

# MINIMAL MODEL PROGRAM WITH SCALING AND ADJUNCTION THEORY

MARCO ANDREATTA

ABSTRACT. Let  $(X, L)$  be a quasi polarized pairs, i.e.  $X$  is a normal complex projective variety and  $L$  is a nef and big line bundle on it. We study, up to birational equivalence, the positivity (nefness) of the adjoint bundles  $K_X + rL$  for high rational number  $r$ . For this we run a Minimal Model Program with scaling relative to the divisor  $K_X + rL$ . We give some applications, namely the classification up to birational equivalence of quasi polarized pairs with sectional genus 0, 1 or 2 and of embedded projective varieties  $X \subset \mathbb{P}^N$  with degree smaller than  $2\text{codim}_{\mathbb{P}^N}(X) + 2$ .

## 1. INTRODUCTION

Let  $X$  be a complex projective normal variety of dimension  $n$  and  $L$  be a nef and big line bundle on  $X$ . The pair  $(X, L)$  is called a **quasi polarized pair**. The goal of Adjunction Theory is to classify quasi polarized pairs via the study of the positivity of the adjunction divisors  $K_X + rL$ , with  $r$  a positive rational number. This has been done extensively in the case in which  $L$  is ample, i.e.  $(X, L)$  is a polarized pair; [BS95] is the best account on this case .

However the set up of quasi polarized pairs is certainly more natural: in particular when passing to a resolution of the singularities and taking the pull back of  $L$ . The classification of quasi polarized pairs will be up to birational equivalence. Quasi polarized pairs were first considered by T. Fujita (see [Fu89]). In that paper he made a connection between this theory and the Minimal Model Program (MMP for short) and he proved some results under the assumption of the existence of the MMP (more precisely under the assumption of existence and termination of flips). In this paper, following T. Fujita ideas as re-proposed by A. Höring in [Ho10], and with the use of the MMP developed in [BCHM10], we describe a MMP with scaling related to divisors of type  $K_X + rL$  (see Section 4).

Using the  $K_X + rL$ -MMP we prove that, either the pair  $(X, L)$  is birational equivalent to some very special quasi polarized pairs, or it is birational equivalent to a pair  $(X', L')$ , which we call a zero reduction, where  $K'_{X'} + rL'$  is nef for  $r \geq (n - 1)$  (Theorem 5.1).

In a further step we prove that there exists a quasi polarized pair  $(X'', L'')$ , which we call a first reduction of the pair  $(X, L)$  and which is related to the original  $(X, L)$  via birational equivalences or blow up of smooth points, such that, a part a finite list of special pairs,  $K_{X''} + rL''$  is nef for  $r \geq (n - 2)$  (Theorem 5.6).

---

1991 *Mathematics Subject Classification*. 14E30, 14J40, 14N30, 14N25.

*Key words and phrases*. Quasi polarized pairs, Adjunction Theory, Minimal Model Program with scaling.

Many thanks to Paolo Cascini for explaining some details of the MMP.

The author was supported by a grant of the Italian Minister of Research - PRIN..

We give then some applications, namely the classification, up to birational equivalence, of quasi polarized pairs with sectional genus 0, 1 or 2 (Corollary 6.1) and, up to first reduction, of embedded projective varieties  $X \subset \mathbb{P}^N$  with degree smaller than  $2\text{codim}_{\mathbb{P}^N}(X) + 2$  (Corollary 6.2).

## 2. NOTATION AND PRELIMINARIES

Our notation is consistent with the books [BS95] and [KM98] and the paper [BCHM10], to which we constantly refer. We give however some basic definition in order to state our main objects.

In general  $X$  will be a normal (complex projective) variety, that is an irreducible and reduced projective scheme over  $\mathbb{C}$ , of dimension  $n$ . Two  $\mathbb{Q}$ -divisors  $D_1, D_2$  are  $\mathbb{Q}$ -linearly equivalent,  $D_1 \sim_{\mathbb{Q}} D_2$ , if there exists a integer  $m > 0$  such that  $mD_i$  are linearly equivalent. A  $\mathbb{Q}$ -divisor  $D$  is  $\mathbb{Q}$ -Cartier if some integral multiple is Cartier. Let  $D$  be an  $\mathbb{R}$ -divisor; it is nef if  $D \cdot C \geq 0$  for any curve  $C \subset X$ . It is big if  $D \sim_{\mathbb{R}} A + B$  where  $A$  is ample and  $B \geq 0$ . It is pseudo-effective if it is in the closure of the cone of effective divisors. Effective or nef divisors are pseudoeffective.

A **quasi polarized variety** is a pair  $(X, L)$  where  $X$  is a (complex projective) variety and  $L$  is a nef and big Cartier divisor.

A log pair  $(X, \Delta)$  is a normal variety  $X$  and an effective  $\mathbb{R}$  divisor  $\Delta$  such that  $K_X + \Delta$  is  $\mathbb{R}$ -Cartier. A log resolution of the pair  $(X, \Delta)$  is projective birational morphism  $g : Y \rightarrow X$  such that  $Y$  is smooth and the exceptional locus is a divisor which, together with  $g^{-1}(\Delta)$ , is simple normal crossing. We can write

$$K_Y + \sum b_i \Gamma_i = g^*(K_X + \Delta).$$

The log pair  $(X, \Delta)$  is Kawamata log terminal (klt) if for every (equivalently for one) log resolution as above  $b_i < 1$  for all  $i$ . If  $\Delta = 0$  and  $b_i < 0$  then  $X$  has terminal singularities.

