ON p-ADIC GIBBS MEASURES FOR HARD CORE MODEL ON A CAYLEY TREE

D. GANDOLFO, U. A. ROZIKOV, J. RUIZ

ABSTRACT. In this paper we consider a nearest-neighbor p-adic hard core (HC) model, with fugacity λ, on a homogeneous Cayley tree of order k (with $k+1$ neighbors). We focus on p-adic Gibbs measures for the HC model, in particular on p-adic "splitting" Gibbs measures generating a p-adic Markov chain along each path on the tree. We show that the p-adic HC model is completely different from real HC model: For a fixed k we prove that the p-adic HC model may have a splitting Gibbs measure only if p divides 2^k-1. Moreover if p divides 2^k-1 but does not divide $k+2$ then there exists unique translational invariant p-adic Gibbs measure. We also study p-adic periodic splitting Gibbs measures and show that the above model admits only translational invariant and periodic with period two (chess-board) Gibbs measures. For $p \geq 7$ (resp. $p = 2, 3, 5$) we give necessary and sufficient (resp. necessary) conditions for the existence of a periodic p-adic measure. For $k = 2$ a p-adic splitting Gibbs measures exists if and only if $p = 3$, in this case we show that if λ belongs to a p-adic ball of radius $1/27$ then there are precisely two periodic (non translational invariant) p-adic Gibbs measures. We prove that a p-adic Gibbs measure is bounded if and only if $p \neq 3$.

Mathematics Subject Classifications (2010). 46S10, 82B26, 12J12 (primary); 60K35 (secondary)

Key words. Cayley trees, hard core interaction, Gibbs measures, translation invariant measures, periodic measures, splitting measures, p-adic numbers.

1. INTRODUCTION

In [40] a hard core (HC) model with nearest neighbor interaction and spin values $0, 1$ on a Cayley tree was studied. In this paper we consider p-adic version of this model.

One of the central problems in the theory of Gibbs measures of lattice systems is to describe infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian. A complete analysis of this set is often a difficult problem. Many papers have been devoted to these studies when the underlying lattice is a Cayley tree [5, 7, 8, 10, 11, 13, 24, 30, 32, 34, 40, 45].

In all these works the models under consideration have a finite set of spin values on the field of real numbers. These models have the following common property: the existence of finitely many translation-invariant and uncountable numbers of non-translation-invariant extreme Gibbs measures. Also for several models it was proved that there exist periodic Gibbs measures (which are invariant with respect to normal subgroups of finite index of the group representation of Cayley tree) and that there are uncountable number of non-periodic Gibbs measures.

On the other hand, various models described in the language of p-adic analysis have been actively studied, see e.g. [3,9,23,42] and numerous applications of p-adic analysis to mathematical physics have been proposed in [4,16–18]. Well-known studies in this area were devoted to quantum mechanical models [41,43]. One of the first applications of p-adic numbers in quantum physics appeared in the framework of quantum logic [6]. This model is of special interest to us because it cannot be described by using conventional real-valued probability.

It is also known [17,21,23,41] that a number of p-adic models in physics cannot be described using ordinary Kolmogorov's probability theory. In [20] an abstract p-adic probability theory was developed by means of the theory of non-Archimedean measures [31].

A non-Archimedean analog of the Kolmogorov theorem was proved in [12]. Such a result allows to construct wide classes of stochastic processes and the possibility to develop statistical mechanics in the context of p-adic theory.

We refer the reader to [14], [19], [26], [27]- [29] where various models of statistical physics in the context of p-adic field are studied.

In the present paper we consider p-adic Gibbs measures of a hard core model on the Cayley tree over the p-adic field (we refer the reader to [40] for the real case).

The paper is organized as follows. Section 2 presents definitions and known results. Section 3 is devoted to the standard construction of (p-adic) Gibbs measures characterized by a functional equation. Section 4 contains conditions of solvability of this equation. Under conditions on p and on the degree k of the tree, we prove in Section 5 the existence and uniqueness of translational-invariant p-adic Gibbs measure. In Section 6 we study p-adic periodic Gibbs measures and show that the HC model admits only translational invariant and periodic with period two (chess-board) Gibbs measures. For $k=2$ a p-adic splitting Gibbs measures exists if and only if $p=3$, in this case we show that if λ belongs to a p-adic ball of radius $1/27$ then there are precisely two periodic (non translational invariant) p-adic Gibbs measures. In Section 7 we prove that a p-adic Gibbs measure is bounded if and only if $p\neq 3$. In the last section devoted to concluding remarks, we present comparisons between real and p-adic Gibbs measures.

2. Preliminaries

2.1. **p-adic numbers and measures.** Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p, every rational number $x\neq 0$ can be represented in the form $x=p^r\frac{n}{m}$, where $r,n\in\mathbb{Z}$, m is a positive integer, and n and m are relatively prime with p: $(p,n)=1$, $(p,m)=1$. The p-adic norm of x is given by

$$|x|_p=\begin{cases} p^{-r} & \text{for } \ x\neq 0\\ 0 & \text{for } \ x=0.\end{cases}$$

This norm is non-Archimedean and satisfies the so called strong triangle inequality

$$|x+y|_p\leq \max\{|x|_p,|y|_p\}.$$

We will often use the following fact: If $|x|_p \neq |y|_p$ then

$$|x+y|_p = \max\{|x|_p, |y|_p\}.$$

The completion of $\mathbb{Q}$ with respect to the p-adic norm defines the p-adic field $\mathbb{Q}_p$. Any p-adic number $x \neq 0$ can be uniquely represented in the canonical form

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \dots), \tag{2.1}$$

where $\gamma(x) \in \mathbb{Z}$ and the integers x_j satisfy: $x_0 > 0$, $0 \leq x_j \leq p-1$ (see [21, 36, 41]). In this case $|x|_p = p^{-\gamma(x)}$.

Theorem 2.2. *[21], [41] The equation $x^2 = a$, $0 \neq a = p^{\gamma(a)}(a_0 + a_1 p + \dots), 0 \leq a_j \leq p-1$, $a_0 > 0$ has a solution $x \in \mathbb{Q}_p$ if and only if the following conditions are fulfilled:*
i) $\gamma(a)$ is even;
ii) a_0 is a quadratic residue modulo p if $p \neq 2$; $a_1 = a_2 = 0$ if $p = 2$.

The elements of the set $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ are called p-adic integers.
The following statement is known as Hensel's lemma [21].

