TOWARDS MOTIVIC QUANTUM COHOMOLOGY OF $\overline{M}_{0,S}$

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Dedicated to Professor V. V. Shokurov, on his 60th birthday

ABSTRACT. We explicitly calculate some Gromov–Witten correspondences determined by maps of labeled curves of genus zero to the moduli spaces of labeled curves of genus zero. We consider these calculations as the first step towards studying the self–referential nature of motivic quantum cohomology.

0. Introduction: Motives and Quantum Cohomology

0.1. A summary. Let Var_k be the category of smooth complete algebaric varieties defined over a field k.

The category of classical motives Mot_k^K, with coefficients in a $\mathbf{Q}$–algebra K, is the target of a functor $h : Var_k^{op} \to Mot_k^K$ which, in the vision of Alexandre Grothendieck, ought to be *a universal cohomology theory,* with values in a tensor K–linear category.

Morphisms $X \to Y$ in Mot_k^K are represented by *classes of correspondences*, algebraic cycles on $X \times Y$ with coefficients in K. Depending on the equivalence relation, imposed on these cycles, one could consider Chow motives, numerical motives, etc.

Besides objects $h(V)$ for $V \in Var_k$, the category Mot_k^K contains their *direct summands*, ("pieces") and their *twists* by Tate's motive. Formally adding these objects one turns the category of motives into a Tannakian category. One can then apply to it the philosophy of motivic fundamental group. Ideally, all inherent structures of cohomology objects can be encoded/replaced by the representations of the respective motivic fundamental group.

What is special about "total motives" $h(V)$, as opposed to pieces and twists?

For example, $h(V)$'s bring with them a natural structure of *commutative algebras* in Mot_k^K. It is not determined only by $h(V)$: isomorphic motives $h(V)$'s may well have different multiplications; but of course, this classical multiplication is motivic in the sense that it is induced by the diagonal map $V \to V \times V$ in Var_k and by the class of its graph in Mot_k^K.

The advent of quantum cohomology from physics to algebraic geometry opened our eyes to the fact that classical cohomology spaces of algebraic varieties, say, over

C, possess an incomparably richer structure: they (or rather their tensor powers) are acted upon by cohomology of moduli spaces of pointed curves $H^*(\overline{M}_{g,n})$, much as Steenrod operations act in topological situation. From the physical perspective, these operations encode "quantum corrections to the classical multiplication".

Grothendieck's vision however turned out prophetic : *this new structure is motivic as well* in the same sense: it is induced by canonical Chow correspondences, Gromov–Witten invariants $I^V_{g,n,\beta}$ in $A_*(\overline{M}_{g,n}\times V^n)$ indexed by effective classes β in $A_1(V)$. This was conjectured in [KoMa1], worked out in more detail in [BehMa], and finally proved in [Beh1], where the virtual fundamental classes in the Chow groups of spaces of stable maps were constructed by algebraic–geometric techniques.

This construction allowed K. Behrend to establish a list of universal identities between the Gromov–Witten invariants that were conjectured earlier.

Taken together, these identities imply that for each total motive $h(V)$, the infinite sum of its copies indexed by the numerical classes β of effective curves on V possesses the canonical structure of an algebra over the cyclic modular operad $\mathcal{HM}$:

$$\mathcal{HM}(n) := \coprod_g h(\overline{M}_{g,n})$$

This is the motivic core of quantum cohomology.

However, this discovery also stressed an inherent tension between the initial Grothendieck vision and the highly non–Tannakian character of the quantum cohomology expressed in the following observations.

First, these structures of $\mathcal{HM}$–algebras are not functorial in any naive sense wrt morphisms in Var_k (except isomorphisms). Notice that the classical multiplication *is* functorial wrt morphisms in Var_k; quantum corrections destroy this. However, as was shown in [LeLW], quantum multiplication *is functorial* wrt at least certain isomorphisms in Mot^K_k (flops) that do not agree with classical multiplication: quantum corrections exactly compensate classical discrepancies. This is a remarkable fact suggesting that motivic functorality might be an important hidden phenomenon.

Second, being total motives, $h(\overline{M}_{g,n})$ *themselves* have quantum cohomology, that is, define algebras over $\mathcal{HM}$.

One aim of this note is to draw attention to this self–referentiality and to start studying the quantum cohomology *of* $\mathcal{HM}$ and its relation to the quantum cohomology action of this operad upon other total motives. Analogies with homotopy theory, in particular, A^1–homotopy formalism, might help to recognize a pattern in algebraic geometry similar to that of iterated loop spaces.

A warning is in order: many meaningful questions cannot be asked and answers cannot be obtained until one extends both parts of the theory, motives and quantum cohomology, from the category Var_k to at least the category of smooth DM–stacks. Some of the arising complications can be avoided if one restricts to the case of genus zero quantum cohomology. We adopt this restriction in this article.

0.2. Results of this paper. This paper is our first installment to the project whose goal is to understand the Gromov–Witten theory of moduli spaces of curves, preferably on the motivic level, that is the level of J– and I–correspondences (cf. [Beh3] for a nice and intutive introduction).

Specifically, the spaces $\overline{M}_{0,S}$ (with variable S) and their products are interrelated by a host of natural morphisms expressing embeddings of boundary strata, forgetting labeled points, relabeling etc: cf. a systematic description in [BehMa].

Gromov–Witten classes that we study in this paper are certain Chow correspondences

$$I(S,\Sigma,\beta)\in A_*(\overline{M}^{\Sigma}_{0,S}\times\overline{M}_{0,\Sigma}) \tag{0.1}$$

where S, Σ are (disjoint) finite sets of labels and β runs over classes of effective curves in $A_1(\overline{M}_{0,S})$.

Our main motivation is the following vague

0.2.1. Guess. *Classes (0.1) are "natural" in the sense that they can be functorially expressed through canonical morphisms in the category of moduli spaces of labeled trees of various combinatorial types.*

This guess is a natural first step to the understanding the self–referential nature of Gromov–Witten theory in motivic algebraic geometry: the fact that components of the basic modular operad "are" algebras over the same operad (if one takes into account twisting and grading by the cones of effective curves).

The main result of this paper is an explicit description, in the spirit of our guess, of those I–correspondences of $\overline{M}_{0,S}$ that correspond *to the classes β of boundary curves:* see Theorem 4.5 in section 4.

This answers a question which is quite natural, in particular, because there is a conjecture that boundary curve classes are generators of the Mori cone: cf. [KeMcK], [FG], [HaT], [CT] for this and related problems.

0.3. From curves to surfaces and further on? One can imaginatively say that quantum cohomology of V reveals hidden geometry that can be seen only when one starts probing V by mapping curves C ("strings") to V. A natural question arises, how to use, say, surfaces ("membranes") in place of curves, and do it in algebraic geometry rather than in symplectic or differential one.

If we expect to discover new universal motivic actions in this way, we must first contemplate the case $V=$ *a point* and pose the question:

What are analogs of moduli spaces $\overline{M}_{g,n}$ (or at least $\overline{M}_{g,0}$) for surfaces in this context?

The experience of stringy case indicates that these analogs must be *rigid* objects, as $\overline{M}_{g,n}$ themselves: see [Hac].

In fact, moduli spaces are only rarely rigid, but according to a brave guess of M. Kapranov, if one starts with an object $X = X(0)$ of dimension n, produces its moduli space of deformations $X(1)$, then moduli space $X(2)$ of deformations of $X(1)$ etc., then $X(n)$ must be rigid. Quoting [Hac], who summarizes philosophy expressed in an unpublished manuscript by M. Kapranov, "one thinks of $X(1)$ as H^1 of a sheaf of non–abelian groups on $X(0)$. Indeed, at least the tangent space to $X(1)$ at $[X]$ is identied with $H^1(\mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf, the sheaf of first order innitesimal automorphisms of X . Then one regards $X(m)$ as a kind of non–Abelian H^m, and the analogy with the usual definition of Abelian H^m suggests the statement above."

Extending this idea, one might guess that an imaginary "membrane quantum cohomology" should define motivic actions of rigid (iterated) moduli spaces of surfaces (endowed with cycles to keep track of incidence conditions) upon certain total (ind–)motives. One motivation of this note is to make some propaganda for this idea.

1. Gromov–Witten correspondences

We start with background terminology and notation.

1.1. Moduli stacks. We will consider schemes over a fixed field k of characteristic zero.

Any scheme W "is the same as" the contravariant functor of its T–points $W(T) = \mathrm{Hom}(T, W)$ with values in *Sets*.

More generally, a stack (of groupoids) $\mathcal{F}$ "is the same as" the class of its T–points $\mathcal{F}_T$, where T runs over schemes. The main new element of the situation is that each $\mathcal{F}_T$ itself and their union $\mathcal{F}$ over "all" T's are not simply sets or classes, but categories. Moreover, they form a sheaf on the étale (or fppf) site of schemes.

So we will think about individual objects of $\mathcal{F}_T$ as schematic T–points of $\mathcal{F}$, whereas nontrivial morphisms between them are functors subject to a list of restrictions specific for stacks. Below we recall this list informally.

As in [Ma], V.3, we will imagine objects of $\mathcal{F}_T$ as "families (of something) over T". In practical terms, one family is usually given by a diagram of schemes and morphisms, in which a part of the data remain fixed, including its " base" T, and the rest is subject to a list of explicit restrictions.

For example, if $\mathcal{F}$ is represented by a scheme W, then "one family over T (of points of W)" is a very simple diagram $T \to W$.

The following requirements must be satisfied.

(i) Each object of $\mathcal{F}$ belongs to an $\mathcal{F}_T$ for a unique scheme T, and the map $b : \mathcal{F} \to Sch$, sending a family to its base, is a functor. Groupoid property requires that if $b(f) = id_T$, then f is an isomorphism between two respective T–points.

(ii) With respect to morphisms $\varphi : T_1 \to T_2$, $\mathcal{F}_T$ must be contravariant: we must be given "base change" functors $\varphi^* : \mathcal{F}_{T_2} \to \mathcal{F}_{T_1}$, together with functor isomorphisms $(\varphi \circ \psi)^* \to \psi^* \circ \varphi^*$ and associativity diagrams for them.