## 3. POLARIZED PAIRS AND ADJUNCTION THEORY

**Definition 3.1.** Two quasi polarized pairs  $(X_1, L_1)$  and  $(X_2, L_2)$  are said to be **birationally equivalent** if there is another variety  $Y$  with birational morphisms  $\varphi_i : Y \rightarrow X_i$  such that  $\varphi_1^* L_1 = \varphi_2^* L_2$ .

**Definition 3.2.** The Hilbert polynomial of the quasi polarized pair  $(X, L)$  is given by  $\chi(X, tL) = \sum_{j=0, \dots, n} \chi_j t^{[j]} / j!$  for some integers  $\chi_0, \dots, \chi_n$ , where  $t^{[j]} = t(t+1)\dots(t+j-1)$  and  $t^{[0]} = 1$ .

By Riemann-Roch Theorem we have  $\chi_n = L^n$  and, if  $X$  is normal,  $-2\chi_{n-1} = (K_X + (n-1)L)L^{n-1}$ , for a canonical divisor  $K_X$  on  $X$ .

The sectional genus of the pair  $(X, L)$  is defined as  $g(X, L) = 1 - \chi_{n-1}$ .

The  $\Delta$ -genus is defined as  $\Delta(X, L) = n + \chi_n - h^0(X, L)$ .

Assume that  $X$  has at most terminal singularities and that  $K_X$  is not nef. Let  $R = \mathbb{R}^+[C]$  be an extremal ray on  $X$ , where  $C \subset X$  is a rational curve with  $-K_X \cdot C > 0$ . Let  $\varphi_R : X \rightarrow Z$  be the contraction associated to  $R$ : that is  $\varphi_R$  is a morphism with connected fibers onto a normal projective variety  $Z$  and  $C \subset X$  is in a fiber of  $\varphi_R$  if and only if it is in the ray  $R$ .

The existence of  $\varphi_R$  is the famous base point free theorem of Kawamata-Shokurov in the MMP theory.

If  $\varphi_R$  is of fiber type (i.e.  $\dim X > \dim Z$ ) then  $\varphi_R : X \rightarrow Z$  is called a **Mori fiber space**.

Otherwise the contraction  $\varphi_R$  is birational; it can be either divisorial or small.

For a normal quasi polarized pair  $(X, L)$  let

$$r(X, L) := \sup\{t \in \mathbb{R} : tK_X + L \text{ is nef}\}.$$

By the Kawamata rationality theorem  $r(X, L)$  is a rational non negative number. If  $r(X, L) \neq 0$  we define  $\tau(X, L) := 1/r(X, L)$ .

**Lemma 3.3.** *Let  $\varphi_R : X \rightarrow Z$  be a Mori fiber space associated to the extremal ray  $R = \mathbb{R}^+[C]$ . A nef and big line bundle  $L$  on  $X$  is  $\varphi_R$ -ample (i.e.  $L \cdot C > 0$ ).*

*Proof.* In fact if by contradiction  $L \cdot C = 0$  then there exists a line bundle  $A$  on  $Z$  such that  $L = \varphi_R^*(A)$  (see Corollary 3.17 in [KM98]). But this implies that  $L^n = 0$ , which is a contradiction since  $L$  is nef and big.

*Remark 3.4.* Let  $\varphi_R : X \rightarrow Z$  be the contraction of the extremal ray  $R = \mathbb{R}^+[C]$ . Assume that  $L$  is  $\varphi$ -ample. By adding the pull back of a sufficiently ample line bundle from  $Z$  we can assume that

- i)  $L$  is ample,
- ii)  $r(X, L) \neq 0$  and
- iii) the intersection of  $(K_X + \tau(X, L)L)$  with curves in  $R$  is zero and positive with all other curves.

(The proof of these remarks is standard in the theory of ample line bundle; use for instance section 1.5, in particular proposition 1.45, of [KM98]).

**Proposition 3.5.** *Let  $\varphi_R : X \rightarrow Z$  be a Mori fiber space associated to the extremal ray  $R = \mathbb{R}^+[C]$  and let  $L$  be a nef and big line bundle on  $X$ .*

*Let  $r$  be a positive rational number such that  $(K_X + rL) \cdot C < 0$ ; note that this implies that  $\tau(X, L) > r$  (possibly adding the pull back of a sufficiently ample line bundle from  $Z$ ).*

A) *If  $r \geq (n - 1)$  then  $(X, L)$  is one of the following pairs:*

- $(\mathbb{P}^n, \mathcal{O}(1))$  and  $r < (n + 1)$ ,
- $(Q, \mathcal{O}(1)|_Q)$ , where  $Q \subset \mathbb{P}^{n+1}$  is a quadric and  $r < n$ ,
- $C_n(\mathbb{P}^2, \mathcal{O}(2))$ , a generalized cone over  $(\mathbb{P}^2, \mathcal{O}(2))$  and  $r < n$ ,
- $\varphi_R$  gives  $X$  the structure of a  $\mathbb{P}^{n-1}$ -bundle over a smooth curve  $C$  and  $L$  restricted to any fiber is  $\mathcal{O}(1)$  and  $r < n$ .

