}$ are decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$, $0<\sum\limits_{n=1}^{\infty} a_n^p<\infty$, $0<\sum\limits_{n=1}^{\infty} b_n^q<\infty$, then the following two inequalities holds:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(m,n)A_mB_n < pqk_\lambda(p)[\sum_{n=1}^{\infty}(na_n)^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(nb_n)^p]^{\frac{1}{q}}, \tag{4.5}$$

And

$$\sum_{n=1}^{\infty}[\sum_{m=1}^{\infty} k_\lambda(m,n)A_m]^p < [qk_\lambda(p)]^p\sum_{n=1}^{\infty}(na_n)^p, \tag{4.6}$$

where the constant factors $pqk_\lambda(p)$ and $[qk_\lambda(p)]^p$ are the best possible.

Theorem 4.4 *Let $0<p<1$,$1/p+1/q=1$, $r+s=\lambda$, a_n, $b_n\geq 0$, $A_n=\sum\limits_{k=1}^{n} a_k$, $B_n=\sum\limits_{k=1}^{n} b_k$, if $k(u,1)u^{r-1}$ and $k(1,u)u^{s-1}$ are decreasing in $(0,\infty)$ and strictly decreasing in a subinterval of $(0,\infty)$, $0<\sum\limits_{n=1}^{\infty} a_n^p<\infty$, $0<\sum\limits_{n=1}^{\infty} b_n^q<\infty$, then the following two inequalities holds:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}-1} k_\lambda(m,n)A_mB_n > [-pqk_\lambda(r)][\sum_{n=1}^{\infty} a_n^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty} b_n^q]^{\frac{1}{q}}, \tag{4.7}$$

And

$$\sum_{n=1}^{\infty}[\sum_{m=1}^{\infty} m^{r-\frac{1}{q}-1} n^{s-\frac{1}{p}} k_\lambda(m,n)A_m]^p > [-qk_\lambda(r)]^p \sum_{n=1}^{\infty} a_n^p, \tag{4.8}$$

where the constant factors $[-pqk_\lambda(r)]$ *and* $[-qk_\lambda(r)]^p$ *are the best possible.*

Theorem 4.5 *Let* $0 < p < 1, 1/p + 1/q = 1$, $\lambda > 0$, a_n, $b_n \geq 0$, $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$, $k_\lambda(p) := \int_0^\infty k(u,1)u^{\frac{1}{p}-1}du$, *if* $k(u,1)u^{\frac{1}{p}-1}$ *and* $k(1,u)u^{\frac{1}{q}-1}$ *are decreasing in* $(0,\infty)$ *and strictly decreasing in a subinterval of* $(0,\infty)$, $0 < \sum_{n=1}^{\infty} a_n^p < \infty$, $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then the following two inequalities holds:*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(m,n)A_mB_n > [-pqk_\lambda(p)][\sum_{n=1}^{\infty}(na_n)^p]^{\frac{1}{p}}[\sum_{n=1}^{\infty}(nb_n)^p]^{\frac{1}{q}}, \tag{4.9}$$

And

$$\sum_{n=1}^{\infty}[\sum_{m=1}^{\infty} k_\lambda(m,n)A_m]^p > [-qk_\lambda(p)]^p \sum_{n=1}^{\infty}(na_n)^p, \tag{4.10}$$

where the constant factors $[-pqk_\lambda(p)]$ *and* $[-qk_\lambda(p)]^p$ *are the best possible.*

5 some particular results

(1) $k_\lambda(x,y) = \frac{1}{|x-y|^\lambda}$, by Lemma 2.3 we have

$$k_\lambda(r) := \int_0^\infty \frac{u^{r-1}}{|1-u|^\lambda}du = B(r, 1-\lambda) + B(s, 1-\lambda),$$

By Theorem 3.1 and 4.1, we have

$$\int_0^\infty\int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}F(x)G(y)}{|x-y|^\lambda}dxdy < pq[B(r,1-\lambda)+B(s,1-\lambda)][\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{5.1}$$

$$\int_0^\infty[\int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}F(x)}{|x-y|^\lambda}dx]^p dy < [q(B(r,1-\lambda)+B(s,1-\lambda))]^p\int_0^\infty f^p(x)dx, \tag{5.2}$$

(2) $k_\lambda(x,y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ by Lemma 2.3 we have

$$k_\lambda(r) := \int_0^\infty \frac{\ln u u^{r-1}}{1-u^\lambda}du = [\tfrac{\pi}{\lambda\sin(r/\lambda)}]^2$$

By Theorem 3.1 and 4.1, we have

$$\int_0^\infty\int_0^\infty \frac{\ln(x/y)x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}F(x)G(y)}{x^\lambda - y^\lambda}dxdy < pq[\tfrac{\pi}{\lambda\sin(r/\lambda)}]^2[\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{5.3}$$

$$\int_0^\infty [\int_0^\infty \frac{\ln(x/y)x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}F(x)}{x^\lambda - y^\lambda}dx]^p dy < [q[\tfrac{\pi}{\lambda\sin(r/\lambda)}]^2]^p \int_0^\infty f^p(x)dx, \tag{5.4}$$

(3) $k_\lambda(x,y) = \frac{1}{|x-y|^\beta(\max\{x,y\})^{\lambda-\beta}}$, $(0 < \beta < 1)$ by Lemma 2.3 we have

$$k_\lambda(r) := \int_0^\infty \frac{u^{r-1}}{|1-u|^\beta(\max\{1,u\})^{\lambda-\beta}}du = B(r, 1-\beta) + B(s, 1-\beta),$$