Moreover, if $F \in Ob\,\mathcal{F}_{T_2}$, then the lifted family $\varphi^*(F) \in Ob\,\mathcal{F}_{T_1}$ must be endowed with a canonical morphism $F \to \varphi^*(F)$ lifting φ and satisfying a set of conditions.

For example, the base change for $T_2 \to W$ is simply the composition $T_1 \to T_2 \to W$.

(iii) $\mathcal{F}$ is a stack, if each T–family is uniquely defined by its restrictions to an étale (or fppf) covering of T and the standard descent data. The same must be true about morphisms of T–families etc.

(iv) Morphisms of stacks are functors between the respective categories of families, identical on bases of families.

Thus, an object $F \in Ob\,\mathcal{F}_T$ can also be treated as a stack, and as such, it is endowed with a morphism of stacks $F \Rightarrow \mathcal{F}$.

1.2. Families of stable maps: preliminaries. We will now describe main classes of families and stacks with which we deal here.

First of all, fix a finite set Σ, a genus $g \geq 0$, a smooth projective manifold W over k , and an effective class $\beta \in A_1(W)$

Then one can define a (smooth proper DM)–stack $\overline{M}_{g,\Sigma}(W, \beta)$.

For a k–scheme T, one object of the groupoid $\overline{M}_{g,\Sigma}(W, \beta)(T)$ of T–points of this stack consists of a diagram of schemes of the following structure:

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "f_T"] \arrow[d, "h_T"] & W \\ T \end{tikzcd} \tag{1.1}$$

and a family of sections $x_{j,T} : T \to \mathcal{C}$, $j \in \Sigma$, $h_T \circ x_{j,T} = id_T$.

They must satisfy the following conditions:

(a) $\mathcal{C}_T \to T$ and $(x_{j,T})$ constitute a flat *prestable* T–family of curves of genus g.

(b) $f_T : (\mathcal{C}_T; (x_{j,T})) \to W$, is a *stable map* of class β.

For precise definitions of (pre)stability and class of the map that we use here, see [BehMa] or [Ma].

Given such a diagram with sections, we call (W, β) its *target*, T its *base*, and the whole setup *a T–family of stable maps.* Isomorphisms of families, lifting id_T, must be identical also on W. Base changes are defined in a rather evident way.

The stack $\overline{M}_{g,\Sigma}(W, \beta)$ is defined as the base of the universal family of this type with given target (W, β):

$$\begin{tikzcd} \overline{C}_{g,\Sigma}(W,\beta) \arrow[r, "f"] \arrow[d, "h"] & W \\ \overline{M}_{g,\Sigma}(W,\beta) \end{tikzcd} \tag{1.2}$$

It is endowed with sections $x_j : \overline{M}_{0,\Sigma}(W, \beta) \to \overline{C}_{0,\Sigma}(W, \beta)$, $j \in \Sigma$.

Naturally, $\overline{C}_{g,\Sigma}(W, \beta)$ is a stack as well.

If W is a point, $\beta = 0$, we routinely omit the target and write simply $\overline{M}_{g,\Sigma}$, $\overline{C}_{g,\Sigma}$ etc.

Moreover, (1.2) produces the *evaluation/stabilization* diagram

$$\begin{tikzcd} \overline{M}_{g,\Sigma}(W,\beta) \arrow[r, "st"] \arrow[d, "ev"] & \overline{M}_{g,\Sigma} \\ W^{\Sigma} \end{tikzcd} \tag{1.3}$$

Here

$$ev = (ev_j = f \circ x_j \,|\, j \in \Sigma) : \quad \overline{M}_{g,\Sigma}(W, \beta) \to W^{\Sigma} \tag{1.4}$$

and, in case $2g + |\Sigma| \geq 3$, the absolute stabilization morphism st discards the map f and stabilizes the remaining prestable family of curves

$$st: \overline{M}_{g,\Sigma}(W,\beta) \to \overline{M}_{g,\Sigma}. \tag{1.5}$$

The *virtual fundamental class*, or the J–class $[\overline{M}_{g,\Sigma}(W,\beta)]^{virt}$, is a canonical element in the Chow ring $A_*(\overline{M}_{0,\Sigma}(W,\beta))$:

$$J_{g,\Sigma}(W,\beta) \in A_D(\overline{M}_{0,\Sigma}(W,\beta))\,, \tag{1.6}$$

where D is the virtual dimension (Chow grading degree)

$$(-K_W,\beta) + |\Sigma| + (\dim W - 3)(1-g). \tag{1.7}$$

The respective *Gromov–Witten correspondence*, defined for $2g + |\Sigma| \geq 3$, is the proper pushforward

$$I_{g,\Sigma}(W,\beta) := (ev, st)_*(J_{g,\Sigma}(W,\beta)) \in A_D(W^{\Sigma} \times \overline{M}_{g,\Sigma}) \tag{1.8}$$

Understanding these correspondences is the content of *motivic* quantum cohomology.

1.3. Example: $g = 0,\ \beta = 0$**.** In this case the universal family (1.2) is

$$\begin{tikzcd} W \times \overline{C}_{0,\Sigma} \arrow[r, "pr_1"] \arrow[d, "id_W \times h"] & W \\ W \times \overline{M}_{0,\Sigma} & \end{tikzcd} \tag{1.9}$$

with structure sections $id_W \times x_j$.

The stabilization morphism is simply the projection

$$st = pr_2: \ W \times \overline{M}_{0,\Sigma} \to \overline{M}_{0,\Sigma}. \tag{1.10}$$

The evaluation morphism is the projection followed by the diagonal embedding Δ_Σ:

$$ev: \ W \times \overline{M}_{0,\Sigma} \to W \to W^{\Sigma}. \tag{1.11}$$

We have ([Beh1], p. 606):

$$J_{0,\Sigma}(W,0) = [\overline{M}_{0,\Sigma}(W,0)] = [W] \otimes [\overline{M}_{0,\Sigma}]. \tag{1.12}$$

The virtual dimension (1.7) is

$$|\Sigma| + \dim W - 3 = \dim (W \times \overline{M}_{0,\Sigma}).$$

Thus, finally, the Gromov–Witten correspondence is the class

$$I_{0,\Sigma}(W,0) = [\Delta_\Sigma(W)] \otimes [\overline{M}_{0,\Sigma}] \in A_*(W^\Sigma \times \overline{M}_{0,\Sigma}). \tag{1.13}$$

1.4. Strategy. In the remaining sections of this paper, we study the Gromov–Witten correspondences *of genus zero* for $W = \overline{M}_{0,S}$, $\beta =$ *class of a boundary curve* in $\overline{M}_{0,S}$ (cf. below). This is the first step of a much more ambitious program in which all components of the stable family diagrams are allowed to be stacks, and and in which we take for targets the stacks $\overline{M}_{g,S}$ and arbitrary β.

Our modest goal here allows us to basically restrict ourselves to the case of schemes, whose geometry is already well known. However, some intermediate constructions require the use of stacks.

In particular , we need to understand the relevant J–classes and the diagrams

$$ev: \overline{M}_{0,\Sigma}(\overline{M}_{0,S},\beta) \to \overline{M}_{0,S}^{\Sigma}, \quad st: \overline{M}_{0,\Sigma}(\overline{M}_{0,S},\beta) \to \overline{M}_{0,\Sigma}. \tag{1.14}$$

We also want to be able to trace various functorialities, in particular, in *both* S and Σ. However, this may result in a rather clumsy notation.

In order to postpone its introduction, in the remaining parts of this section we describe a somewhat more general situation. Afterwards we will show that our main problem is contained in it.

1.5. Setup, part I. Consider a morphism of smooth irreducible projective manifolds $b: E \to W$. Let β_E be an effective curve class on E, and $\beta := b_*(\beta_E)$ its pushforward to W. Any stable map $\mathcal{C}_T/T \to E, (x_j : T \to \mathcal{C}_T \,|\, j \in \Sigma)$, of class β_E, induces, after composition with b and stabilization, a stable map with target (W,β). Thus, we get a map

$$\widetilde{b}: \overline{M}_{0,\Sigma}(E,\beta_E) \to \overline{M}_{0,\Sigma}(W,\beta)$$

that clearly fits into the commutative diagram

$$\begin{array}{ccc} \overline{M}_{0,\Sigma}(E,\beta_E) & \xrightarrow{\widetilde{b}} & \overline{M}_{0,\Sigma}(W,\beta) \\ \Big\downarrow{\scriptstyle (ev_E, st_E)} & & \Big\downarrow{\scriptstyle (ev_W, st_W)} \\ E^\Sigma \times \overline{M}_{0,\Sigma} & \xrightarrow{b^\Sigma \times id} & W^\Sigma \times \overline{M}_{0,\Sigma} \end{array} \tag{1.15}$$

If $|\Sigma| \leq 2$, the space $\overline{M}_{0,\Sigma}$ is not a DM–stack; discarding it and stabilization morphisms in (1.15), we still get a commutative diagram. Whenever $\overline{M}_{0,\Sigma}$ appears, we assume that $|\Sigma| \geq 3$.

1.5.1. Proposition. *(i) Assume that*

$$J_{0,\Sigma}(W,\beta) = \widetilde{b}_*(J_{0,\Sigma}(E,\beta_E)). \tag{1.16}$$

Then

$$I_{0,\Sigma}(W,\beta) = (b^\Sigma \times id)_*(I_{0,\Sigma}(E,\beta_E)). \tag{1.17}$$

(ii) Let $\gamma_j \in H^*(W)$, $j \in \Sigma$, be a family of cohomology classes marked by Σ. Then from (1.16) it follows that

$$pr_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap I_{0,\Sigma}(W,\beta) =$$

$$= (b^\Sigma \times id)_*[pr_E^*(\otimes_{j\in\Sigma} b^*(\gamma_j)) \cap I_{0,\Sigma}(E,\beta_E)]. \tag{1.18}$$

Here we denote by $pr_W : W^\Sigma \times \overline{M}_{0,\Sigma} \to W^\Sigma$ and $pr_E : E^\Sigma \times \overline{M}_{0,\Sigma} \to E^\Sigma$ the respective projection morphisms, and H^* can be any standard cohomology theory.