B) *If  $r \geq (n - 2)$  then  $(X, L)$  is one of the following pairs:*

- one of the pair in the previous list,
- a del Pezzo variety, that is  $-K_X \sim_{\mathbb{Q}} (n - 1)L$  with  $L$  ample,  $r < (n - 1)$ ,
- $(\mathbb{P}^4, \mathcal{O}(2))$ ,
- $(\mathbb{P}^3, \mathcal{O}(3))$ ,
- $(Q, \mathcal{O}(2)|_Q)$ , where  $Q \subset \mathbb{P}^4$  is a quadric,
- $\varphi_R$  gives  $X$  the structure of a quadric fibration over a smooth curve  $C$  and  $L$  restricted to any fiber is  $\mathcal{O}(1)|_Q$ ,  $r < (n - 1)$ ,
- $\varphi_R$  gives  $X$  the structure of a  $\mathbb{P}^{n-2}$ -bundle over a normal surface  $S$  and  $L$  restricted to any fiber is  $\mathcal{O}(1)$ ,  $r < (n - 1)$ ,
- $n = 3$ ,  $Z$  is a smooth curve and the general fiber of  $\varphi_R$  is  $\mathbb{P}^2$  and  $L$  restricted to it is  $\mathcal{O}(1)$ .

*Proof.* Using Lemma 3.3 and Remark 3.4 we can assume that  $L$  is ample. The proposition follows then by the "classic" adjunction theory developed by T. Fujita and by A. Sommese and his school: more precisely the results are summarized in section 7.2 and 7.3 of [BS95]. One of my personal contribution to this theory is its extension to the case with terminal or even log-terminal singularities in the papers [An94] and [An95]

**Proposition 3.6.** *Let  $\varphi_R : X \rightarrow Z$  be a birational contraction associated to the extremal ray  $R = \mathbb{R}^+[C]$ ; let  $L$  be a nef and big line bundle on  $X$ . Let  $r$  be a rational number such that  $(K_X + rL) \cdot C < 0$ .*

*Assume that  $r \geq (n - 2)$  and that  $L \cdot C \neq 0$ .*

*Then  $\tau(X, L) = (n - 1) > r$ ,  $Z$  has at most terminal singularities and  $\varphi_R : X \rightarrow Z$  is the blow-up of a smooth point. Moreover on  $Z$  there exists a nef and big line bundle  $L'$  such that  $\varphi_R^* L' = L + E$ , where  $E \simeq \mathbb{P}^{n-1}$  is the exceptional divisor of the blow-up.*

*Proof.* By the assumption, using the Remark 3.4, we can assume that  $L$  is ample, that the intersection of  $(K_X + \tau(X, L)L)$  with the curves in  $R$  is zero and positive on all other curves and that  $\tau(X, L) > r \geq (n - 2)$ .

If  $\tau(X, L) \geq (n - 1)$  Theorem 3.1 of [An95] applies (see also the Theorem 2.1 there): we get that  $\varphi_R$  is the blow up of a smooth point, and everything is as stated in the proposition.

The case  $(n - 2) < \tau(X, L) < (n - 1)$  cannot happen: one can use for instance Theorem 7.3.4 in [BS95], which says that under this assumption  $\varphi_R$  has to be of fiber type.

#### 4. MINIMAL MODEL PROGRAM WITH SCALING

Let  $(X, L)$  be a quasi polarized variety and assume that  $X$  has at most terminal singularities. Let also  $r$  be a rational positive number.

**Lemma 4.1.** *Under the above assumption (in particular  $L$  is nef and big) there exists an effective  $\mathbb{Q}$ -divisor  $\Delta^r$  on  $X$  such that*

$$\Delta^r \sim_{\mathbb{Q}} rL \quad \text{and} \quad (X, \Delta^r) \text{ is Kawamata log terminal.}$$

*Proof.* This lemma is well known to specialists and it can be proved in different ways. Since  $L$  is nef and big the asymptotic multiplier ideal of  $rL$  is trivial, i.e.  $\mathcal{J}(X, \lfloor rL \rfloor) = \mathcal{O}_X$  (Proposition 11.2.18 in [La04] in the smooth case or Corollary 5.2 in [CD11] in the terminal case and under the weaker assumption that  $L$  is nef and abundant). Take  $D$  a generic divisor in  $mrL$  for sufficiently large  $m$  and let  $\Delta^r := \frac{1}{m}D$ .  $\Delta^r$  is effective and  $\mathbb{Q}$  linearly equivalent to  $rL$ .

Moreover, for  $m$  sufficiently large,  $\mathcal{J}(X, \Delta^r) = \mathcal{J}(X, \frac{1}{m}(\lfloor mrL \rfloor)) = \mathcal{J}(X, \lfloor rL \rfloor) = \mathcal{O}_X$ , i.e.  $(X, \Delta^r)$  is Kawamata log terminal.

Consider the pair  $(X, \Delta^r)$  and the  $\mathbb{Q}$ -Cartier divisor  $K_X + \Delta^r \sim_{\mathbb{Q}} K_X + rL$ .

By the Theorem 1.2 and the Corollary 1.3.3 of [BCHM10] we can run a

**$K_X + \Delta^r$ - Minimal Model Program with scaling:**

$$(X_0, \Delta_0^r) = (X, \Delta^r) \rightarrow (X_1, \Delta_1^r) \rightarrow \dots \rightarrow (X_s, \Delta_s^r)$$

such that:

1) each map  $\varphi_i : X_i \rightarrow X_{i+1}$  is a birational map which is either a divisorial contraction or a flip associated to an extremal ray  $R_i$ ,

2) if  $K_X + \Delta^r$  is pseudoeffective then  $K_{X_s} + \Delta_s^r$  is nef,  
 if  $K_X + \Delta^r$  is not pseudoeffective then  $X_s$  is a Mori fiber space relatively to  $K_{X_s} + \Delta_s^r$ .

The next proposition has been proved by T. Fujita in section 4 of [Fu89], under the assumption of the existence of Minimal Models (more precisely subordinated to the Flip conjecture). A. Höring ([Ho10]) has adapted Fujita argument to the notations and the spirit of [BCHM10]; see the Claim in the course of the proof of his Proposition 1.3.