Theorem 2.3. *Let $F(x) = \sum_{i=0}^{n} c_i x^i$ be a polynomial whose coefficients are p-adic integers. Let $F'(x) = \sum_{i=0}^{n} i c_i x^{i-1}$ be the derivative of $F(x)$. Let a_0 be a p-adic integer such that $F(a_0) \equiv 0 \,(\operatorname{mod} p)$ and $F'(a_0) \not\equiv 0 \,(\operatorname{mod} p)$. Then there exists a unique p-adic integer a such that $F(a) = 0$ and $a = a_0 \,(\operatorname{mod} p)$.*

Given $a \in \mathbb{Q}_p$ and $r > 0$ put

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}.$$

The p-adic *logarithm* is defined by the series

$$\log_p(x) = \log_p(1 + (x-1)) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n},$$

which converges for $x \in B(1, 1)$; the p-adic exponential is defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, p^{-1/(p-1)})$.

Lemma 2.4. *[21, 36]. Let $x \in B(0, p^{-1/(p.1)}$, then*

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p, \quad |\log_p(1+x)|_p = |x|_p,$$

$$\log_p(\exp_p(x)) = x, \quad \exp_p(\log_p(1+x)) = 1 + x.$$

We refer the reader to [21, 36, 41] for the basics of p-adic analysis and p-adic mathematical physics.

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets of X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is said to be a p-adic measure if for any $A_1, ..., A_n \in \mathcal{B}$ such that $A_i \cap A_j = \emptyset$, $i \neq j$, the following holds:

$$\mu(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure is called a probability measure if $\mu(X) = 1$, see, e.g. [15], [31].

2.5. **Cayley tree.** The Cayley tree (Bethe lattice [5]) Γ^k of order $k \geq 1$ is an infinite tree, i.e., a graph without cycles, such that exactly $k+1$ edges originate from each vertex. Let $\Gamma^k = (V, L)$ where V is the set of vertices and L the set of edges. Two vertices x and y are called *nearest neighbors* if there exists an edge $l \in L$ connecting them. We will use the notation $l = \langle x, y \rangle$. A collection of nearest neighbor pairs $\langle x, x_1 \rangle, \langle x_1, x_2 \rangle, ..., \langle x_{d-1}, y \rangle$ is called a *path* from x to y. The distance $d(x, y)$ on the Cayley tree is the number of edges of the shortest path from x to y.

For a fixed $x^0 \in V$, called the root, we set

$$W_n = \{x \in V | d(x, x^0) = n\}, \qquad V_n = \bigcup_{k=1}^{n} W_k$$

and denote

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in W_n,$$

the set of *direct successors* of x.

2.6. **Hard Core model.** We consider HC model with nearest neighbor interactions on a Cayley tree where the spins assigned to the vertices of the tree take values in the set $\Phi := \{0, 1\}$. A configuration σ on $A \subset V$ is then defined as a function $x \in A \mapsto \sigma(x) \in \Phi$. The set of all configurations is Φ^A. A site x is called "occupied" if $\sigma(x) = 1$ and "vacant" if $\sigma(x) = 0$.

A configuration is called *admissible* if the product $\sigma(x)\sigma(y) = 0$ for all nearest neighbor pair $\langle x, y \rangle$. We denote Ω_A the set of all admissible configurations on $A \subset V$ and set $\Omega = \Omega_V$.

Let p be a fixed prime number. The (formal) p-adic Hamiltonian of the HC model is the mapping $H : \Omega \to \mathbb{Q}_p$ given by

$$H(\sigma) = -J \sum_{x \in V} \sigma(x), \tag{2.2}$$

where $J \in \mathbb{Q}_p$ is a constant such that

$$|J|_p < p^{-1/(p-1)}. \tag{2.3}$$

Note that such a condition provides the existence of a p-adic Gibbs measure defined through the p-adic exponential. As it was mentioned above the set of values of a p-adic norm $|\cdot|_p$ is $\{p^m : m \in \mathbb{Z}\}$, consequently the condition (2.3) is equivalent to the condition $|J|_p \leq \frac{1}{p}$.

3. Construction of p-adic Gibbs measure

Let us construct p-adic Gibbs measures of this HC model. Since we use $\exp_p(x)$ to define the p-adic Gibbs measure, all quantities which arise below must belong to the set:

$$\mathcal{E}_p = \{x \in Q_p : |x|_p = 1, |x-1|_p < p^{-1/(p-1)}\}.$$

As in classical (real) case we consider a special class of Gibbs measures. We call them p-adic splitting Gibbs measures, a formal definition follows.

Write $x < y$ if the pathes from x^0 to y go through x. By this notation a vertex y is a direct successor of x if $y > x$ and x, y are nearest neighbors. Note that the root x^0 has $k+1$ direct successors and any vertex $x \neq x^0$ has k direct successors.

Let $z : x \to z_x = (z_{0,x}, z_{1,x}) \in \mathcal{E}_p^2$ be a vector-valued function on V, we will consider the p-adic probability measures on Ω_{V_n} defined by

$$\mu^{(n)}(\sigma_n) = \frac{1}{Z_n} \exp_p\left(J \sum_{x \in V_n} \sigma(x)\right) \prod_{x \in W_n} z_{\sigma(x),x} \tag{3.1}$$

where Z_n is the corresponding partition function:

$$Z_n = \sum_{\varphi \in \Omega_{V_n}} \exp_p\left(J \sum_{x \in V_n} \varphi(x)\right) \prod_{x \in W_n} z_{\varphi(x),x}. \tag{3.2}$$

Let us mention that function z plays the role of a generalized boundary condition.

One of the central results of probability theory concerns the construction of an infinite-volume distribution with given finite-dimensional distributions. In this paper we consider this problem in p-adic context. More precisely, we want to define a p-adic probability measure μ on the set Ω of admissible configurations. In general, the existence of such a measure is not known, since there is not enough information on the topological properties of the set of all p-adic measures defined even on compact spaces. Therefore, we can only use the p-adic Kolmogorov extension theorem (see [12, 22]) based on the so-called compatibility condition.

We say that the p-adic probability measures $\mu^{(n)}$ are compatible if for all $n \geq 1$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$:

$$\sum_{\omega_n \in \Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \mathbf{1}(\sigma_{n-1} \vee \omega_n \in \Omega_{V_n}) = \mu^{(n-1)}(\sigma_{n-1}). \tag{3.3}$$

where the symbol $\vee$ denotes concatenation of configurations.

This condition implies the existence of a unique p-adic measure μ defined on Ω such that, for all n and $\sigma_n \in \Omega_{V_n}$, $\mu(\{\sigma|_{V_n} = \sigma_n\}) = \mu^{(n)}(\sigma_n)$.