Proof. (i) This follows directly from (1.16) and commutativity of (1.15).

(ii) We have, using the projection formula

$$(b^\Sigma \times id)_*[pr_E^*(\otimes_{j\in\Sigma} b^*(\gamma_j)) \cap I_{0,\Sigma}(E,\beta_E)] =$$

$$= (b^\Sigma \times id)_*[(b^\Sigma \times id)^* \circ pr_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap I_{0,\Sigma}(E,\beta_E)] =$$

$$= pr_W^*(\otimes_{j\in\Sigma}\gamma_j) \cap (b^\Sigma \times id)_*(I_{0,\Sigma}(E,\beta_E))$$

The last expression coincides with l.h.s. of (1.18) in view of (1.17). This completes the proof.

1.5.2. Remark. In our applications to the case $W = \overline{M}_{0,S}$ (cf. section 4), E will be a boundary stratum containing the boundary curve representing β, and the virtual fundamental classes $J_{0,\Sigma}$ will coincide with the usual fundamental classes since the relevant deformation problem will be unobstructed. Moreover, E will have a very special additional structure. We will axiomatize the relevant geometry in the next subsections.

1.6. Setup, part II. Keeping notation of 1.5, we make the following additional assumptions:

(a) E is explicitly represented as $E = B \times C$ where C is isomorphic to $\mathbf{P}^1$. This identification, including the projections $p = pr_B : E \to B$ and $pr_C : E \to C$, constitutes a part of structure.

(b) β_E is the (numerical) class of any fiber of p.

(c) The deformation problem for any fiber C_0 of p embedded via b_0 in W is trivially unobstructed in the sense of [Beh3]:

$$H^1(C_0, b_0^*(\mathcal{T}_W)) = 0\,. \tag{1.19}$$

(d) The map $\widetilde{b}$ in (1.15) is an isomorphism.

These assumptions are quite strong. In particular, from (b) – (d) it follows that (1.16) holds since the relevant virtual fundamental classes coincide with the ordinary ones. Thus, we can complete the explicit computation of $I_{0,\Sigma}(W,\beta)$ starting with the right hand side of (1.17). We will do it in the remaining part of the section.

First of all, we have

$$pr_{B*}(\beta_E) = 0, \quad pr_{C*}(\beta_E) = \mathbf{1}$$

where $\mathbf{1}$ is the fundamental class $[C]$ in the Chow ring of C.

Thus, the two projections induce the map

$$(\widetilde{pr}_B, \widetilde{pr}_C) : \overline{M}_{0,\Sigma}(E,\beta_E) \to \overline{M}_{0,\Sigma}(B,0) \times \overline{M}_{0,\Sigma}(C,\mathbf{1})\,.$$

Stabilization maps embed this morphism into the commutative diagram

$$\begin{tikzcd} \overline{M}_{0,\Sigma}(E,\beta_E) \arrow[r] \arrow[d, "st_E"'] & \overline{M}_{0,\Sigma}(B,0) \times \overline{M}_{0,\Sigma}(C,\mathbf{1}) \arrow[d, "st_B \times st_C"'] \\ \overline{M}_{0,\Sigma} \arrow[r, "\Delta_{\overline{M}_{0,\Sigma}}"] & \overline{M}_{0,\Sigma} \times \overline{M}_{0,\Sigma} \end{tikzcd} \tag{1.20}$$

where the lower line is the diagonal embedding (cf. [Beh2], Proposition 5).

Similarly, evaluation maps embed this morphism into the commutative diagram

$$\begin{tikzcd} \overline{M}_{0,\Sigma}(E,\beta_E) \arrow[r] \arrow[d, "ev_E"'] & \overline{M}_{0,\Sigma}(B,0)\times\overline{M}_{0,\Sigma}(C,\mathbf{1}) \arrow[d, "ev_B\times ev_C"'] \\ E^\Sigma \arrow[r, "s"] & B^\Sigma\times C^\Sigma \end{tikzcd} \tag{1.21}$$

where the lower line is now the evident permutation isomorphism induced by $E = B\times C$.

Combining these two diagrams, we get

$$\begin{tikzcd} \overline{M}_{0,\Sigma}(E,\beta_E) \arrow[r] \arrow[d, "(ev_E,st_E)"'] & \overline{M}_{0,\Sigma}(B,0)\times\overline{M}_{0,\Sigma}(C,\mathbf{1}) \arrow[d, "(ev_B,st_B)\times(ev_C,st_C)"] \\ E^\Sigma\times\overline{M}_{0,\Sigma} \arrow[r, "\widetilde{\Delta}"] & B^\Sigma\times\overline{M}_{0,\Sigma}\times C^\Sigma\times\overline{M}_{0,\Sigma} \end{tikzcd} \tag{1.22}$$

Here the lower line is an obvious composition of permutations and the diagonal embedding of $\overline{M}_{0,\Sigma}$.

From (1.22) and [Beh2] it follows that

$$I_{0,\Sigma}(E,\beta_E) = \widetilde{\Delta}^!(I_{0,\Sigma}(B,0)\otimes I_{0,\Sigma}(C,\mathbf{1}))\,. \tag{1.23}$$

(Notice that for $x\in A_*(X), y\in A_*(Y)$ we often denote simply by $x\otimes y\in A_*(X\times Y)$ the image of $x\otimes y\in A_*(X)\otimes A_*(Y)$ wrt the canonical map $A_*(X)\otimes A_*(Y)\to A_*(X\times Y)$).

Furthermore, according to (1.13),

$$I_{0,\Sigma}(B,0) = [\Delta_\Sigma(B)\times\overline{M}_{0,\Sigma}]\in A_*(B^\Sigma\times\overline{M}_{0,\Sigma}). \tag{1.24}$$

Finally, the space $\overline{M}_{0,\Sigma}(C,\mathbf{1})$ and the class $I_{0,\Sigma}(C,\mathbf{1})$ can be described as follows.

Recall a construction from [FuMPh]. Let V be a smooth complete algebraic manifold. For a finite set Σ, let V^Σ be the direct product of a family of V's labeled by elements of Σ. Denote by $\widetilde{V}^\Sigma$ the blow up of the (small) diagonal in V^Σ. Finally, define $V^{\Sigma,0}$ as the complement to all partial diagonals in V^Σ.

The Fulton–MacPherson's *configuration space* $V\langle\Sigma\rangle$ (for curves it was earlier introduced by Beilinson and Ginzburg) is the closure of $V^{\Sigma,0}$ naturally embedded in the product

$$V^{\Sigma} \times \prod_{\Sigma' \subset \Sigma, |\Sigma'| \geq 2} \widetilde{V}^{\Sigma'}.$$

In [FuPa], it was shown that $\overline{M}_{0,\Sigma}(C, \mathbf{1})$ can be identified with $C\langle\Sigma\rangle$ in such a way that the birational morphism ev_C becomes the tautological open embedding when restricted to $C^{\Sigma,0}$.

Therefore, denoting by $D_{\Sigma} \subset C^{\Sigma} \times \overline{M}_{0,\Sigma}$ the closure of the graph of the canonical surjective map $C^{\Sigma,0} \to M_{0,\Sigma}$, we get

$$I_{0,\Sigma}(C, \mathbf{1})) = [D_{\Sigma}]\,. \tag{1.25}$$

Now we can state the main result of this section:

1.6.1. Proposition. *Assuming 1.6 (a)–(d), we have*

$$I_{0,\Sigma}(E, \beta_E) = \widetilde{\Delta}^!([\Delta_{\Sigma}(B) \times \overline{M}_{0,\Sigma} \times D_{\Sigma}])\,. \tag{1.26}$$

and

$$I_{0,\Sigma}(W, \beta) = (b^{\Sigma} \times id)_* \circ \widetilde{\Delta}^!([\Delta_{\Sigma}(B) \times \overline{M}_{0,\Sigma} \times D_{\Sigma}])\,. \tag{1.27}$$

This is a straightforward combination of (1.23) – (1.25) and (1.17).

2. Target space $\overline{M}_{0,4}$

2.1. Notation. Stressing functoriality wrt labeling sets, and having in mind further developments, we denote in this section by S a set *of cardinality 4*, with a marked point $\bullet$. We put $S = P \sqcup \{\bullet\}$. Thus, we are considering the moduli space $\overline{M}_{0,P\sqcup\{\bullet\}}$. It is a projective line endowed with three pairwise distinct points D_{σ} labeled by unordered partitions $\sigma\,:\; P \sqcup \{\bullet\} = S_1 \sqcup S_2$, $|S_i| = 2$. They are exactly those points over which the universal stable curve $\overline{C}_{0,P\sqcup\{\bullet\}}$ splits into two components, and labeled points are redistributed among them according to σ. Now, the set of such partitions is naturally bijective to P: $j \in P$ corresponds to the partition $\{\bullet, j\}\sqcup(P\setminus\{\bullet, j\})$. Hence finally $\overline{M}_{0,P\sqcup\{\bullet\}}$ is a projective line $\mathbf{P}^1$ stabilized by three points labeled by P. This identification is functorial wrt pointed bijections of S.

The only boundary class of curves in $A_1(\overline{M}_{0,P\sqcup\{\bullet\}})$ is the fundamental class $\beta = \mathbf{1} := [\overline{M}_{0,P\sqcup\{\bullet\}}]$. We have already invoked the description of the universal family of stable maps with this target and the relevant I–class at the end of 1.6, see (1.25). But now, for the sake of a future generalization, we will use a slightly different family and an alternative description of $I_{0,\Sigma} \in A_*((\mathbf{P}^1)^\Sigma \times \overline{M}_{0,\Sigma})$ that will better fit the passage to target spaces $\overline{M}_{0,S}$ with $|S| > 4$.

2.2. An alternative family. Consider the moduli space $\overline{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$.

Recall that for any finite set R and its subset $Q \subset R$ with complement of cardinality ≥ 3, the space $\overline{M}_{0,R}$ is the source of the standard *forgetful morphism* $\psi_Q : \overline{M}_{0,R} \to \overline{M}_{0,R\setminus Q}$: "forget the subset of sections labeled by Q and stabilize".