**Proposition 4.2.** *Under the above notations and assumptions suppose moreover that  $r \geq (n-1)$ .*

*For every  $i = 0, \dots, s$ , we have  $\Delta_i^r \cdot R_i = 0$  and therefore there exist nef and big Cartier divisors  $L_i$  on  $X_i$  such that  $\varphi_i^*(L_{i+1}) = L_i$  and  $\Delta_i^r \sim_{\mathbb{Q}} rL_i$ .*

*Since the  $K_{X_i} + \Delta_i^r$  negative contraction  $\varphi_i$  is  $K_{X_i}$  negative,  $X_{i+1}$  has at most terminal singularities.*

*Thus at every step of the MMP we have a quasi polarized variety  $(X_i, L_i)$  with at most terminal singularities. Note also that  $H^0(K_{X_i} + tL_i) = H^0(K_{X_{i+1}} + tL_{i+1})$  for any  $t = 0, \dots, r$ .*

*Proof.* The proposition follows by induction on  $i$ .

Each map  $\varphi_i : X_i \rightarrow X_{i+1}$  is a birational map associated to an extremal ray  $R_i = \mathbb{R}^+[C_i]$  with

$$(K_{X_i} + rL_i) \cdot C_i = (K_{X_i} + \Delta_i^r) \cdot C_i < 0.$$

Since  $r \geq (n-1)$ , by Proposition 3.6, we have that  $L_i \cdot C_i = \Delta_i^r \cdot C_i = 0$ .

Let  $\varphi_{R_i} : X_i \rightarrow Z$  be the contraction of the extremal ray  $R_i$ ; since  $L_i \cdot C_i = 0$  there exists a nef and big line bundle  $L$  on  $Z$  such that  $\varphi_{R_i}^* L = L_i$  (Corollary 3.17 of [KM98]).

If  $\varphi_{R_i}$  is birational then  $\varphi_{R_i} = \varphi_i$  and we take  $L_{i+1}$  to be  $L$  itself. If  $\varphi_{R_i}$  is small let  $\varphi^+ : X_{i+1} \rightarrow Z$  be its flip; define then  $L_{i+1}$  to be  $\varphi^{+*}(L)$ .

Note that  $\Delta_{i+1}^r = \varphi_* \Delta_i^r \sim_{\mathbb{Q}} \varphi_*(rL_i) = rL_{i+1}$ .

Let us prove the last statement, namely  $H^0(K_{X_i} + tL_i) = H^0(K_{X_{i+1}} + tL_{i+1})$  for any  $t = 0, \dots, r$ . This is obvious if  $\varphi_i$  is a flip, since  $X_i$  and  $X_{i+1}$  are isomorphic in codimension 1. If  $\varphi_i$  is birational then  $\varphi_i^*(K_{X_{i+1}}) = K_{X_i} - E_i$ , where  $E_i$  is an effective  $\mathbb{Q}$ -divisor. For any  $t = 0, \dots, r$  we have  $(K_{X_i} + tL_i) \cdot R_i < 0$ , which implies that  $H^0(E_i, K_{X_i} + tL_i) = 0$ . Thus the claim follows from the exact sequence

$$0 \rightarrow \varphi_i^*(K_{X_{i+1}} + tL_{i+1}) = K_{X_i} + tL_i - E_i \rightarrow K_{X_i} + tL_i \rightarrow (K_{X_i} + tL_i)|_{E_i} \rightarrow 0.$$

**Corollary 4.3.** *Let  $(X, L)$  be a quasi polarized variety such that  $X$  has at most terminal singularities.*

*A  $K_X + \Delta^{(n-1)}$ -Minimal Model Program with scaling is a  $K_X + \Delta^r$ -Minimal Model Program with scaling for any  $r \geq (n-1)$ , with possibly a difference in the last step. Namely if  $X_s$  is a Mori fiber space relative to  $K_{X_s} + \Delta^{(n-1)}$  it can be that, for  $r > (n-1)$ , the divisor  $K_{X_s} + \Delta^r$  is nef.*

*Proof.* In the spirit of [BCHM10] take  $A$  be an ample line bundle on  $X$  and run a  $K_X + \Delta^{(n-1)}$  minimal model with scaling  $A$ . This means that at each step  $i = 0, \dots, s$  we take  $\lambda_i$  and  $R_i = \mathbb{R}^+[C_i]$  such that  $\lambda_i = \min\{l : K_{X_i} + \Delta_i^{(n-1)} + lA_i\}$  is nef and  $(K_{X_i} + \Delta_i^{(n-1)} + \lambda_i A_i) \cdot C_i = 0$ .

The Proposition says that  $L_i \cdot R_i = \Delta_i^r \cdot R_i = 0$ . Therefore  $\lambda_i = \min\{l : K_{X_i} + \Delta_i^r + lA_i\}$  is nef and  $(K_{X_i} + \Delta_i^r + \lambda_i A_i) \cdot C_i = 0$  for every  $r \geq (n-1)$ .

If  $K_{X_s} + \Delta_s^{(n-1)}$  is nef then, for  $r \geq (n-1)$ ,  $K_{X_s} + \Delta_s^r$  is nef as well. However if  $X_s$  is a Mori fiber space relative to the ray  $R = \mathbb{R}^+[C]$  such that  $(K_{X_s} + \Delta_s^{(n-1)}) \cdot C < 0$ , it can be that  $K_{X_s} + \Delta_s^r$  is positive or zero on  $C$ , i.e. it is nef.