Definition 3.1. *Measure μ defined by (3.1), (3.3) is called a p-adic splitting (hard core) Gibbs measure, corresponding to the function z.*

The following statement describes conditions on the function z that ensure compatibility of measures $\mu^{(n)}$.

Proposition 3.2. *Probability measures $\mu^{(n)}, n = 1, 2, ...,$ in (3.1) are compatible iff for any $x \in V$ the following equation holds:*

$$z'_x = \prod_{y \in S(x)} \frac{\lambda + z'_y}{z'_y}, \tag{3.4}$$

here $z'_x = \frac{z_{0,x}}{z_{1,x}}$, $\lambda = \exp_p(J)$.

Proof. The proof consists in checking condition (3.3) for the measures (3.1). It is analogous to that of Proposition 2.1 in [40]. □

Without loss of generality, we set hereafter $z_{1,x} = 1$ and $z_x = z'_x = z_{0,x} \in \mathcal{E}_p$. Then condition (3.4) reads

$$z_x = \prod_{y \in S(x)} \frac{\lambda + z_y}{z_y}. \tag{3.5}$$

4. Conditions of solvability of equation (3.5)

In this section, we examine the conditions on the parameters $k \geq 1$, p and λ for the existence of solutions of equation (3.5). Notice that by Lemma 2.4 we have $\lambda = \exp_p(J) \in \mathcal{E}_p$.

Theorem 4.1. *If p does not divide $2^k - 1$, then the equation (3.5) has no solution $z_x \in \mathcal{E}_p$, $x \in V$.*

Proof. Let $z_x \in \mathcal{E}_p$, $x \in V$ be a solution then from (3.5) we get

$$|z_x|_p = \prod_{y \in S(x)} \left| \frac{\lambda + z_y}{z_y} \right|_p = \prod_{y \in S(x)} |\lambda + z_y|_p =$$

$$\prod_{y \in S(x)} |\lambda - 1 + z_y - 1 + 2|_p = \begin{cases} 1, & \text{if } \ p \neq 2, \\ < p^{-k/(p-1)}, & \text{if } \ p = 2. \end{cases}$$

We shall use the following (see Lemma 4.6 of [27])

Lemma 4.2. *If $a_i \in \mathcal{E}_p$ for all $i = 1, \ldots, m$. Then*

$$\prod_{i=1}^{m} a_i \in \mathcal{E}_p.$$

Assume $S(x) = \{x_1, \ldots, x_k\}$ then from (3.5) we get

$$|z_x - 1|_p = \left| \prod_{i=1}^{k} \frac{\lambda + z_{x_i}}{z_{x_i}} - 1 \right|_p =$$

$$\left| \prod_{i=1}^{k} (\lambda + z_{x_i}) - \prod_{i=1}^{k} z_{x_i} \right|_p =$$

$$\left|\lambda\sum_{i=1}^{k}\prod_{\substack{j=1\\ j\neq i}}^{k} z_{x_j}+\lambda^2\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\prod_{\substack{q=1\\ q\neq i,j}}^{k} z_{x_q}+\cdots+\lambda^{k-1}\sum_{i=1}^{k} z_{x_i}+\lambda^k\right|_p=$$

$$\left|\sum_{i=1}^{k}\left(\lambda\prod_{\substack{j=1\\ j\neq i}}^{k} z_{x_j}-1\right)+\sum_{i=1}^{k}\sum_{\substack{j=1\\ j\neq i}}^{k}\left(\lambda^2\prod_{\substack{q=1\\ q\neq i,j}}^{k} z_{x_q}-1\right)+\dots\right.$$
$$\left.+\sum_{i=1}^{k}\left(\lambda^{k-1}z_{x_i}-1\right)+\left(\lambda^k-1\right)+\left(2^k-1\right)\right|_p. \tag{4.1}$$

Now using Lemma 4.2, we get from (4.1)

$$\text{RHS of } (4.1)=\begin{cases}1, & \text{if } p\nmid 2^k-1,\\ <p^{-1/(p-1)}, & \text{if } p\mid 2^k-1.\end{cases}$$

Thus the solution z_x belongs to $\mathcal{E}_p$ iff p divides 2^k-1. □

Using this theorem, for given $k\geq 1$, one can find values of the prime number p for which the equation (3.5) may have solutions (see the following table for small values of k).

k	p
1	$\emptyset$
2	3
3	7
4	3, 5
5	31
6	3, 7
7	127
8	3, 5, 17
9	7, 73
10	3, 11, 31

The following theorem gives a sufficient condition for the existence of a solution.

Theorem 4.3. *If p divides 2^k-1 and p does not divide $k+2$ then the equation (3.5) has at least one solution $z_x\in\mathcal{E}_p$, $x\in V$.*

Proof. We shall prove that under conditions of theorem the equation (3.5) has a constant (translational-invariant) solution $z_x=z$, $\forall x\in V$. In this case from (3.5) we get

$$z=\left(\frac{\lambda+z}{z}\right)^k. \tag{4.2}$$

This equation can be written as $F(z) = 0$ with $F(z) = z^{k+1} - (\lambda + z)^k$. Since $|\lambda|_p = 1$ we have that the polynomial $F(z)$ has only p-adic integer coefficients. Hence we shall check other conditions of Hensel's lemma. Take $a_0 = 1$ then we have

$$F(1) = -(\lambda^k + k\lambda^{k-1} + \cdots + k\lambda) = -\Big((\lambda^k - 1) + k(\lambda^{k-1} - 1) + \cdots + k(\lambda - 1) + (2^k - 1)\Big).$$

Since $|\lambda - 1|_p \le \frac{1}{p}$, using Lemma 4.2 we get $|\lambda^m - 1|_p \le \frac{1}{p}$ for each $m = 1, 2, \ldots$
This means that p divides all $\lambda^m - 1$ and also divides $2^k - 1$. Consequently p divides $F(1)$, i.e. $F(1) \equiv 0 \,(\mathrm{mod}\, p)$. Now let us check that if p does not divide $k + 2$, then $F'(1) \ne 0 \,(\mathrm{mod}\, p)$. We have

$$F'(1) = k + 1 - k(\lambda + 1)^{k-1} = k + 1 - k\left((\lambda - 1) + 2\right)^{k-1} =$$

$$k + 1 - k\Big((\lambda - 1)^{k-1} + (k-1)(\lambda - 1)^{k-2}2 + \cdots + (k-1)(\lambda - 1)2^{k-2} + 2^{k-1}\Big).$$