Thus we get the diagram

$$\begin{tikzcd} \overline{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}} \arrow[r, "\psi_\Sigma"] \arrow[d, "\psi_{\{\bullet\}}"] & \overline{M}_{0,P\sqcup\{\bullet\}} \\ \overline{M}_{0,\Sigma\sqcup P} & \end{tikzcd} \tag{2.1}$$

Another standard morphism identifies the vertical arrow in (2.1) with the projection of the universal $(\Sigma \sqcup P)$–labeled curve $\overline{C}_{0,\Sigma\sqcup P}$ to its base: cf. e.g. [Ma], Ch. V, Theorem 4.5.

From the explicit form of this identification, one easily sees that the image in $\overline{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ of the section $x_j : \overline{M}_{0,\Sigma\sqcup P} \to \overline{C}_{0,\Sigma\sqcup P}$ for $j \in \Sigma \sqcup P$ is precisely the boundary divisor of $\overline{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ corresponding to the stable 2–partition

$$\Sigma \sqcup P \sqcup \{\bullet\} = \{\bullet, j\} \sqcup ((\Sigma \sqcup P) \setminus \{j\}) \tag{2.2}$$

Here we will denote this divisor by D_j.

Consider now (2.1) as the family of maps of class $\mathbf{1}$, in which *only the sections x_j for $j \in \Sigma$ are counted as structure sections,* whereas those labeled by P are discarded. Then the family will not be stable anymore: if an irreducible component of fiber curve contains only three special points and one of them corresponds to the section labeled by an element of P, then this component will be contracted by ψ_Σ. We can stabilize this new family and get a diagram

$$\begin{tikzcd} \overline{C} \arrow[r, "\overline{\psi}_\Sigma"] \arrow[d] & \overline{M}_{0,P\sqcup\{\bullet\}} \\ \overline{M}_{0,\Sigma\sqcup P} & \end{tikzcd} \tag{2.3}$$

endowed additionally with sections labeled by Σ and the stabilizing morphism

$$\chi : \overline{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}} \to \overline{C}, \quad \psi_\Sigma = \overline{\psi}_\Sigma \circ \chi. \tag{2.4}$$

For each $j \in \Sigma$, denote by $\xi_j : \overline{M}_{0,\Sigma\sqcup P} \to \overline{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ the section of $\psi_{\{\bullet\}}$ identifying $\overline{M}_{0,\Sigma\sqcup P}$ with $D_j \subset \overline{M}_{0,\Sigma\sqcup P\sqcup\{\bullet\}}$ from (2.2). Consider the map

$$\overline{ev} := (\psi_\Sigma \circ \xi_j \,|\, j \in \Sigma) : \overline{M}_{0,\Sigma\sqcup P} \to (\overline{M}_{0,P\sqcup\{\bullet\}})^\Sigma. \tag{2.5}$$

The stable family (2.3) may be obtained by a base change from the universal family of stable maps of class β. Let

$$\mu : \overline{M}_{0,\Sigma\sqcup P} \to \overline{M}_{0,P\sqcup\{\bullet\}}\langle\Sigma\rangle \tag{2.6}$$

be the respective morphism of bases.

Dimensions of the two smooth irreducible schemes in (2.6) coincide. It is not difficult to see that morphism μ is birational and hence surjective. In fact, consider a generic fiber of $\overline{C}_{0,\Sigma\sqcup P}$. It is simply $\mathbf{P}^1$ with pairwise distinct $(\Sigma\sqcup P(\Pi))$–labeled points. When we discard Σ–labeled ones, we land in $\mathbf{P}^1$ endowed with three points labeled by $P(\Pi)$; inverse images of them are just missing section that we discarded when constructing (2.3) from (2.1); so in fact at a generic point we neither loose, nor gain any information passing from (2.1) to (2.3).

We can now prove the main result of this section.

2.3. Proposition. *We have for $|\Sigma| \geq 3$:*

$$I_{0,\Sigma}(\overline{M}_{0,P\sqcup\{\bullet\}}, \mathbf{1}) = (\overline{ev}, \psi_P)_*([\overline{M}_{0,\Sigma\sqcup P}]) \in A_{|\Sigma|}((\overline{M}_{0,P\sqcup\{\bullet\}})^\Sigma \times \overline{M}_{0,\Sigma}). \tag{2.7}$$

Proof. Since μ is birational and surjective, we can identify the relevant J–class with

$$\mu_*([\overline{M}_{0,\Sigma\sqcup P}]) = [\overline{M}_{0,P\sqcup\{\bullet\}}\langle\Sigma\rangle].$$

In order to prove (2.7), it remains to check that

$$ev \circ \mu = \overline{ev}, \quad st \circ \mu = \psi_P. \tag{2.8}$$

Both facts follow from the discussion in 2.2 above.

3. Boundary curve classes in $\overline{M}_{0,S}$

In this section, after recalling some basic facts about boundary of $\overline{M}_{0,S}$ following [Ma] and [BehMa], we summarize relevant parts of [KeMcK] and fix our notation.

3.1. Boundary strata of $\overline{M}_{0,S}$. The main combinatorial invariant of an S–pointed stable curve C is its *dual graph* $\tau = \tau_C$. Its set of vertices V_τ is (bijective to) the set of irreducible components of C. Each vertex v is a boundary point of the set of *flags* $f \in F_\tau(v)$ which is (bijective to) the set consisting of singular points and S–labeled points on this component. We put $F_\tau = \cup_{v\in V_\tau} F_\tau(v)$. If two components of C intersect, the respective two vertices carry two flags that are grafted to form an *edge* e connecting the respective vertices; the set of edges is denoted E_τ. The flags that are not pairwise grafted are called *tails.* They form a set T_τ which is naturally bijective to the set of S–labeled points and therefore itself is labeled by S. Stable curves of genus zero correspond to trees τ whose each vertex carries at least three flags.

The space $\overline{M}_{0,S}$ is a disjoint union of locally closed strata M_τ indexed by stable S–labeled trees. Each such stratum M_τ represents the functor of families consisting of curves of combinatorial type τ. In particular, the open stratum $M_{0,S}$ classifies irreducible smooth curves with pairwise distinct S–labeled points. Its graph is a star: tree with one vertex, to which all tails are attached, and having no edges.

Generally, a stratum M_τ lies in the closure $\overline{M}_\sigma$ of M_σ, iff σ can be obtained from τ by contracting a subset of edges. Closed strata $\overline{M}_\sigma$ corresponding to trees with nonempty set of edges, are called *boundary* ones. The number of edges is the codimension of the stratum.

3.1.1. Boundary divisors and $A^1(\overline{M}_{0,S})$. The classes of boundary divisors generate the whole Chow ring, but are not linearly independent. The following useful basis is constructed in [FG].

For $s \in S$, let $\mathcal{L}_s$ be the line bundle on $\overline{M}_{0,S}$ whose fiber over a stable curve $(C, (x_t \,|\, t \in S))$ is $T^*_{x_s}C$. Put $\psi_s := c_1(\mathcal{L}_s)$

3.1.2. Proposition. *The classes of boundary divisors D_S with $|S_1|, |S_2| \geq 3$ and classes $\psi_i := c_1(L_s)$, $s \in S$, constitute a basis of the group $A^1(\overline{M}_{0,S})$.*

The rank of this group is $2^{n-1} - \dfrac{n(n-1)}{2} - 1$.

This is Lemma 2 in [FG]. An expression of ψ_s through boundary classes is given in Lemma 1 of [FG].

Below we give some details on one–dimensional strata.

3.2. Boundary curves and $A_1(\overline{M}_{0,S})$: preparatory combinatorics. We start with the following combinatorial construction.

We will use here the term *an unordered partition* of a set S as synonymous to *an equivalence relation* on S. A component of a partition is the same as an equivalence class of the respective relation; in particular, all components are non–empty.

Call an unordered partition Π of S *distinguished*, if it consists of precisely *four components*. Denote by the $S(\Pi)$ the set of the components, that is, the quotient of S wrt the respective equivalence relation.

Distinguished partitions are in a natural bijection with isomorphism classes of *distinguished stable S–labeled trees* π. By definition, such a tree is endowed with one *distinguished vertex* v_0, the set of flags at this vertex $F_\pi(v_0)$ being (labeled by) elements of $S(\Pi)$. Clearly this vertex is of multiplicity four. The flags labeled by one–element components $\{s\}$ of Π are tails, carrying the respective labels $s \in S$. The remaining flags are halves of edges; the second vertex of an edge, whose one half is labeled by a component S_i carries tails labeled by elements of S_i.

We will routinely identify $F_\pi(v_0)$ with $S(\Pi)$.

3.2.1. Definition. *(i) Given a distinguished partition* Π*, denote by* $P = P(\Pi)$ *the set of those stable 2–partitions* σ *of* S*, each component of which is a union of two different components of* Π*. For* $|S| \geq 4$ *we have* $|P(\Pi)| = 3$.

(ii) $N = N(\Pi)$ *is the set of those stable 2–partitions of* S *whose one component coincides with one component of* Π*. We have for* $|S| \geq 5$*:* $1 \leq |N(\Pi)| \leq 4$.

3.2.2. Lemma. Π *can be uniquely reconstructed from* $P(\Pi)$*; hence* $P(\Pi)$ *uniquely determines* $N(\Pi)$ *as well.*

Proof. In fact, if $\Pi = (S_1, S_2, S_3, S_4)$ (numeration arbitrary), then by definition $P(\Pi)$ must consist of partitions

$$\sigma_1 = (S_1 \cup S_2, S_3 \cup S_4),\ \sigma_2 = (S_1 \cup S_3, S_2 \cup S_4),\ \sigma_3 = (S_1 \cup S_4, S_2 \cup S_3)$$

Hence conversely, knowing $P(\Pi)$, we can unambiguously reconstruct Π: its components are exactly non–empty pairwise intersections of components of different $\sigma_i \in P(\Pi)$.