**Definition 4.4.** Let  $(X, L)$  be a quasi polarized variety such that  $X$  has at most terminal singularities. Let  $(X_s, L_s)$  be a quasi polarized pair where  $X_s$  is the last variety in a  $K_X + \Delta^{(n-1)}$ - Minimal Model Program with scaling and  $L_s$  be the corresponding nef and big line bundle on  $X_s$  coming from Proposition 4.2. We will call  $(X', L') = (X_s, L_s)$  a **zero reduction of the pair**  $(X, L)$ .

*Remark 4.5.* i) A zero reduction is birationally equivalent to the original pair.  
ii) Long ago with A. Sommese we studied the surface case ( $n = 2$ ) in [AS89]; in particular Proposition 1.7 in that paper gives the construction of the zero reduction for Gorenstein surfaces (note that for  $n = 2$  terminal singularities are actually smooth).

## 5. ADJUNCTION THEORY VIA MMP WITH SCALING

**5.1. Adjunction on the zero reduction.** The following theorem is the first step in the Adjunction Theory of quasi polarized pairs; Part 3) was first proved by A. Höring ([Ho10], Proposition 1.3).

**Theorem 5.1.** *Let  $(X, L)$  be a quasi polarized variety such that  $X$  has at most terminal singularities.*

- 1)  $K_X + (n+1)L$  is pseudoeffective and on a zero reduction  $(X', L')$  the  $\mathbb{Q}$ -Cartier divisor  $K_{X'} + (n+1)L'$  is nef.
- 2)  $K_X + nL$  is not pseudoeffective if and only any zero reduction  $(X', L')$  is  $(\mathbb{P}^n, \mathcal{O}(1))$ . If  $K_X + nL$  is pseudoeffective then on a zero reduction  $(X', L')$  the  $\mathbb{Q}$ -Cartier divisor  $K_{X'} + nL'$  is nef.
- 3)  $K_X + (n-1)L$  is not pseudoeffective if and only if any zero reduction  $(X', L')$  is one of the pairs in 3.5 A). If  $K_X + (n-1)L$  is pseudoeffective then on a zero reduction  $(X', L')$  the  $\mathbb{Q}$ -Cartier divisor  $K_{X'} + (n-1)L'$  is nef.

*Proof.* We use the construction in Section 4 and Proposition 3.5.

Assume, by contradiction, that  $K_X + (n+1)L$  is not pseudoeffective. Run a  $K_X + (n+1)L$ -Minimal Model Program on  $(X, L)$  as in Section 4 and let  $(X_s, L_s)$  be the last pair of the process (i.e., by Corollary 4.3, a zero reduction of the pair  $(X, L)$  as in Definition 4.4).  $X_s$  is a Mori fiber space associated to an extremal ray  $R = \mathbb{R}^+[C]$  such that  $(K_{X_s} + (n+1)L_s) \cdot C < 0$ . This cannot exist by Proposition 3.5. Therefore  $K_X + (n+1)L$  has to be pseudoeffective and, on a zero reduction  $(X', L')$ , the divisor  $K_{X'} + (n+1)L' \sim_{\mathbb{Q}} K_{X_s} + \Delta_s^{n+1}$  is nef.

Points 2) and 3) can be proved similarly; let us prove for instance point 3). Let  $(X', L')$  be a zero reduction of  $(X, L)$  defined in 4.4: if  $K_X + (n-1)L$  is not pseudoeffective.  $X'$  is a Mori fiber space associated to an extremal ray  $R = \mathbb{R}^+[C]$  such that  $(K_{X'} + (n-1)L') \cdot C < 0$ . By Proposition 3.5, it has to be one of the pairs in 3.5 A).

If  $K_X + (n-1)L$  is pseudoeffective then  $K_{X'} + (n-1)L'$  is nef.

**Corollary 5.2.** *On the zero reduction  $(X', L')$  there are no extremal rays  $R = \mathbb{R}^+[C]$  such that  $L' \cdot C = 0$ .*

*Proof.* In fact  $K_{X'} + (n + 1)L'$  is nef and therefore for every curve  $C \subset X'$  such that  $-K_{X'} \cdot C < 0$  it must be  $L' \cdot C > 0$ .

*Remark 5.3.* The zero reduction is related to the almost holomorphic map constructed in [BCEKPRSW00], a reduction map for nef line bundles. Actually their map will factor through the zero reduction; this last in fact 'reduces' the curves on which  $L$  is zero and which are extremal rays. In particular on a Fano variety  $X$  a zero reduction is a map as in [BCEKPRSW00].

**5.2. First reduction for quasi polarized pairs.** Let  $(X, L)$  be a quasi polarized pair with at most terminal singularities and let  $(X', L')$  be the zero reduction of  $(X, L)$ . We proceed with a further step in Adjunction theory.

Namely let  $r \geq (n - 2)$  and, as in Lemma 4.1, take  $\Delta'^r$  an effective  $\mathbb{Q}$ -divisor on  $X$  such that:

$\Delta'^r \sim_{\mathbb{Q}} rL'$  and  $(X', \Delta'^r)$  is Kawamata log terminal.