Since p divides $\lambda - 1$ we must have $p \nmid (k + 1 - k2^{k-1})$. Using $p|2^k - 1$ one can see that $p \nmid (k + 1 - k2^{k-1})$ is equivalent to $p \nmid (k + 2)$. Thus conditions of Hensel's lemma are satisfied for $F(z)$ hence there exists a unique p-adic integer a such that $F(a) = 0$ and $a \equiv a_0 \,(\mathrm{mod}\, p)$, i.e. $F(z) = 0$ has a solution $z = a$. Since $a_0 = 1$ and $a \equiv 1 \,(\mathrm{mod}\, p)$ we conclude $a \in \mathcal{E}_p$. This proves the theorem. □

As a corollary of this Theorem, we have the following

Theorem 4.4. *If p divides $2^k - 1$ and p does not divide $k + 2$ then for the p-adic HC model on Cayley tree of order $k \ge 1$ there exists at least one p-adic (splitting) Gibbs measure.*

5. Uniqueness of translational-invariant measure

In the previous section under conditions of Theorem 4.3 we have shown that equation (4.2) has at least one solution. Consequently by Proposition 3.2 there exists at least one translational invariant p-adic Gibbs measure. The following theorem asserts that such a measure is unique.

Theorem 5.1. *If p divides $2^k - 1$ and p does not divide $k + 2$ then there exists a unique translational invariant p-adic Gibbs measure.*

Proof. We shall prove that the equation (4.2) has a unique solution $z = a \in \mathcal{E}_p$. Assume that there are two such solutions a and b, $a \ne b$. Then we have $F(a) = F(b) = 0$. Hence

$$F(a) - F(b) = (a - b)\Big((a^k + a^{k-1}b + \cdots + b^k) -$$

$$((a + \lambda)^{k-1} + (a + \lambda)^{k-2}(b + \lambda) + \cdots + (b + \lambda)^{k-1})\Big) = 0.$$

Since $a \ne b$ we get

$$a^k + \cdots + b^k = (a + \lambda)^{k-1} + \cdots + (b + \lambda)^{k-1}$$

which can be written

$$(a^k - 1) + \cdots + (b^k - 1) + k + 1 = [(a-1) + (\lambda - 1)]^{k-1} + \cdots + (k-1)[(a-1) + (\lambda - 1)]2^{k-2} +$$

$$[(b-1)+(\lambda-1)]^{k-1}+\cdots+(k-1)[(b-1)+(\lambda-1)]2^{k-2}+k2^{k-1}.$$

Consequently,

$$2(a^k-1)+\cdots+2(b^k-1)-2[(a-1)+(\lambda-1)]^{k-1}-\cdots-2(k-1)[(a-1)+(\lambda-1)]2^{k-2}-$$
$$2[(b-1)+(\lambda-1)]^{k-1}-\cdots-2(k-1)[(b-1)+(\lambda-1)]2^{k-2}=k(2^k-1)-(k+2).$$

This equality is not satisfied for any $a,b\in\mathcal{E}_p$ since the p-adic norm of its LHS is $\leq\frac{1}{p}$ while the p-adic norm of the RHS is 1. □

6. Periodic p-adic measures

In this section, we shall consider periodic measures and use the group structure of the Cayley tree. It is known (see [11]) that there exists a one-to-one correspondence between the set of vertices V of a Cayley tree of order $k\geq 1$ and the group G_k, free product of $k+1$ second-order cyclic groups with generators $a_1,a_2,...,a_{k+1}$.

Definition 6.1. *Let $\tilde{G}$ be a normal subgroup of the group G_k. The set $z=\{z_x : x\in G_k\}$ is said to be $\tilde{G}$-periodic if $z_{yx}=z_x$ for any $x\in G_k$ and $y\in\tilde{G}$.*

Definition 6.2. *The (p-adic) Gibbs measure corresponding to a $\tilde{G}$-periodic set of quantities z is said to be $\tilde{G}$-periodic.*

It is easy to see that a G_k-periodic measure is translational invariant. Denote

$$G^{(2)}=\{x\in G_k:\ \text{the length of word}\, x\, \text{is even}\}.$$

This set is a normal subgroup of index two [11, 32].

The following theorem characterizes the set of all periodic measures.

Theorem 6.3. *Let $\tilde{G}$ be a normal subgroup of finite index in G_k. Then each $\tilde{G}$- periodic p-adic Gibbs measure for HC model is either translation-invariant or $G^{(2)}$- periodic.*

Proof. Denote $f(z)=\frac{\lambda+z}{z}$. It is easy to check that $f(z)=f(t)$ if and only if $z=t$. This property together with arguments similar to the ones given in the proof of Theorem 2 in [24] lead to the statement. □

Let $\tilde{G}$ be a normal subgroup of finite index in G_k. Let us state condition on $\tilde{G}$ for each $\tilde{G}$-periodic p-adic Gibbs measure to be translation invariant.
Set $I(\tilde{G})=\tilde{G}\cap\{a_1,...,a_{k+1}\}$, where the a_i are the generators of G_k.

Theorem 6.4. *If $I(\tilde{G})\neq\emptyset$ then each $\tilde{G}$- periodic p-adic Gibbs measure is translational-invariant.*

Proof. Similar to proof of Theorem 3 in [24]. □

By Theorems 6.3 and 6.4, the description of a $\tilde{G}$-periodic p-adic Gibbs measure for $I(\tilde{G})\neq\emptyset$ reduces to finding that of fixed points of the map $(f(z))^k$ (these fixed points correspond to translational invariant p-adic Gibbs measures).
For $I(\tilde{G})=\emptyset$, it reduces to the solutions of system (6.1) below. This system describes

periodic measures with period two, more precisely, $G^{(2)}$-periodic p-adic measures. They correspond to functions

$$z_x = \begin{cases} z_1, & \text{if } x \in G^{(2)}, \\ z_2, & \text{if } x \in G_k \setminus G^{(2)}. \end{cases}$$

In this case, we have from (3.5):

$$z_1 = \left(\frac{z_2+\lambda}{z_2}\right)^k, \quad z_2 = \left(\frac{z_1+\lambda}{z_1}\right)^k. \tag{6.1}$$

Namely, z_1 and z_2 satisfy

$$z = g(g(z)), \quad \text{where} \ \ g(z) = ((z+\lambda)/z)^k. \tag{6.2}$$

Note that to get periodic (non translational invariant) measure we must find solutions $z_1, z_2 \in \mathcal{E}_p$ of (6.1) with $z_1 \neq z_2$. Obviously, such solutions are roots of the equation

$$\frac{g(g(z))-z}{g(z)-z} = 0, \tag{6.3}$$

which is equivalent to the equation

$$\frac{L(z)}{M(z)} = 0, \ \text{ with } \ L(z) = \left(\lambda z^k + (\lambda+z)^k\right)^k - z(\lambda+z)^{k^2}; \ \ M(z) = (\lambda+z)^k - z^{k+1}. \tag{6.4}$$