3.3. Boundary curves and $A_1(\overline{M}_{0,S})$: geometry. Each distinguished partition Π of S determines the following boundary stratum of $\overline{M}_{0,S}$:

$$b_\Pi : \quad \overline{M}_\Pi := \cap_{\sigma \in N(\Pi)} D_\sigma \hookrightarrow \overline{M}_{0,S}\,. \tag{3.1}$$

Equivalently, $\overline{M}_\Pi$ is the stratum, corresponding to the special tree π associated to Π. In other words, now all components of Π are indexed by the flags $f \in F_\pi(v_0)$ at the special vertex v_0, whereas components of cardinality ≥ 2 are also naturally indexed by the remaining vertices of π:

$$\overline{M}_\Pi = \overline{M}_{0,F_\pi(v_0)} \times \prod_{v \neq v_0} \overline{M}_{0,F_\pi(v)}. \tag{3.2}$$

Here the equality sign refers to the canonical isomorphism that is defined for any stable marked tree: it produces from such a tree the product of moduli spaces corresponding to the stars of all vertices.

The information about edges determines the embedding morphism (3.1) of such a product as a boundary stratum. On the level of universal curves, it is defined by merging the pairs of sections labeled by halves of an edge.

Codimension of $\overline{M}_\Pi$ is $|N(P)|$, and $1 \leq |N(\Pi)| \leq 4$. Since $|F_\pi(v_0)| = 4$, the moduli space $\overline{M}_{0,F_\pi(v_0)}$ is $\mathbf{P}^1$ with three points naturally labeled by the set of stable partitions of $F_\pi(v_0)$ which in turn is canonically bijective to $P(\Pi)$, cf. 2.1.

Hence the representation (3.2) allows us to define the projection map

$$p = p_\Pi : \overline{M}_\Pi \to B_\Pi := \prod_{v \neq v_0} \overline{M}_{0,F_\pi(v)} \tag{3.3}$$

having three canonical disjoint sections canonically labeled by $P(\Pi)$.

Clearly, all fibers of p_Π are rationally equivalent so that they define a class $\beta = \beta(\Pi) \in A_1(\overline{M}_{0,S})$.

3.3.1. Lemma ([KeMcK]). *(i) For $n := |S| \geq 4$, each boundary curve (one–dimensional boundary stratum) C_τ is a fiber of one of the projections (3.3).*

(ii) $[C_{\tau_1}] = [C_{\tau_2}] \in A_1(\overline{M}_{0,S})$ iff these curves are fibers of one and the same projection (3.3).

We reproduce the proof for further use.

Proof. (i) Since C_τ is a curve, the S–labeled stable tree τ is a tree with $|E_\tau| = n-4$ and hence $|V_\tau| = n-3$. Since the tree is stable, all except one of its vertices must have multiplicity 3. The exceptional vertex denoted $v_0 = v_0(\tau)$ has multiplicity 4.

If we delete from the geometric tree τ the vertex v_0, it will break into 4 connected components. Thus, the set S of labels of tails will be broken into 4 non–empty subsets. Among them there are $|T_\tau(v_0)|$ one–element sets (labels of tails adjacent to v_0), and $|E_\tau(v_0)|$ sets of cardinality ≥ 2: each part consist of labels of those tails that can be reached from the critical vertex by a path (without backtracks) starting with the respective flag. We will denote this partition $\Pi(\tau)$. Hence if we contract all edges of τ excepting those that are attached to v_0, we will get the distinguished tree associated with a distinguished partition $\Pi = \Pi(\tau)$. It determines the required projection.

(ii) Now consider two sets of stable 2–partitions of S produced from $\Pi = \Pi(\tau)$ as in the Definition 3.2.1, and denote them respectively $P(\tau)$ and $N(\tau)$.

First of all, we will check that

$$\begin{gathered}(D_\sigma, C_\tau) = 1, \text{ if } \sigma \in P(\tau),\\ (D_\sigma, C_\tau) = -1, \text{ if } \sigma \in N(\tau), \\ (D_\sigma, C_\tau) = 0 \quad \text{otherwise.}\end{gathered} \tag{3.4}$$

Now we will use formulas and facts proved in [Ma1], III.3 and [KoMaKa], Appendix. In particular, we use the notion of *good monomials*, elements of the commutative polynomial ring freely generated by symbols $m(\sigma)$ where σ runs over stable 2–partitions of S. These monomials form a family indexed by stable S–labeled trees τ: $m(\tau) := \prod_{e\in E_\tau} m(\sigma_e)$ where σ_e is the 2–partition of S obtained by cutting e.

Assume first that $m(\sigma)m(\tau)$ is a good monomial so that $(D_\sigma, C_\tau) = 1$. Then it is of the form $m(\rho)$ where ρ is a stable S–labeled tree with all vertices of multiplicity 3 and an edge e such that $m(\sigma) = m(\rho_e)$. This edge is unambiguously characterized by the fact that after collapsing e in ρ to one vertex, we get the labeled tree (canonically isomorphic to) τ. But the vertex to which e collapses must then have multiplicity larger than 3. It follows that e must collapse precisely to the exceptional vertex v_0 of τ. Conversely, the set of ways of putting e back is clearly in a bijection with $P(\tau)$: the 4 flags adjacent to v_0 must be distributed in two groups, 2 flags in each, that will be adjacent to two ends of e.

Assume now that $m(\sigma)$ divides $m(\tau)$. Using Proposition 1.7.1 of [KoMaKa], one sees that $m(\sigma)m(\tau)$ represents zero in the Chow ring (and so $(D_\sigma, C_\tau) = 0$) unless $\sigma = \tau_e$ where e is an edge adjacent to v_0. In this latter case Kaufmann's formula (1.9) from [KoMaKa] implies $(D_\sigma, C_\tau) = -1$. The set of such σ's is in a bijection with $N(\tau)$.

Finally, for any other stable 2–partition σ there exists an $e \in E_\tau$ such that we have $a(\sigma, \tau_e) = 4$ in the sense of [Ma1], III.3.4.1. In this case, $(D_\sigma, C_\tau) = 0$ in view of [Ma], III.3.4.2.

Now, we have $[C_{\tau_1}] = [C_{\tau_2}]$ iff $(D_\sigma, C_{\tau_1}) = (D_\sigma, C_{\tau_2})$ for all stable 2–partitions σ, because boundary divisors generate A^1. In view of (3.4), this latter condition means precisely that

$$P(\tau_1) = P(\tau_2), \quad N(\tau_1) = N(\tau_2).$$

But lemma 3.2.2 shows that in this case $\Pi(\tau_1) = \Pi(\tau_2)$. This completes the proof.

3.4. Proposition. *Denote the canonical class of $\overline{M}_{0,S}$ by K_S. Using notation of 3.3, we have*

$$(-K_S, \beta(\Pi)) = 2 - |N(\Pi)|. \tag{3.5}$$

Proof. For $2 \le j \le [n/2]$, denote by B_j the sum of all divisors D_σ such that one part of the partition σ is of cardinality j, and by B the sum of all boundary divisors. We have

$$-K_S = 2B - \sum_{j=2}^{[n/2]} \frac{j(n-j)}{n-1} B_j \tag{3.6}$$

(cf. [KeMcK], [FG], and references therein).

For a stable 2–partition $\sigma = (S_1, S_2)$ of S, put $c(\sigma) := |S_1||S_2|$. Then, combining (1.4) and (1.6), we get:

$$(-K_S, \beta(\Pi)) = 2(|P(\tau)| - |N(\tau)|) - \sum_{\sigma \in P(\tau)} \frac{c(\sigma)}{n-1} + \sum_{\sigma \in N(\tau)} \frac{c(\sigma)}{n-1}. \tag{3.7}$$

The most straightforward way to pass from (3.7) to (3.5) is to consider the four cases $|N(\Pi)| = 1, 2, 3, 4$ separately. Here is the calculation for $|N(\Pi)| = 3$; it demonstrates the typical cancellation pattern. We leave the remaining cases to the reader.

We have $2(|P(\Pi)| - |N(\Pi)|) = 0$. Let $(1, a, b, c)$ be the cardinalities of the components of Π, where $a, b, c \ge 2$, $a + b + c = n - 1$. Then $P(\Pi)$ consists of three partitions, of the following cardinalities respectively

$$(a+1, b+c), \ (b+1, a+c), \ (c+1, a+b).$$

Hence

$$\sum_{\sigma \in P(\Pi)} c(\sigma) = 2(ab + ac + bc) + 2(a + b + c).$$

Similarly, partitions in $N(\Pi)$ produce the list

$$(a, 1+b+c),\ (b, 1+a+c),\ (c, 1+a+b)$$

so that

$$\sum_{\sigma\in N(\Pi)} c(\sigma) = 2(ab+ac+bc) + (a+b+c).$$

Combining all together, we get $(-K_S, \beta(\Pi)) = -1 = 2 - |N(\Pi)|$.

3.5. Proposition. *Each class of a boundary curve β is indecomposable in the cone of effective curves.*

Proof. This follows from (3.5) and [KeMcK], Lemma 3.6: $(K_S + B, \beta(\Pi)) = 1$, and the divisor $K_S + B$ is ample.

3.6. Examples: $\overline{M}_{0,4}$ and $\overline{M}_{0,5}$. If $|S| = 4$, there is one distinguished partition Π, with all components of cardinality 1. The respective "boundary" curve is in fact the total space $\overline{M}_{0,S}$.

If $|S| = 5$, the boundary curves are 10 exceptional curves on the del Pezzo surface $\overline{M}_{0,S}$ corresponding to 10 different distinguished partitions of S whose components have cardinalities $(1, 1, 1, 2)$. They define 10 different Chow classes.

3.7. Example: $\overline{M}_{0,6}$. There are two combinatorial types of unlabeled trees τ corresponding to boundary curves. Below we draw their subgraphs consisting of all vertices and edges, and mark them with the numbers of tails at each vertex.

$$3 \bullet - \bullet\, 1 - \bullet 2 \qquad\qquad 2 \bullet - \bullet\, 2 - \bullet 2$$

If we take into account possible labellings by S, we will get 60 boundary curves of the first type and 45 boundary curves of the second type. They form two different S_6–orbits.