Consider a  $K_{X'} + \Delta'^r$ - Minimal Model Program with scaling as in the first part of Section 4.

$$(X'_0, \Delta'^r_0) = (X', \Delta'^r) \rightarrow (X'_1, \Delta'^r_1) \rightarrow \dots \rightarrow (X'_s, \Delta'^r_s)$$

**Proposition 5.4.** *Under the above notations and assumptions, at every step  $i = 0, \dots, s$ , the morphism  $\varphi_i : X'_i \rightarrow X'_{i+1}$  is the blow-up of a smooth point; in particular  $X'_{i+1}$  has at most terminal singularities.*

*On  $X'_{i+1}$  there exists a nef and big line bundle  $L'_{i+1}$  such that  $\varphi_i^*(L'_{i+1}) = L'_i + E_i$ , where  $E_i \simeq \mathbb{P}^{n-1}$  is the exceptional divisor of the blow-up.*

*In particular  $\varphi_i^*(K_{X_{i+1}} + (n - 1)L_{i+1}) = K_{X_i} + (n - 1)L_i$  and  $H^0(K_{X_i} + tL_i) = H^0(K_{X_{i+1}} + tL_{i+1})$  for any  $t = 0, \dots, r$ .*

*Moreover  $\Delta'^r_i \sim_{\mathbb{Q}} rL'_i$ .*

*Proof.* The proof is by induction on  $i$ . Each map  $\varphi_i : X_i \rightarrow X_{i+1}$  is a birational map associated to an extremal ray  $R_i = \mathbb{R}^+[C_i]$  with

$$(K_{X_i} + rL_i) \cdot C_i = (K_{X_i} + \Delta_i) \cdot C_i < 0.$$

The Proposition will follow directly from Proposition 3.6 if we prove that  $\Delta'_i \cdot R_i = rL'_i \cdot R_i \neq 0$ .

By Corollary 5.2 this is the case for  $i = 0$ . Assume by contradiction that, at a further step  $k$ , we have a ray  $R_k = \mathbb{R}^+[C_k]$  with  $L'_k \cdot C_k = 0$ . At the previous step, by induction,  $\varphi_{k-1} : X'_{k-1} \rightarrow X'_k$  is the blow up at a smooth point  $p$  and  $L'_{k-1} = \varphi_{k-1}^*L'_k + E_k$ . Therefore we have that  $L'_{k-1} \cdot \bar{C} = -E_k \cdot \bar{C}$ , where  $\bar{C}$  is the strict transform of  $C_k$ . Since  $L'_{k-1}$  is nef and  $E_k$  effective this implies that this intersection is zero and  $\bar{C}$  doesn't pass through  $p$ . We have a diagram

$$(5.2.1) \quad \begin{array}{ccc} Bl_p X'_k & & Bl_p X'_{k+1} \\ \varphi_{k-1} \downarrow & & \downarrow \varphi \\ X'_k & \xrightarrow{\varphi_k} & X'_{k+1} \end{array}$$

where the vertical arrows are blow ups at  $p$ . By the universal property of the blow up there exists a map  $Bl_p X'_k \rightarrow Bl_p X'_{k+1}$  closing the diagram. This will be the contraction of the curves numerically equivalent to  $\bar{C}$  and therefore  $\mathbb{R}^+[\bar{C}]$  will be an extremal ray on  $X'_k$ . But it has zero intersection with  $L'_k$ , which is a contradiction.



The proof that  $H^0(K_{X_i} + tL_i) = H^0(K_{X_{i+1}} + tL_{i+1})$  for any  $t = 0, \dots, r$  is similar to the one in Proposition 4.2, it can be find in the proof of Proposition 7.6.1 in [BS95].

As for the last claim recall that  $\Delta'_{i+1} := \varphi_* \Delta'_i$  and  $rL'_{i+1} = (\varphi_*(rL'_i))^*$ . By the inductive assumption, we can assume  $\Delta'_i \sim_{\mathbb{Q}} rL'_i$ , i.e. that there exists an integer  $m$  such that  $m\Delta'_i \in |mrL'_i|$ . Since  $\varphi_i$  is a blow up of a smooth point  $m\Delta'_{i+1}$  is Cartier and  $m\Delta'_{i+1} \in |mrL'_{i+1}|$ , i.e.  $\Delta'_{i+1} \sim_{\mathbb{Q}} rL'_{i+1}$ .

**Definition 5.5.** Let  $(X, L)$  be a quasi polarized variety such that  $X$  has at most terminal singularities. and let  $(X', L')$  be a zero reduction. Let  $(X'_s, L'_s)$  be a quasi polarized pair where  $X'_s$  is the last variety in a  $K'_{X'} + \Delta'^{(n-2)}$ - Minimal Model Program and  $L'_s$  be the corresponding nef and big line bundle on  $X'_s$  coming from Proposition 5.4. Let  $\rho : X' \rightarrow X''$  be the composition  $\rho = \varphi_{s-1} \circ \dots \circ \varphi_0$ . We will call  $(X'', L'') = (X'_s, L'_s)$ , together with a zero reduction  $X'$  and the map  $\rho : X' \rightarrow X''$ , a **first reduction of the pair**  $(X, L)$ .

Using the first reduction we can push adjunction theory a step further.

**Theorem 5.6.** *Let  $(X, L)$  be a quasi polarized variety such that  $X$  has at most terminal singularities.*

- 1)  $K_X + (n-2)L$  is not pseudoeffective if and only if any first reduction  $(X'', L'')$  is one of the pairs in 3.5 A) or B).
- 2) If  $K_X + (n-2)L$  is pseudoeffective then on any first reduction  $(X'', L'')$  the divisor  $K_{X''} + (n-2)L''$  is nef.

*Proof.* The proof is similar to the one of Theorem 5.1. Take a  $K_{X'} + \Delta'^{(n-2)}$ - Minimal Model Program ending in the first reduction  $(X'', L'')$ . If  $K_X + (n-2)L$  is not pseudoeffective then  $(X'', L'')$  is a Mori fiber space and, by Proposition 3.5, we are as in point 1). Otherwise  $K_{X''} + (n-2)L''$  is nef.