We have

$$L(z) = \left((\lambda+z)z^k + M(z)\right)^k - z\left(z^{k+1}+M(z)\right)^k = (\lambda+z)^k z^{k^2} +$$

$$\sum_{i=1}^{k}\binom{k}{i}M^i(z)((\lambda+z)z^k)^{k-i} - z^{k^2+k+1} - z\sum_{j=1}^{k}\binom{k}{j}M^j(z)(z^{k+1})^{k-j} = M(z)U(z),$$

where

$$U(z) = (1-k)z^{k^2} + k\left((\lambda+z)z^k\right)^{k-1} +$$

$$\sum_{i=2}^{k}\binom{k}{i}M^{i-1}(z)z^{k(k-i)}\left((\lambda+z)^{k-i} - z^{k-i+1}\right).$$

Hence in order to get $G^{(2)}$-periodic (not translation invariant) solutions of (6.1) we must find solutions of equation $U(z) = 0$. Conditions for existence of such solutions are given in the following theorem.

Theorem 6.5. *Let $p \neq 3, 5$. The equation $U(z) = 0$ has a solution $z \in \mathcal{E}_p$ if and only if p divides $2^k - 1$ and p divides $k - 2$.*

Proof. Necessity: from above, it follows that if p divides $2^k - 1$ then $|M(z)|_p \leq \frac{1}{p}$. The function $U(z)$ can be written as

$$U(z) = (1-k)(z^{k^2}-1) + k\left((\lambda-1+z-1)z^k\right)^{k-1} + k(2^{k-1}-1) + 1 +$$

$$\sum_{i=2}^{k}\binom{k}{i}M^{i-1}(z)z^{k(k-i)}\left((\lambda+z)^{k-i}-z^{k-i+1}\right).$$

Since p divides $M(z)$, it must divide $(k(2^{k-1}-1)+1)$. By using $p \mid 2^k-1$ one can see that $p \mid (k(2^{k-1}-1)+1)$ is equivalent to $p \mid (k-2)$.

Sufficiency: Since $|\lambda|_p=1$, the polynomial $U(z)$ has only p-adic integer coefficients. Hence we shall check other conditions of Hensel's lemma. Take $a_0=1$ then it is easy to see that $U(1)\equiv 0\,(\mathrm{mod}\,p)$. Now we shall check that $U'(1)\neq 0\,(\mathrm{mod}\,p)$. We have

$$U'(1)=(1-k)k^2+k(k-1)(k\lambda+k+1)(\lambda+1)^{k-1}+$$
$$\frac{k(k-1)}{2}\left(k(\lambda+1)^{k-1}-(k+1)\right)\left((\lambda+1)^{k-2}-1\right)+pN,$$

where $N\in\mathbb{N}$. Now using hypothesis of theorem, we get

$$U'(1)=15+pN_1,\quad N_1\in\mathbb{N}.$$

Hence if $p\neq 3,5$ all conditions of Hensel's lemma are satisfied. This completes the proof. □

For given $k\geq 1$, one can easily find values of prime number p for which the equation $U(z)=0$ has a solution (see the following table for small values of k)

k	p
1	$\emptyset$
2	3
3	$\emptyset$
4	$\emptyset$
5	$\emptyset$
6	$\emptyset$
7	$\emptyset$
8	3
9	7
10	$\emptyset$

Now we are going to give all $G^{(2)}$-periodic solutions for $k=2$. In this case the equation (6.3) has the following form:

$$z^2-(\lambda^2-2\lambda)z+\lambda^2=0. \tag{6.5}$$

The solutions of this quadratic equation are

$$z_{1,2}=\frac{\lambda}{2}\left(\lambda-2\pm\sqrt{\lambda(\lambda-4)}\right). \tag{6.6}$$

We must check the existence of $\sqrt{\lambda(\lambda-4)}$ and additionally that $z_{1,2}\in\mathcal{E}_p$. For $k=2$, following Theorem 4.1, only the case $p=3$ has to be considered. Since $\lambda\in\mathcal{E}_3$ we have its following representation

$$\lambda = 1 + \lambda_1 \cdot 3 + \lambda_2 \cdot 3^2 + \lambda_3 \cdot 3^3 + \cdots .$$

It is easy to see that λ satisfies hypothesis of Theorem 2.2. Hence $\sqrt{\lambda}$ exists in $\mathbb{Q}_3$. So we must check the existence of $\sqrt{\lambda - 4}$ in $\mathbb{Q}_3$. We have

$$-3 = 3 \cdot \frac{2}{1-3} = 2 \cdot 3 + 2 \cdot 3^2 + 2 \cdot 3^2 + \cdots .$$

Consequently

$$\lambda - 4 = -3 + \lambda_1 \cdot 3 + \lambda_2 \cdot 3^2 + \lambda_3 \cdot 3^3 + \cdots = 3\left((\lambda_1 + 2) + (\lambda_2 + 2) \cdot 3 + (\lambda_3 + 2) \cdot 3^2 + \cdots\right).$$

From this equality and Theorem 2.2, it follows that $\lambda_1 = 1$ ensures the existence of $\sqrt{\lambda - 4}$. Then we have

$$\lambda - 4 = 3^2\left(\lambda_2 + \lambda_3 \cdot 3 + \lambda_4 \cdot 3^2 + \lambda_5 \cdot 3^3 + \cdots\right)$$

which implies that λ_2 must be a quadratic residue modulo 3 by refering again to Theorem 2.2. This leads to $\lambda_2 = 1$ only, therefore

$$\lambda - 13 = 3^3\left(\lambda_3 + \lambda_4 \cdot 3 + \lambda_5 \cdot 3^2 + \cdots\right), \quad 0 \le \lambda_i \le 2,\ i = 3, 4, 5, \cdots .$$

Now we check that $z_{1,2} \in \mathcal{E}_3$. We have

$$|z_{1,2}|_3 = \left|\lambda - 2 \pm \sqrt{\lambda(\lambda - 4)}\right|_3 = \left|(\lambda - 1) - 1 \pm \sqrt{\lambda((\lambda - 1) - 3)}\right|_3 = 1. \tag{6.7}$$

$$|z_{1,2} - 1|_3 = \left|(\lambda - 1)^2 - 3 \pm \lambda\sqrt{\lambda((\lambda - 1) - 3)}\right|_3 < 3^{\frac{-1}{2}}. \tag{6.8}$$