If τ is of the first type, then $c(\sigma) = 8$ for all 3 partitions $\sigma \in P(\tau)$. The set $N(\tau)$ contains unique partition σ, with $c(\sigma) = 9$. Applying Proposition 3.4, we get

$$(-K_6, C_\tau) = 1.$$

If τ is of the second type, we have respectively $c(\sigma) = 8, 9, 9$ for $\sigma \in P(\tau)$. The set $N(\tau)$ consists of 2 partitions σ, with $c(\sigma) = 8$. Applying Proposition 3.4, we get

$$(-K_6, C_\tau) = 0.$$

Chow classes of the boundary curves for $n = 6$ are extremal rays of the Mori cone. There are 20 classes of the first type and 45 classes of the second type.

3.8. Example: $\overline{M}_{0,7}$. Similarly, there are four combinatorial types of unlabeled trees τ corresponding to boundary curves.

$$A: \quad 3\bullet - \bullet 1 - \bullet 1 - \bullet 2 \qquad\qquad B: \quad 2\bullet - \bullet 2 - \bullet 1 - \bullet 2$$

and

$$C: \; 3\bullet - \bullet\!\begin{smallmatrix} /\bullet 2 \\ \backslash\bullet 2 \end{smallmatrix} \qquad\qquad D: \quad 2\bullet - 1\bullet\!\begin{smallmatrix} /\bullet 2 \\ \backslash\bullet 2 \end{smallmatrix}$$

Here the numerology looks as follows.

Type A. We have $c(\sigma) = 10$ for all $\sigma \in P(\tau)$; $|N(\tau)| = 1, c(\sigma) = 12$ for $\sigma \in N(\tau)$. Hence

$$(-K_7, C_\tau) = 1.$$

Finally, there are 420 labeled trees/boundary curves of this type.

Type B. We have $c(\sigma) = 10, 12, 12$ for $\sigma \in P(\tau)$; $|N(\tau)| = 2, c(\sigma) = 10, 12$ for $\sigma \in N(\tau)$. Hence

$$(-K_7, C_\tau) = 0.$$

There are 630 boundary curves of this type.

Type C. We have $c(\sigma) = 10$ for all $\sigma \in P(\tau)$; $|N(\tau)| = 1, c(\sigma) = 12$ for $\sigma \in N(\tau)$. Hence

$$(-K_7, C_\tau) = 1.$$

There are 105 boundary curves of this type.

Type D. Finally, here $c(\sigma) = 12$ for all $\sigma \in P(\tau)$; $|N(\tau)| = 3, c(\sigma)=10$ for $\sigma \in N(\tau)$, and

$$(-K_7, C_\tau) = -1.$$

There are 105 boundary curves of this type.

In the Chow group, there are 35 classes of types A and C altogether, 210 classes of type B, and 105 classes of type D.

4. Gromov–Witten correspondences for boundary curves in $\overline{M}_{0,S}$.

In this section we will state and prove the main theorem of this paper. We start with some preparation.

4.1. Preparation: combinatorics. In this section, we choose and fix two disjoint finite sets S and Σ. Assume that $|S| \geq 4$, $|\Sigma| \geq 3$.

Fix one element $s_0 \in S$. Choose and fix a distinguished partition Π of S into four disjoint nonempty subsets (cf. 3.2 above). Denote by $S(\Pi)$ the set, elements of which are components of Π. Thus, $|S(\Pi)| = 4$. Denote by $\bullet \in S(\Pi)$ the component of Π that contains the marked element $s_0 \in S$.

The sets $P(\Pi)$ and $N(\Pi)$ are defined as in 3.2.1. In our setup, the three–element set $P(\Pi)$ is canonically bijective to two more sets:

a) The set of stable unordered partitions of $S(\Pi)$ into two parts (each consisting of two elements).

b) The set $S(\Pi) \setminus \{\bullet\}$: any $j \in S(\Pi) \setminus \{\bullet\}$ corresponds to the partition $S(\Pi) = (\{\bullet, j\} \sqcup S(\Pi) \setminus \{\bullet, j\})$. We have already used this trick in sec. 2.1, and here we will use it again transporting the results of sec. 2 to a new context.

Slightly abusing notation, we will sometimes consider these last identifications as identical maps.

Being more fussy, we can say that our constructions are functorial on the category of pointed finite sets S with bijections. Eventually, they must be extended to the category of marked trees (and more general modular graphs) encoding boundary combinatorial types of curves and maps. Dependence of our geometric construction on the target boundary curve class β is reflected in the dependence of its combinatorial side on Π.

4.2. Preparation: geometry. We intend to show that results of sec. 1.5–1.6 are applicable in the present situation.

More precisely, specialize the objects, introduced in 1.5 in the following way:

$$W := \overline{M}_{0,S}, \quad E := \overline{M}_{\Pi}, \quad b := b_{\Pi}, \quad \beta := \beta(\Pi) \tag{4.1}$$

(cf. (3.1)).

Furthermore, specialize the objects described in 1.6:

$$B := B_{\Pi}, \quad C := \overline{M}_{0,F_{\pi}(v_0)}, \quad p := p_{\Pi} \tag{4.2}$$

(cf. (3.2), (3.3)).

4.3. Proposition. *The assumptions 1.6 (a)–(d) hold for (4.1)–(4.2).*

Proof. The assumptions 1.6 (a) and (b) hold by definition. We start with checking the critical assumption 1.6 (d).

In our context it says that the structure embedding $b_\Pi : \overline{M}_\Pi \to \overline{M}_{0,S}$ induces a canonical isomorphism

$$\widetilde{b}_\Pi : \quad \overline{M}_{0,\Sigma}(\overline{M}_\Pi, \beta_\Pi) \to \overline{M}_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi)). \tag{4.3}$$

where β_Π is the Chow class of a fiber of $p_\Pi : \overline{M}_\Pi \to B_\Pi$.

One T–point of $\overline{M}_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi))$ is a family of prestable curves $\mathcal{C}_T/T$ together with a stable map f_T of the class $\beta(\Pi)$ and labeled sections

$$f_T : \mathcal{C}_T \to \overline{M}_{0,S}, \quad x_{j,T} : T \to \mathcal{C}_T,\ j \in \Sigma. \tag{4.4}$$

The crucial fact is that any such map f_T can be factored through b_Π. More precisely, we will show that f_T determines and is determined by a unique pair of maps

$$\varphi : T \to B_\Pi = \prod_{v \neq v_0} \overline{M}_{0,F_\pi(v)}, \quad \mathcal{C}_T \to \varphi^*(\overline{M}_\Pi) \tag{4.5}$$

In particular, all effective curves in the class $\beta(\Pi)$ are represented by the fibers of the family $p_\Pi : \overline{M}_\Pi \to B_\Pi$.

In order to produce (4.5) from f_T, notice that the morphism $f_T : \mathcal{C}_T \to \overline{M}_{0,S}$ is essentially the same thing as a flat family of S–labeled stable curves over $\mathcal{C}_T$ together with its explicit identification with $f_T^*(\overline{C}_{0,S})$, where $\overline{C}_{0,S} \to \overline{M}_{0,S}$ is the universal family of S–labeled stable curves of genus zero.

Consider first a boundary divisor $D_\sigma \subset \overline{M}_{0,S}$ such that $(D_\sigma, \beta(\Pi)) = -1$. Over D_σ, fibers of the universal family $\overline{C}_{0,S}$ are exactly those S–labeled curves whose dual graphs contain an edge producing the partition σ.

Since $\beta(\Pi)$ is indecomposable (Proposition 3.5), for any fiber of $\mathcal{C}_T/T$, f_T must contract to a point each its component excepting one.

If there is a point x on such un–contracted component over which the fiber of the induced family $f_T^*(\overline{C}_{0,S})$ has a dual graph containing no edge producing σ, then the fiber of $\mathcal{C}_T \to T$ containing x must have intersection index ≥ 0 with D_σ (cf. [Ma], III.3.6).

This is impossible, so that f_T must factor through the closed embedding

$$\mathcal{C}_T \to D_\sigma \hookrightarrow \overline{M}_{0,S},$$

and hence through the embedding

$$\mathcal{C}_T \to \cap_{\sigma \in N(\Pi)} D_\sigma = \overline{M}_{0,F_\pi(v_0)} \times \prod_{v \neq v_0} \overline{M}_{0,F_\pi(v)} \hookrightarrow \overline{M}_{0,S}, \tag{4.6}$$

cf. (3.1), (3.2).

We will now establish that the first arrow in (4.6), which we will denote $\overline{f}_T$, can be embedded into a commutative diagram:

$$\begin{tikzcd} \mathcal{C}_T \arrow[r, "\overline{f}_T"] \arrow[d] & \overline{M}_\Pi \arrow[d, "p_\Pi"] \\ T \arrow[r, "\varphi"] & B_\Pi \end{tikzcd} \tag{4.7}$$

In fact, assume that $p_\Pi \circ \overline{f}_T$ does not factor through $\mathcal{C}_T \to T$. We will show that this leads to a contradiction.

Choose a geometric fiber X of $\mathcal{C}_T \to T$. Denote by σ the dual graph of the curve from the universal family $\overline{C}_{0,S}$ over a generic point of X. We know that σ admits a contraction onto π. If $p_\Pi \circ \overline{f}_T(X)$ is not a point, then X must contain a point over which the dual graph σ' of the universal family is not isomorphic to σ. In this case it must admit a non–trivial contraction $\sigma' \to \sigma$. Compose it with the canonical contraction $\sigma \to \pi$.

One of the following two alternatives must hold:

(A) There is an edge of σ' that contracts onto one of the vertices $v \neq v_0$ of π.

(B) No edge of σ' contracts onto one of the vertices $v \neq v_0$, but there is an edge contracting to v_0.

Consider the stable 2–partition ρ of S corresponding to the contracting edge, and the respective boundary divisor D_ρ in $\overline{M}_{0,S}$. Geometrically, our assumption (A) implies that $f(X)$ is a curve that does not lie in D_ρ but intersects D_ρ, hence we must have

$$(D_\rho, \beta) = (D_\rho, f_*([X])) > 0.$$

But from (3.4) it follows that if ρ contracts onto a vertex $v \neq v_0$, then $(D_\rho, \beta) = 0$. Hence this possibility is excluded.