*Remark 5.7.* i) The definition of first reduction is in agreement with the Sommesè's definition for the polarized case (see [BS95], section 7.3).

ii) The pairs  $(X, L)$  and  $(X'', L'')$  are not birationally equivalent.

However the morphism  $\rho : X' \rightarrow X''$  is very simple, namely it consists of a series of blow-up at smooth points. It is actually possible to prove other feature of  $\rho$ : for instance that at each stage the smooth point to be blown up has to be either outside the exceptional locus of the previous blow ups or in a component of it isomorphic to  $\mathbb{P}^{n-1}$ . That is the exceptional locus of  $\rho : X' \rightarrow X''$  consist of a finite set of disjoint divisors  $D_j = \cup_{k=0, \dots, s_j} D_j^k$  and the components of  $D_j$  are as follows:  $D_j^0 = \mathbb{P}^{n-1}$  and  $D_j^k = Bl_p(\mathbb{P}^{n-1})$  for  $k > 0$ . Moreover  $D_j^k \cap D_j^{k+1} \neq \emptyset$  and they intersect along a  $\mathbb{P}^{n-1}$  (which is a section of the  $\mathbb{P}^1$ -bundle structure of  $D_j^{k+1}$ ); the other intersections between the components are empty.

We do not give a proof of these facts as they are pretty straightforward.

iii) We could of course run directly a  $K_X + \Delta^{(n-2)}$ - Minimal Model Program with scaling on  $(X, L)$ . In this case, with the help of Proposition 3.6, we have at each step  $i$  two possibilities. Either  $\Delta_i \cdot R_i = 0$ , and we define a nef and big line bundle on  $L_{i+1}$  on  $X_{i+1}$  such that  $\varphi_i^*(L_{i+1}) = L_i$ .

Or  $\Delta_i \cdot R_i \neq 0$ : in this case  $\varphi_i : X_i \rightarrow X_{i+1}$  is the blow-up of a smooth point and we can define a nef and big line bundle  $L_{i+1}$  on  $X_{i+1}$  (with  $\varphi_i^*(L_{i+1}) = L_i + E_i$ , where  $E_i$  is the exceptional divisor of the blow-up).



At the end we will reach a quasi polarized pair  $(X_s, L_s)$  which has the same property of the first reduction  $(X'', L'')$  in the Theorem 5.6.

The above construction, which splits the Program in two parts, namely a first part contracting all rays with zero intersection with the polarization and a second with all the blow up of smooth points, is more accurate and useful.

## 6. APPLICATIONS

Parts 1) and 2) of the next corollary were proved by T. Fujita ([Fu89]), under the assumption of the existence of a Minimal Model for  $X$ , and later by A. Höring ([Ho10]).

**Corollary 6.1.** *Let  $(X, L)$  be a quasi polarized variety.*

- 1) ([Ho10])  $g(X, L) \geq 0$
- 2) ([Ho10])  $g(X, L) = 0$  if and only if  $(X, L)$  is birational equivalent to one of the following quasi polarized pairs:

- $(\mathbb{P}^n, \mathcal{O}(1))$ , or
- $(Q, \mathcal{O}(1)|_Q)$ , where  $Q \subset \mathbb{P}^{n+1}$  is a quadric, or
- $C_n(\mathbb{P}^2, \mathcal{O}(2))$ , a generalized cone over  $(\mathbb{P}^2, \mathcal{O}(2))$ , or
- $X$  has the structure of a  $\mathbb{P}^{n-1}$ -bundle over a smooth rational curve  $C$  and  $L$  restricted to any fiber is  $\mathcal{O}(1)$  (a scroll over a rational curve).

- 3) )If  $X$  normal then  $g(X, L) = 1$  if and only if  $(X, L)$  is birational equivalent to one of the following quasi polarized pairs:

- a del Pezzo variety, i.e.  $-K_{X'} \sim_{\mathbb{Q}} (n-1)L'$  with  $L'$  ample,
- $X'$  has the structure of a  $\mathbb{P}^{n-1}$ -bundle over an elliptic curve  $C$  and  $L'$  restricted to any fiber is  $\mathcal{O}(1)$  (a scroll over an elliptic curve).

*Proof.* Let  $\nu : X' \rightarrow X$  be the normalization of  $X$ ; it is straightforward to see that  $g(X', \nu^*L) \leq g(X, L)$  (see for instance [Ho10], p. 128). Therefore we can assume that  $X$  is normal also in 1) and 2).

By the Lemma 1.8 in [Fu89] the sectional genus is a birational invariant of normal quasi polarized pairs, so we can replace  $(X, L)$  first with its resolution and then with its zero reduction. Call this new pair  $(X', L')$ .

By the Theorem 5.1 if  $K'_X + (n-1)L'$  is not nef then  $(X', L')$  is one of the pair in 3.5 A). They give the first three cases in 2) and the case in which  $(X, L)$  is a scroll over a curve  $C$ . In this last  $g(X, L) = g(C)$  and we get the fourth case in 2) and the second in 3).

We can thus assume that  $K'_X + (n-1)L'$  is nef; therefore  $2g(X', L') - 2 = (K'_X + (n-1)L')L'^{n-1} \geq 0$ , i.e  $g(X, L) \geq 1$ .