Hence we have proven the following

Theorem 6.6. *If $k = 2$, $p = 3$ and $\lambda \in \{x \in \mathbb{Q}_3 : |x - 13|_3 \le \frac{1}{27}\}$ then there exist precisely two $G^{(2)}$-periodic p-adic Gibbs measures.*

Remark 6.7. *In classical (real) models of statistical mechanics, a phase transition is said to occur whenever varying a parameter leads to a change in the number of Gibbs states. For example, on a Cayley tree, Ising and Potts models exhibit a phase transition at some critical temperature T_c. Similar phenomena also occurs for real HC model at some λ_c. This is not the case for p-adic models since the field of p-adic numbers $\mathbb{Q}_p$ is not ordered. However in the case $k = 2, p = 3$ the sphere $\{x \in \mathbb{Q}_p : |x - 13|_p = \frac{1}{27}\}$ can be considered as a critical "curve".*

Note that in p-adic case the geometry of balls and spheres are more complicated than in real case [15–17, 21, 31, 36, 41].

7. Boundedness of p-adic Gibbs measures

Now we are interested to find out whether a p-adic Gibbs measure is bounded.

For a set A we denote by $|A|$ its number of elements and recall that Ω_n is the set of all admissible configurations $\sigma_n : V_n \to \{0, 1\}$. We need the following

Lemma 7.1. *The number of admissible configurations is given by*

$$|\Omega_n| = 2^{(k+1)\frac{k^n-1}{k-1}} + 1.$$

Proof. We first compute the number of non admissible configurations. It is known that if a connected subset M of a tree contains m vertices then it contains $m-1$ edges. Thus V_n contains $|V_n|-1$ edges. Note that a configuration σ_n is non admissible if there exists at least one edge $\langle x, y\rangle$ such that $\sigma(x) = \sigma(y) = 1$. Therefore, the number of non admissible configurations on V_n is equal to

$$\sum_{m=1}^{|V_n|-1} \binom{|V_n|-1}{m} = 2^{|V_n|-1} - 1.$$

Consequently

$$|\Omega_n| = 2^{|V_n|} - \left(2^{|V_n|-1} - 1\right) = 2^{|V_n|-1} + 1.$$

This, together with the following formula

$$|V_n| = 1 + (k+1)\frac{k^n-1}{k-1}$$

completes the proof □

Theorem 7.2. *Assume $p \mid 2^k - 1$, then the p-adic Gibbs measure μ corresponding to the p-adic HC-model on the Cayley tree of order $k \geq 1$ is bounded if and only if $p \neq 3$.*

Proof. It suffices to show that the values of μ on cylindrical subsets are bounded.

Denote

$$\tilde{H}(\sigma_n) = J\sum_{x\in V_n}\sigma(x) + \sum_{x\in W_n}\log_p z_{\sigma(x),x}.$$

Let us estimate $\left|\mu^{(n)}(\sigma_n)\right|_p$:

$$\left|\mu^{(n)}(\sigma_n)\right|_p = \left|\frac{\exp_p\left(\tilde{H}(\sigma_n)\right)}{\sum_{\varphi_n\in\Omega_n}\exp_p\left(\tilde{H}(\varphi_n)\right)}\right|_p =$$

$$\frac{1}{\left|\sum_{\varphi_n\in\Omega_n}\left[\exp_p\left(\tilde{H}(\varphi_n)\right) - 1\right] + |\Omega_n|\right|_p}. \tag{7.1}$$

Using Lemma 7.1 we get

$$|\Omega_n| = 2^{k\mathbf{K}+1} + 1 = 2\left[(2^k-1)^{\mathbf{K}} + \mathbf{K}(2^k-1)^{\mathbf{K}-1} + \cdots + (2^k-1)\right] + 3,$$

where $\mathbf{K} = 2 + 2k + \cdots + 2k^{n-2} + k^{n-1}$. Consequently,

$$||\Omega_n||_p = \begin{cases} \leq \frac{1}{p} & \text{if } \ p = 3 \\ 1 & \text{if } \ p \neq 3. \end{cases}$$

Now from (7.1) we get

$$\left|\mu^{(n)}(\sigma_n)\right|_p = \begin{cases} \geq p & \text{if } \; p=3 \\ 1 & \text{if } \; p \neq 3. \end{cases}$$

Thus boundedness is proved for $p \neq 3$.

Now we shall prove that μ is not bounded if $p=3$. Put

$$p_{ij}^{x,y} = \begin{cases} \frac{\exp_p(J(i+j)+z_{i,x}+z_{j,y})}{\sum_{\substack{u,v\in\{0,1\}\\ u+v\neq 2}} \exp_p(J(u+v)+z_{u,x}+z_{v,y})} & \text{if } \; i,j=0,1;\; i+j\neq 2 \\ 0 \qquad \text{if } \; i=j=1. \end{cases}$$

In order to show that the measure μ is not bounded at $p=3$, it is enough to show that its marginal measure is not bounded. Let $\pi = \{..., x_{-1}, x_0, x_1, ...\}$ be an arbitrary infinite path in Γ^k. From (3.1) we can see that a marginal measure μ_π on admissible configurations on $\{0,1\}^\pi$ has the form

$$\mu_\pi(\omega_n) = p_{\omega(x_{-n})}^{x_{-n},x_{-n+1}} \prod_{m=-n}^{n-1} p_{\omega(x_m)\omega(x_{m+1})}^{x_m,x_{m+1}} \tag{7.2}$$

Here $\omega_n : \{x_{-n}, ..., x_0, ..., x_n\} \to \{0,1\}$ is a configuration on $\{x_{-n}, ..., x_0, ..., x_n\}$ and p_i^{xy} is a coordinate of the invariant vector of the matrix $\left(p_{ij}^{x,y}\right)_{i,j=0,1}$.

We have

$$\left|p_{ij}^{x,y}\right|_3 = \frac{1}{\left|\sum_{\substack{u,v\in\{0,1\}\\ u+v\neq 2}} \left[\exp_3\left(J(u+v)+z_{u,x}+z_{v,y}\right)-1\right]+3\right|_3} > 3 \tag{7.3}$$

for all i,j. From (7.2) and (7.3) we find that μ_n is not bounded.

The theorem is proven. □

8. Concluding remarks

To conclude, we will give a brief description of the differences of behavior between classical (real) models and p-adic models on Cayley trees.