By Theorem 3.1 and 4.1, we have

$$\int_0^\infty \int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}F(x)G(y)}{|x-y|^\beta(\max\{x,y\})^{\lambda-\beta}}dxdy < pq[B(r,1-\beta)+B(s,1-\beta)][\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{5.5}$$

$$\int_0^\infty [\int_0^\infty \frac{x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}F(x)}{|x-y|^\beta(\max\{x,y\})^{\lambda-\beta}}dx]^p dy < [q(B(r,1-\beta) + B(s,1-\beta))]^p \int_0^\infty f^p(x)dx, \tag{5.6}$$

(4) $k_\lambda(x,y) = \frac{(\min\{x,y\})^{\beta-\lambda}}{|x-y|^\beta}$, $(0 < \beta < 1)$ by Lemma 2.3 we have

$$k_\lambda(r) := \int_0^\infty \frac{(\min\{1,u\})^{\beta-\lambda}u^{r-1}}{|1-u|^\beta}du = B(\beta - r, 1-\beta) + B(\beta - s, 1-\beta),$$

By Theorem 3.1 and 4.1, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{(\min\{x,y\})^{\beta-\lambda}x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}F(x)G(y)}{|x-y|^\beta}dxdy \\ &< pq[(\beta - r, 1-\beta) + B(\beta - s, 1-\beta)][\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \end{aligned} \tag{5.7}$$

$$\int_0^\infty [\int_0^\infty \frac{(\min\{x,y\})^{\beta-\lambda}x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}}F(x)}{|x-y|^\beta}dx]^p dy < [q((\beta-r,1-\beta)+B(\beta-s,1-\beta))]^p \int_0^\infty f^p(x)dx, \tag{5.8}$$

(5) $k_\lambda(x,y) = \frac{|x^\beta - y^\beta|}{(\max\{x,y\})^{\lambda+\beta}}$, $(\beta > -\min\{r,s\})$ by Lemma 2.3 we have

$$k_\lambda(r) := \int_0^\infty \frac{|1-u^\beta|u^{r-1}}{(\max\{1,u\})^{\lambda+\beta}}du = \frac{|\beta|(r(r+\beta) + s(s+\beta))}{rs(r+\beta)(s+\beta)},$$

By Theorem 3.1 and 4.1, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|x^\beta - y^\beta|x^{r-\frac{1}{q}-1}y^{s-\frac{1}{p}-1}F(x)G(y)}{(\max\{x,y\})^{\lambda+\beta}}dxdy \\ &< pq\frac{|\beta|(r(r+\beta) + s(s+\beta))}{rs(r+\beta)(s+\beta)}[\int_0^\infty f^p(x)dx]^{\frac{1}{p}}[\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \end{aligned} \tag{5.9}$$

$$\int_0^\infty [\int_0^\infty \frac{|x^\beta - y^\beta| x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}} F(x)}{(\max\{x,y\})^{\lambda+\beta}} dx]^p dy < [q \frac{|\beta|(r(r+\beta)+s(s+\beta))}{rs(r+\beta)(s+\beta)}]^p \int_0^\infty f^p(x)dx, \tag{5.10}$$

(6) $k_\lambda(x,y) = \frac{|\ln(x/y)|}{(\max\{x,y\})^\lambda}$, $(0 < \beta < 1)$ by Lemma 2.3 we have

$$k_\lambda(r) := \int_0^\infty \frac{|\ln u| u^{r-1}}{(\max\{1,u\})^\lambda} du = \frac{1}{r^2} + \frac{1}{s^2},$$

By Theorem 3.1 and 4.1, we have

$$\int_0^\infty \int_0^\infty \frac{|\ln(x/y)| x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1} F(x)G(y)}{(\max\{x,y\})^\lambda} dxdy < pq(\frac{1}{r^2}+\frac{1}{s^2})[\int_0^\infty f^p(x)dx]^{\frac{1}{p}} [\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \tag{5.11}$$

$$\int_0^\infty [\int_0^\infty \frac{|\ln(x/y)| x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}} F(x)}{(\max\{x,y\})^\lambda} dx]^p dy < [q(\frac{1}{r^2}+\frac{1}{s^2})]^p \int_0^\infty f^p(x)dx, \tag{5.12}$$

(7) $k_\lambda(x,y) = \frac{|\ln(x/y)|}{x^\lambda + y^\lambda}$, $(0 < \beta < 1)$ by Lemma 2.3 we have

$$k_\lambda(r) := \int_0^\infty \frac{|\ln u| u^{r-1}}{1+u^\lambda} du = \sum_{n=1}^\infty (-1)^n \frac{2}{(\lambda n + r)^2},$$

By Theorem 3.1 and 4.1, we have

$$\begin{aligned} &\int_0^\infty \int_0^\infty \frac{|\ln(x/y)| x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}-1} F(x)G(y)}{x^\lambda + y^\lambda} dxdy \\ &< pq(\sum_{n=1}^\infty (-1)^n \frac{2}{(\lambda n+r)^2})^{\frac{1}{p}} (\sum_{n=1}^\infty (-1)^n \frac{2}{(\lambda n+s)^2})^{\frac{1}{q}} [\int_0^\infty f^p(x)dx]^{\frac{1}{p}} [\int_0^\infty g^q(x)dx]^{\frac{1}{q}}, \end{aligned} \tag{5.13}$$

$$\int_0^\infty [\int_0^\infty \frac{|\ln(x/y)| x^{r-\frac{1}{q}-1} y^{s-\frac{1}{p}} F(x)}{x^\lambda + y^\lambda} dx]^p dy < q^p (\sum_{n=1}^\infty (-1)^n \frac{2}{(\lambda n+r)^2})^{p-1} \sum_{n=1}^\infty (-1)^n \frac{2}{(\lambda n+s)^2} \int_0^\infty f^p(x)dx, \tag{5.14}$$

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