Assume that  $g(X', L') = 1$ . By the previous equality, the facts that  $(K'_X + (n-1)L')$  is nef and  $L'$  is nef and big, we get that  $K'_X + (n-1)L'$  is numerically trivial. It is straightforward to see that  $K'_X + (n-1)L'$  is effective (see for instance [Fu89], p. 115). Therefore  $K'_X + (n-1)L'$  is trivial and we are in the first case of 3).

The following application extends the main result in [Io85] from the case of smooth embedded varieties to the singular ones.

**Corollary 6.2.** *Let  $X \subset \mathbb{P}^N$  be a projective variety of dimension  $n \geq 3$  and of degree  $d$ . Assume that  $d < 2\text{codim}_{\mathbb{P}^N}(X) + 2$  (equivalently that  $d > 2\Delta(X, \mathcal{O}_X(1))$ ). Then either  $(X, \mathcal{O}(1))$  is birational equivalent to one of the quasi polarized pair in*

*Proposition 3.5 A) or the first reduction of the resolution of  $X$  is one of the quasi polarized variety in Proposition 3.5 B).*

*Proof.* Let  $\pi : \tilde{X} \rightarrow X$  be a resolution of the singularities of  $X$ : let also  $\tilde{L} := \pi^*L$ .  $\tilde{L}$  is globally generated and  $h^0(\tilde{X}, \tilde{L}) \geq N + 1$ . Take  $L_1, \dots, L_{n-1}$  general members in  $|\tilde{L}|$  such that

$$X_i := L_1 \cap \dots \cap H_i, \quad i = 1, \dots, n-1$$

is irreducible, smooth and of dimension  $n - i$ ; let also  $X_0 := \tilde{X}$ . For each  $i = 1, \dots, n-2$ , from the exact sequence

$$0 \rightarrow \mathcal{O}_{X_i} \rightarrow \tilde{L}_{X_i} \rightarrow \tilde{L}_{X_{i+1}} \rightarrow 0$$

we find that  $\dim|\tilde{L}_{X_{i+1}}| \geq \dim|\tilde{L}_{X_i}| - 1$ , therefore  $\dim|\tilde{L}_{X_i}| \geq N - i$ . In particular for the smooth curve  $X_{n-1} := C$  we have:

$$\deg(\tilde{L}_C) - 2\dim|\tilde{L}_C| \leq d - 2(N - (n-1)) = d - 2\text{codim}_{\mathbb{P}^N}(X) - 2 < 0.$$

Thus, by Clifford's theorem, we must have  $h^1(\tilde{L}_C) = 0$ , which gives

$$\chi(\tilde{L}_C) = h^0(\tilde{L}_C) \geq N - (n-1) + 1 = N - n + 2.$$

Therefore on the smooth surface  $X_{n-2} := S$ , by the Riemann-Roch theorem and the short exact sequence above with  $i = n-2$ , we get

$$K_S \cdot C = C^2 - 2(\chi(\tilde{L}_S) - \chi(\mathcal{O}_S)) = C^2 - 2\chi(\tilde{L}_C) \leq d - 2(N - n + 2) < -2.$$

By adjunction this implies that  $(K_{\tilde{X}} + (n-2)\tilde{L}) \cdot C = K_S \cdot C < -2$ .

By the Theorem 0.2 in [BDPP04] and the definition of  $C$  we have that  $(K_{\tilde{X}} + (n-2)\tilde{L})$  is not pseudoeffective.

The Corollary follows then by Theorem 5.6 1).

#### REFERENCES

- [An94] M. Andreatta. Contractions of Gorenstein polarized varieties with high nef value, *Math. Ann.*, volume 300, 1994, p. 669-679.
- [An95] M. Andreatta. Some remarks on the study of good contractions, *Manuscripta Math.*, volume 87, 1995, p. 359-367.
- [AS89] M. Andreatta, A.J. Sommese. Generically ample divisors on normal Gorenstein surfaces, *Contemporary Mathematics*, volume 90, 1989, p. 1-19.
- [BCEKPRSW00] T. Bauer, F. Campana, T. Eckl, S. Kebekus, T. Peternell, S. Rams, T. Szemberg, L. Wotzlaw. A reduction map for nef line bundles, *Complex geometry (Gttingen, 2000)*, Springer- Berlin, 2002, p. 27 - 36.
- [BS95] M. Beltrametti, A.J. Sommese. *The Adjunction Theory of Complex Projective Varieties*, volume 16 of *Expositions in Mathematics*. De Gruyter, Berlin-New York, 1995.
- [BCHM10] C. Birkar, P. Cascini, C. Hacon, J. McKernan. Existence of minimal models for varieties of log general type, *J. Amer. Math. Soc.*, volume 23 n. 2, 2010, p. 405-468.
- [BDPP04] S. Boucksom, J.P. Demailly, M. Paun, T. Peternell. The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension., *arXiv:math/0405285*, May 2004.
- [CD11] S. Cacciola, L. Di Biagio. Asymptotic base loci on singular varieties, *arXiv:math/1105.1253*, May 2011.
- [Fu89] T. Fujita. Remarks on quasi-polarized varieties, *Nagoya Math. J.* Vol. 115, 1989, 105-123.
- [Ho10] A. Höring. The sectional genus of quasi-polarized varieties, *Arch. Math.* vol. 95, 2010, 125133.
- [Io85] P. Ionescu. On varieties whose degree is small with respect to codimension, *Math. Ann.* vol. 271, 1985, 339-348.

- [La04] *Positivity in algebraic geometry. II*, volume 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge*. Springer-Verlag, Berlin, 2004. Positivity for vector bundles, and multiplier ideals.
- [KM98] János Kollár, Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI TRENTO, I-38050 POVO (TN)  
*E-mail address:* marco.andreatta@unitn.it