Hard core models. *Real case:* In this model (see [40]), for all $\lambda > 0$ and $k \geq 1$, there exists a unique translational invariant splitting Gibbs measure μ_0. Let $\lambda_c = \frac{1}{(k-1)}\left(\frac{k}{k-1}\right)^k$, then:

(i) for $\lambda \leq \lambda_c$, the Gibbs measure is unique (and coincides with the above measure μ_0),

(ii) for $\lambda > \lambda_c$, in addition to μ_0, there exist two distinct extreme periodic measures, μ^+ and μ^-. In addition, there are a continuum set of distinct, extreme, non-translational-invariant, Gibbs measures.

For $\lambda > \frac{1}{(\sqrt{k}-1)}\left(\frac{\sqrt{k}}{\sqrt{k}-1}\right)^k$, the measure μ_0 is not extreme.

p-adic case: In this paper we have shown that the p-adic HC model is completely different from real HC model. For a fixed k, the p-adic HC model may have a splitting Gibbs measure only if p divides 2^k-1. Moreover if p divides 2^k-1 and does not

divide $k+2$ then there exists unique translation invariant p-adic Gibbs measure. The HC model admits only translation invariant and periodic with period two (chess-board) Gibbs measures. For $p \geq 7$, a periodic p-adic Gibbs measure exists iff p divides both $2^k - 1$ and $k-2$. For $k=2$, a p-adic splitting Gibbs measure exists if and only if $p=3$; in this case we have shown that if λ belongs to a p-adic ball of radius $1/27$ then there are precisely two periodic (non translation invariant) p-adic Gibbs measures. Finally, we have proven that a p-adic Gibbs measure is bounded if and only if $p \neq 3$.

Potts model. *Real case:* The ferromagnetic q states Potts model for any $q \geq 2$ exhibits possibly $q+1$ distinct translation invariant Gibbs measures. Namely, there exist two critical temperatures $0 < T'_{\mathrm{c}} < T_{\mathrm{c}}$ such that:
(i) for $T \in (T'_{\mathrm{c}}, T_c]$ there are $q+1$ extreme Gibbs measures, one of them is called unordered;
(ii) for $T \leq T'_{\mathrm{c}}$, q extreme Gibbs measures coexist: there is the unordered which is not extreme;
(iii) for $T > T_{\mathrm{c}}$ there is one Gibbs measure, [10, 25].

p-adic case: The model exhibits a phase transition whenever $k=2, q \in p\mathbb{N}$ and $p \geq 3$ (resp. $q \in 2^2\mathbb{N}, p=2$) [27]. Whenever $k \geq 3$ a phase transition may occur only at $q \in p\mathbb{N}$ if $p \geq 3$ and $q \in 2^2\mathbb{N}$ if $p=2$. Moreover for the p-adic Ising model ($q=2$) there is no phase transition. This is one interesting difference between real and p-adic Ising model on trees.

λ-model. *Real case:* A nearest-neighbor λ-model with two spin values on Cayley tree is considered in [33]. There, it was proven that this model has similar properties like Ising model.

p-adic case: (see [14]) For p-adic non homogeneous λ-model there is no phase transition and as well as being unique, the p-adic Gibbs measure is bounded if and only if $p \geq 3$. If $p=2$, a condition asserting the non existence of a phase transition was given.

This result shows that, in p-adic case, even non homogeneous interactions do not lead to the ocurence of a phase transition.

From the above given results it follows that the set of p-adic Gibbs measures is sparse with respect to the set of real Gibbs measures. The main reasons for this could be explained by the following:

(i) The set of values of real norm $|x|$ is continuous $[0, +\infty)$, but the set of values of p-adic norm is discrete $\{p^m : m \in \mathbb{Z}\}$.

(ii) Real function e^x is defined for any $x \in R$ but p-adic function $\exp_p(x)$ is defined only for $x \in \mathbb{Q}_p$ with $|x|_p \leq \frac{1}{p}$.

(iii) The set of values of real function e^x and its norm $|e^x|$ is continuous $(0, +\infty)$, but the set of values of p-adic function $\exp_p(x)$ is $\{x : |x-1|_p \leq \frac{1}{p}\}$ and the set of values of its norm $|\exp_p(x)|_p$ contains unique point 1, i.e., $|\exp_p(x)|_p = 1$ for all x with $|x|_p \leq \frac{1}{p}$.

Nevertheless, we believe that p-adic Gibbs measures might have interesting applications.

Acknowledgements

UAR thanks the université du Sud Toulon Var for supporting his visit to Toulon and the Centre de Physique Théorique de Marseille for kind hospitality. He also acknowledge IMU/CDE-program for a travel support and the TWAS Research Grant: 09-009 RG/Maths/As-I; UNESCO FR: 3240230333.

References

[1] Albeverio S., Karwowski W. A Random Walk on p-adics–the Generator and Its Spectrum,*Stochastic Processes Appl.* **53** (1994), 1-22.

[2] Albeverio S., Zhao X. Measure-Valued Branching Processes Associated with Random Walks on p-adics,*Ann. Probab.* **28** (2000), 1680-1710.

[3] Aref'feva I.Ya, Dragovich B., Frampton P.H., Volovich I.V., The Wave Function of the Universe and p-adic Gravity, *Int. J. Mod. Phys.* A 6 (24),(1991), 4341-4358.

[4] Avetisov V.A, Bikulov A.H., Kozyrev S.V., Application of p-adic Analysis to Models of Breaking of Replica Symmetry, *J. Phys. A: Math. Gen.* **32** (50), (1999), 8785-8791.

[5] Baxter R.J. *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).

[6] Beltrametti E.G., Cassinelli G., Quantum Mechanics and p-adic Numbers,*Found. Phys.* **2**, (1972), 1-7.

[7] Bleher P.M., Ganikhodjaev N.N. On pure phases of the Ising model on the Bethe lattice. *Theor. Probab. Appl.* **35** (1990), 216-227.

[8] Bleher P.M., Ruiz J., Zagrebnov V.A. On the purity of the limiting Gibbs state for the Ising model on the Bethe lattice. *Journ. Statist. Phys.* **79** (1995), 473-482.

[9] Freund P. G. O., Olson M., Non-Archimedean Strings, *Phys. Lett.* B **199**, (1987), 186-190.

[10] Ganikhodjaev N.N. On pure phases of the ferromagnet Potts with three states on the Bethe lattice of order two. *Theor. Math. Phys.* **85** (1990), 163175.

[11] Ganikhodjaev N.N., Rozikov U.A. Description of periodic extreme Gibbs measures of some lattice model on the Cayley tree. *Theor. Math. Phys.* **111** (1997), 480-486.