Consider now the alternative (B). Then we must have $\rho \in P(\Pi)$. This implies the following degeneration pattern of the induced family of curves parametrized by X. At a generic point, the tree of the curve consists of one irreducible component C to which trees are attached at $|N(\tau)|$ different points of this component. When the degeneration at a point of D_ρ occurs, C breaks down into two components, say, C_1 and C_2, and the attached trees are distributed among them: some become attached to C_1, and remaining ones to C_2. What is important here, is that the *labeled combinatorial type of each of the attached trees does not change* – otherwise we could have used the option (A) which was already excluded.

But in this case the image $p_\Pi \circ \overline{f}_T(X)$ must land in the product of the *open strata* $\prod_{v \neq v_0} M_{F_\pi(v)}$. This is possible only if this image is a point because such a product is an affine scheme.

It order to complete the proof that (4.3) is an isomorphism, it remains to perform several simple checks:

(a) The described map on T–points naturally extends to morphisms of T–points, and we get a functor.

(b) Following our construction in reverse direction, we can construct a functor from the rhs of (4.3) to the lhs.

(c) The constructed two functors are (quasi)inverse to each other.

We leave them as an exercise to the reader.

It remains to check the unobstructedness condition (1.19) implying, in particular, that

$$[\overline{M}_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi))]^{virt} = [\overline{M}_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi))].$$

Let C be a geometric fiber of $p : \overline{M}_\Pi \to B_\Pi$. We have already used the fact that it is isomorphic to $\mathbf{P}^1$. Let $j : C \to \overline{M}_{0,S}$ be the natural closed embedding. We assert that

$$j^*(\mathcal{T}_{\overline{M}_{0,S}}) \cong \mathcal{O}(2) \oplus \mathcal{O}^{n-4-|N(\Pi)|} \oplus \mathcal{O}(-1)^{|N(\Pi)|} \tag{4.8}$$

where $\mathcal{T}_{\overline{M}_{0,S}}$ is the tangent sheaf and $\mathcal{O} := \mathcal{O}_C$.

In fact, consider the embedding $i : C \to \overline{M}_\Pi$ and the natural filtration

$$\{0\} \subset \mathcal{T}_C \subset i^*(\mathcal{T}_{\overline{M}_\Pi}) \subset j^*(\mathcal{T}_{\overline{M}_{0,S}}). \tag{4.9}$$

The consecutive summands in (4.8) correspond to the consecutive quotients of (4.9). Namely, $\mathcal{T}_C \cong \mathcal{O}(2)$; $i^*(\mathcal{T}_{\overline{M}_\Pi})/\mathcal{T}_C$ is trivial of the rank

$$\dim B_\Pi = \sum_{v\in V_\pi} (|F_\pi(v)| - 2) = |S| - 4 - |N(\Pi)|, \tag{4.10}$$

finally, the last isomorphism follows from (3.1) and (3.4).

From (4.8) we see that $H^1(C, j^*(\mathcal{T}_{\overline{M}_{0,S}})) = 0$. It follows that B_Π is the base of universal deformation of any fiber C.

The virtual fundamental class is simply the fundamental class in this "locally convex" case. The virtual dimension (1.7) of our moduli stack, in view of (3.5), is

$$2 - |N(\Pi)| + |\Sigma| + |S| - 6.$$

It coincides with actual dimension (cf. (4.10)):

$$\dim B_\Pi + \dim \overline{M}_{0,F_\pi(v_0)}\langle\Sigma\rangle = |S| - 4 - |N(\Pi)| + |\Sigma|. \tag{4.11}$$

4.4. The final summary. We will now briefly restate the results of stepwise calculations of sec. 1 and 2 in our current situation (4.1) – (4.2).

4.4.1. Step I: Gromov–Witten correspondences for the target space $\overline{M}_{0,S(\Pi)}$. We reproduce here the main result of sec. 2 applied to the target space $\overline{M}_{0,S(\Pi)}$ and its fundamental class $\mathbf{1}$. Notice that the sets denoted S (resp. P) in sec. 2 are now $S(\Pi)$ (resp. $P(\Pi)$), and $S(\Pi) = P(\Pi) \sqcup \{\bullet\}$.

According to Proposition 2.3, we have:

$$I_{0,\Sigma}(\overline{M}_{0,P(\Pi)\sqcup\{\bullet\}}, \mathbf{1}) = (\overline{ev}, \psi_{P(\Pi)})_*([\overline{M}_{0,\Sigma\sqcup P(\Pi)}]) \in$$
$$\in A_{|\Sigma|}((\overline{M}_{0,P(\Pi)\sqcup\{\bullet\}})^\Sigma \times \overline{M}_{0,\Sigma}). \tag{4.12}$$

4.4.2. Step II: Gromov–Witten correspondences for the target space B_Π and zero beta–class. Acccording to the Example 1.3, we have:

$$I_{0,\Sigma}(B_\Pi, 0) = [\Delta_\Sigma(B_\Pi) \times \overline{M}_{0,\Sigma}] \in A_*(B_\Pi^\Sigma \times \overline{M}_{0,\Sigma}). \tag{4.13}$$

Here $\Delta_\Sigma(B_\Pi)$ is the diagonal in the cartesian product B_Π^Σ of Σ copies of B_Π.

4.4.3. Step III: Gromov–Witten correspondences for the target space M_Π and fiber beta–class. In this subsection, β_Π is the Chow class of a fiber of the projection $\overline{M}_\Pi \to B_\Pi$. We now have a canonical splitting

$$\overline{M}_\Pi = B_\Pi \times \overline{M}_{0,P(\Pi)\sqcup\{\bullet\}} \tag{4.14}$$

since $F_\pi(v_0)$ is identified with $S(\Pi) = P(\pi) \sqcup \{\bullet\}$ (cf. 3.3).

Thus using (4.12) and (4.13), we have

$$I_{0,\Sigma}(\overline{M}_\Pi, \beta_\Pi) = \widetilde{\Delta}^!([\Delta_\Sigma(B_\Pi) \times \overline{M}_{0,\Sigma}] \otimes (\overline{ev}, \psi_{P(\Pi)})_*([\overline{M}_{0,\Sigma\sqcup P(\Pi)}]))\,. \tag{4.15}$$

To summarize, we have proved our final theorem, a specialization of Proposition 1.6.1:

4.5. Theorem. *The structure embedding $b_\Pi : \overline{M}_\Pi \to \overline{M}_{0,S}$ induces a canonical isomorphism*

$$\widetilde{b}_\Pi : \ \overline{M}_{0,\Sigma}(\overline{M}_\Pi, \beta_\Pi) \to \overline{M}_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi)). \tag{4.16}$$

where β_Π is the Chow class of a fiber of $p_\Pi : \overline{M}_\Pi \to B_\Pi$.

This isomorphism $\widetilde{b}_\Pi$ is compatible with evaluation/stabilization morphisms for both moduli spaces and induces the identity

$$I_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi)) = (b_\Pi^\Sigma \times id)_*(I_{0,\Sigma}(\overline{M}_\Pi, \beta_\Pi)) \tag{4.17}$$

where

$$b_\Pi^\Sigma \times id : \quad \overline{M}_\Pi^\Sigma \times \overline{M}_{0,\Sigma} \to (\overline{M}_{0,S})^\Sigma \times \overline{M}_{0,\Sigma}.$$

The rhs of (4.17) is given by (4.15).

4.6. Gromov–Witten numbers. In this subsection, we will specialize formula (1.18) to our situation in order to calculate numerical invariants of Chow classes of boundary curves.

Let $\gamma_j \in H^{2d_j}(\overline{M}_{0,S})$ be a family of cohomology classes indexed by $j \in \Sigma$. If $\sum_{j\in\Sigma} d_j = \dim B_\Pi$, then the correspondence

$$I_{0,\Sigma}(\overline{M}_{0,S}, \beta(\Pi)) \in A_*((\overline{M}_{0,S})^\Sigma \times \overline{M}_{0,\Sigma})$$

maps $\otimes_{j\in\Sigma}\gamma_j \in (H^*(\overline{M}_{0,S}))^{\otimes\Sigma}$ to a class of maximal dimension in $H^*(\overline{M}_{0,\Sigma})$. The degree of this class is denoted

$$\langle I^{\overline{M}_{0,S}}_{0,\Sigma,\beta(\Pi)}\rangle(\otimes_{j\in\Sigma}\gamma_j).$$

Generally, this degree is the virtual number of stable maps of pointed curves of class $\beta(\Pi)$ satisfying the incidence conditions $f(x_j) \in \Gamma_j$, where (Γ_j) are cycles in general position whose dual classes are γ_j:

$$f : (C; (x_j \,|\, j \in \Sigma)) \to \overline{M}_{0,S},$$

whenever such incidence conditions are strong enough to enforce existence only of finite (virtual) number of such maps. In our unobstructed case, this virtual number is the actual number of such maps whenever the incidence cycles are in general position.

Recall also that this number is polylinear in (γ_j).

4.6.1. Proposition. *We have*

$$\langle I^{\overline{M}_{0,S}}_{0,\Sigma,\beta(\Pi)}\rangle(\otimes_{j\in\Sigma}\gamma_j) = \deg\left(\cap_{j\in\Sigma} pr_{B_\Pi *} \circ b^*_\Pi(\gamma_j)\right) . \tag{4.18}$$

Sketch of proof. Skipping a clumsy but straightforward formal derivation of (4.18) from (1.18), we describe the geometric content of this counting formula in the general situation axiomatized in 1.6.

First of all, (1.18) reduces the count to the case of an incidence condition represented by some cycles in $E = M_\Pi$: in fact, $b^*_\Pi(\gamma_j)$ are represented by $\Gamma_j \cap \overline{M}_\Pi$ in the case of transversal intersections.

Now, in $\overline{M}_\Pi$ the incidence cycles can be replaced by ones of the form $\Delta_j \times c_j + \Delta'_j \times C$ where c_j are points on a projective line C as in (4.2) corresponding to the decomposition $\overline{M}_\Pi = B_\Pi \times C$.

Assume first that $\Delta'_j \neq 0$ for some $j = j_0$. If for such an incidence condition there is a fiber C_0 of $\overline{M}_\Pi \to B_\Pi$ satisfying it at all, then the number of relevant pointed stable maps must be infinite, because x_{j_0} can be chosen arbitrarily along this fiber. Hence decomposable cycles containing at least one factor of the form $\Delta'_j \times C$ give zero contributions to (4.18).