[12] Ganikhodjaev N.N., Mukhamedov F.M., Rozikov U.A., Phase Transitions in the Ising Model on Z over the p-adic Number Field, *Uzb. Mat. Zh.*, No. 4, (1998), 23-29.

[13] Georgii H.-O., Gibbs Measures and Phase Transitions (W. de Gruyter, Berlin, 1988).

[14] Khamraev M., Mukhamedov F.M., Rozikov U.A. On the uniqueness of Gibbs measures for $p-$adic non homogeneous $\lambda-$ model on the Cayley tree. *Letters in Math. Phys.* **70** (2004), 17-28.

[15] Khrennikov A. Yu., p-Adic Valued Probability Measures, *Indag. Math.*, New Ser. **7** (1996), 311-330.

[16] Khrennikov A. Yu., p-Adic Valued Distributions in Mathematical Physics (Kluwer, Dordrecht, 1994).

[17] Khrennikov A. Yu., Non-Archimedean Analysis: Quantum Paradoxes, Dynamical Systems and Biological Models (Kluwer, Dordrecht, 1997).

[18] Khrennikov A. Yu., Kozyrev S.V., Replica Symmetry Breaking Related to a General Ultrametric Space. I: Replica Matrices and Functionals,*Physica A* **359**, (2006) 222-240; II: RSB Solutions and the $n \to 0$ Limit, *Physica A* **359**, (2006), 241-266; III: The Case of General Measure, *Physica A* **378**,(2007), 283-298.

[19] Khrennikov A. Yu., Mukhamedov F.M., Mendes J.F.F., On p-adic Gibbs Measures of the Countable State Potts Model on the Cayley Tree, *Nonlinearity* **20**, (2007), 2923-2937.

[20] Khrennikov A. Yu., Yamada S., van Rooij A., The Measure-Theoretical Approach to p-adic Probability Theory, *Ann. Math. Blaise Pascal* **6**, (1999), 21-32.

[21] Koblitz N., p-Adic Numbers, p-adic Analysis, and Zeta-Functions (Springer, Berlin, 1977).

[22] Ludkovsky S., Khrennikov A. Yu., Stochastic Processes on Non-Archimedean Spaces with Values in Non-Archimedean Fields, *Markov Processes Relat. Fields* **9**, (2003), 131-162.

[23] Marinary E., Parisi G., On the p-adic Five-Point Function, *Phys. Lett.* B **203**, (1988) 52-54.

[24] Martin J.B, Rozikov U.A., Suhov Yu.M. A three state hard-core model on a Cayley tree. *Jour. Nonlinear Math. Phys* **12**(3) (2005), 432-448.
[25] Mossel E. Survey: information flow on trees. In: Graphs, Morphisms and Statistical Physics. DIMACS *Series in Discrete Mathematics and Theoretical Computer Science*, vol. 63, pp. 155170. AMS, Providence (2004)
[26] Mukhamedov F.M. On the Existence of Generalized Gibbs Measures for the One-Dimensional p-adic Countable State Potts Model. *Proc. Steklov Inst. Math.*, **265** (2009), 165-176.
[27] Mukhamedov F.M., Rozikov U.A., On Gibbs Measures of p-adic Potts Model on the Cayley Tree, *Indag. Math.*, New Ser. **15** (2004), 85-100.
[28] Mukhamedov F.M., Rozikov U.A., On Inhomogeneous p-adic Potts Model on a Cayley Tree, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **8**, (2005), 277-290.
[29] Mukhamedov F.M., Rozikov U.A., Mendes J.F.F., On Phase Transitions for p-adic Potts Model with Competing Interactions on a Cayley Tree, *in p-Adic Mathematical Physics*: Proc. 2nd Int. Conf., Belgrade, 2005 (Am. Inst. Phys., Melville, NY, 2006), AIP Conf. Proc. 826, pp. 140-150.
[30] Preston C. *Gibbs states on countable sets* (Cambridge University Press, London 1974).
[31] van Rooij A. C. M., Non-Archimedean Functional Analysis (M. Dekker, New York, 1978).
[32] Rozikov U.A. Partition structures of the Cayley tree and applications for describing periodic Gibbs distributions. *Theor. Math. Phys.* **112** (1997), 929-933.
[33] Rozikov U.A. Description of limit Gibbs measures for $\lambda-$ models on the Bethe lattice. *Siberan Math. Jour.* **39** (1998), 373-380.
[34] Rozikov U.A. Description of periodic Gibbs measures of the Ising model on the Cayley tree. *Russian Math. Surv.* **56** (2001), 172-173.
[35] Rozikov U.A. A contour method on Cayley trees. *Jour. Stat. Phys.* **130** (2008), 801-813.
[36] Schikhof W.H., Ultrametric Calculus (Cambridge Univ. Press, Cambridge, 1984).
[37] Sinai Ya.G., Theory of phase transitions: Rigorous Results (Pergamon, Oxford, 1982).
[38] Shiryaev A.N., Probability (Nauka, Moscow, 1980; Springer, New York, 1984).
[39] Spitzer F. Markov random fields on an infinite tree, *Ann. Prob.* **3** (1975), 387–398.
[40] Suhov Y.M., Rozikov U.A. A hard - core model on a Cayley tree: an example of a loss network, *Queueing Syst.* **46** (2004), 197–212.
[41] Vladimirov V.S., Volovich I. V., Zelenov E. V., p-Adic Analysis and Mathematical Physics (Nauka, Moscow, 1994; World Sci., Singapore, 1994).
[42] Volovich I. V., Number Theory as the Ultimate Physical Theory,*Preprint* No. TH 4781/87 (CERN, Geneva, 1987).
[43] Volovich I. V., p-Adic String, *Class. Quantum Grav.* **4**, (1987), L83-L87.
[44] Yasuda K., Extension of Measures to Infinite Dimensional Spaces over p-adic Field,*Osaka J. Math.* **37**, (2000), 967-985.
[45] Zachary S. Countable state space Markov random fields and Markov chains on trees, *Ann. Prob.* **11** (1983), 894–903.

D. Gandolfo and J. Ruiz, Centre de Physique Théorique, UMR 6207, Universités Aix-Marseille et Sud Toulon-Var, Luminy Case 907, 13288 Marseille, France.
E-mail address: gandolfo@cpt.univ-mrs.fr ruiz@cpt.univ-mrs.fr

U. A. Rozikov, Institute of mathematics and information technologies, 29, Do'rmon Yo'li str., 100125, Tashkent, Uzbekistan.
E-mail address: rozikovu@yandex.ru