Now consider the case of incidence conditions of the form $\Delta_j \times c_j$ for all $j \in \Sigma$. Let $\Delta_j = pr_{B_\Pi}(\Delta_j \times c_j)$ be in a general position in B_Π so that the intersection cycle $\cap_{j\in\Sigma}\Delta_j$ is a sum of points $y_a \in B_\Pi$, of multiplicity one each. We can also lift Δ_j arbitrarily to M_Π, that is choose all $c_j \in C$ pairwise distinct, and consider $\Delta_j \times c_j$ as a geometric incidence condition representing the initial cohomological incidence condition (γ_j).

After that the geometric count becomes straightforward: each point y_a produces one fiber of the class $\beta(\Pi)$ intersecting each $\Delta_j \times c_j$ at one point corresponding to c_j.

The number of (y_a) is the right hand side of (4.18), and the curve count interprets the left hand side of (4.18).

5. Examples and remarks

5.1. "Naturality" of Gromov–Witten correspondences. In this subsection we try to make somewhat more precise our guess 0.2.1. To this end we recall first, that natural objects in the relevant category are moduli spaces $\overline{M}_\tau$, and natural morphisms/correspondences are those ones that are produced from morphisms in the category of modular graphs. The latter include contractions, forgetful morphisms, relabeling morphisms etc., cf. [BehMa].

The least controllable characteritic of GW–correspondences is their dependence on the argument β in the relevant Mori cone. So far we have considered only boundary β's, and they are, of course, "natural" by definition.

In this subsection we will show that, keeping notation of section 4, we may naturally encode most of the relevant combinatorial and geometric information in one moduli space $\overline{M}_{0,\Sigma\times(S\setminus\{s_0\})}$ and a configuration of certain of its boundary strata. This is only a tentative suggestion, we do not develop it fully, because we still lack even a conjectural description of the situation for more general β's.

5.1.1. The tree T. The tree $\mathbf{T}$ has one special vertex that will be called *central* one and denoted v_c. Its flags are bijectively labeled by Σ: we will use the notation

$$F_{\mathbf{T}}(v_c) := \{\langle j\rangle \,|\, j \in \Sigma\}\ . \tag{5.1}$$

The remaining vertices constitute a set bijective to $\Sigma \times \{s_0\}$. Together with (5.1), this bijection is a part of structure, and we may refer to a non–central vertex $v \in V_{\mathbf{T}}$ as $v_j := \langle j, s_0\rangle,\ j \in \Sigma$.

Furthermore, we put

$$F_{\mathbf{T}}(v_j) := \{j\} \times S = \{(j,s) \,|\, s \in S\}\ . \tag{5.2}$$

Thus, the standard identification of $\overline{M}_{\mathbf{T}}$ with the product of moduli spaces $\prod_{v\in V_{\mathbf{T}}} \overline{M}_{0,F_{\mathbf{T}}(v)}$ corresponding to stars of all vertices, can be rewritten as

$$\overline{M}_{\mathbf{T}} = (\overline{M}_{0,S})^{\Sigma} \times \overline{M}_{0,\Sigma} \tag{5.3}$$

where the last factor corresponds to the central vertex.

(ii) Edges. The flag $\langle j\rangle$ attached to the central vertex (see (5.1)) is grafted to the flag (j,s_0) incident to the vertex v_j (see (5.2)). There are no more edges.

Thus, the central vertex carries no tails, and the set of edges $E_{\mathbf{T}}$ is naturally bijective to Σ. The set of tails is

$$T_{\mathbf{T}} = \coprod_{j\in\Sigma} (F_{\mathbf{T}}(\langle j, s_0\rangle) \setminus (j,s_0)) = \coprod_{j\in\Sigma} (\{j\} \times (S\setminus\{s_0\})) \cong \Sigma\times(S\setminus\{s_0\}). \tag{5.4}$$

If we interpret the last set in (5.4) as the set of labels of tails , then the described above set of edges of $\mathbf{T}$ determines the canonical embedding of $\overline{M}_{\mathbf{T}}$ as a boundary stratum:

$$\overline{M}_{\mathbf{T}} \hookrightarrow \overline{M}_{0,\Sigma\times(S\setminus\{s_0\})} \tag{5.5}$$

This embedding corresponds to full contraction of all edges of $\mathbf{T}$ to the star with flags T_{τ}.

We will now encode information about Π into another tree $\mathbf{T}(\Pi)$, together with its contraction onto $\mathbf{T}$.

5.1.2. The tree $\mathbf{T}(\Pi)$. Briefly, to get $\mathbf{T}(\Pi)$, we replace each non–central vertex v_j, $j\in\Sigma$, by a copy π_j of the tree π described in 3.2.

More precisely, the special vertex of π_j denoted $v_{0,j}$ now carries tails (5.2) distributed among other vertices of π_j according to Π, and its tail (j,s_0) is grafted in $\mathbf{T}(\Pi)$ to the same flag $\langle j\rangle$ of its central vertex as it was in $\mathbf{T}$.

The contraction $\mathbf{T}(\Pi)\to\mathbf{T}$ contracts each π_j to the star of v_j, and is identical on the stars of the central vertices. Combining the relevant boundary morphism with (5.5), we get the diagram of strata embedding

$$\overline{M}_{\mathbf{T}(\Pi)} \hookrightarrow \overline{M}_{\mathbf{T}} \hookrightarrow \overline{M}_{0,\Sigma\times(S\setminus\{s_0\})} \tag{5.5}$$

The intermediate and final correspondences considered in sec. 4, can be expressed using the geometry of (5.5).

5.2. Using the Reconstruction Theorems. For a general target W, if the Chow ring $A^*(W)$ (with coeficients in $\mathbf{Q}$) coincides with the whole $H^*(W)$ and is generated by $A^1(W)$, then the total motivic quantum cohomology of W of genus zero understood as the family of I–correspondences is completely determined by triple correlators (3–point GW–invariants) of codimension zero. This follows from the First and the Second Reconstruction Theorems of [KoMa1].

In any case, these triple correlators are precisely coefficients of small quantum cohomology as a formal series in q^β. Hence in the same assumptions the total quantum cohomology is completely determined by the small quantum multiplication in $H^*(V)[[q^\beta]]$:

$$\Delta_a \cdot \Delta_b = \Delta_a \cup \Delta_b + \sum_{\beta\neq 0}\sum_{c\neq 0}\langle \Delta_a\Delta_b\Delta_c\rangle_\beta \Delta^c q^\beta .$$

Here (Δ_a) is a basis of H^* such that Δ_0 is identity, $g_{ab} = (\Delta_a, \Delta_b)$, (g^{ab}) the inverse matrix to (g_{ab}), and and $\Delta^a := \sum_b g^{ab}\Delta_b$.

This is applicable to all $\overline{M}_{0n}$.

In turn, the associativity equations allow one to express all triple correlators through a part of them. We will now make explicit this subset for $\overline{M}_{0n}$.

5.3. A generating subset of triple correlators. Put $|\Delta| = i$ for $\Delta \in H^{2i}(\overline{M}_{0n})$. (No confusion with cardinality $|S|$ of a set S should arise). Then all invariants can be recursively calculated through 3–point invariants $\langle \Delta_a\Delta_b\Delta_c\rangle_\beta$ with Δ_c divisorial, $|\Delta_a|, |\Delta_b| \geq 1$, $\beta \neq 0$, and

$$|\Delta_a| + |\Delta_b| = (-K_n, \beta) + n - 4.$$

where K_n is the canonical class of $\overline{M}_{0n}$. Hence, β are restricted by

$$2 - (n - 3) \leq (-K_n, \beta) - 1 \leq n - 3.$$

See [KoMa1], Theorem 3.1, with the following easy complements. If $|\Delta_a|$ or $|\Delta_b| = 0$, $\beta \neq 0$, then the respective GW–invariant is 0 because of [KoMa1], (2.7). If $\beta = 0$, we can use [KoMa1], (2.8). It remains to consider the following list of parameters:

$$6 - n \leq (-K_n, \beta) \leq n - 2,$$

$$2 \leq |\Delta_a| + |\Delta_b| = (-K_n, \beta) + n - 4 \leq 2n - 6$$

Finally, if Δ is a divisorial class with $(\Delta, \beta) = 0$, then $\langle \Delta'\Delta''\Delta\rangle_\beta = 0$ for any $\Delta,' \Delta''$ due to the Divisor Axiom.

5.3.1. Tables for the first values of n.

$(-K_5,\beta)$	1	2	3
$(\lvert\Delta_a\rvert,\lvert\Delta_b\rvert)$	(1,1)	(1,2)	(2,2)

$(-K_6,\beta)$	0	1	2	3	4
$(\lvert\Delta_a\rvert,\lvert\Delta_b\rvert)$	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)
			(1,3)		

$(-K_7,\beta)$	-1	0	1	2	3	4	5
$(\lvert\Delta_a\rvert,\lvert\Delta_b\rvert)$	(1,1)	(1,2)	(2,2)	(2,3)	(3,3)	(3,4)	(4,4)
			(1,3)	(1,4)	(2,4)		

Notice that $\overline{M}_{05}$ is the del Pezzo surface of degree 5, in particular, its anticanonical class is ample and hence the generating subset of triple correlators is finite. In fact, generating sets for del Pezzo surfaces are collected in [BaMa1]. It is known also that all del Pezzo surfaces have generically semisimple quantum cohomology, and more generally, this remains true for blow ups of any finite set of points on or over $\mathbf{P}^2$ (A. Bayer).

Already for $\overline{M}_{06}$ the situation is more mysterious. For 45 out of 105 generators of the cone of β's we have $(-K_6,\beta)=0$. Hence our generating list above is in principle infinite. Semisimplicity is an open question as well. For $n\geq 7$ the difficulties grow.

5.4. Strategies of computation. A possible way to compute some Gromov–Witten invariants of $\overline{M}_{0,n}$ with non–boundary β's consists in choosing a birational morphism $p_n:\overline{M}_{0,n}\to X_n$ such that

a) (Sufficiently many) GW–invariants of X_n are known/computable.

b) Morphism p_n is such that there exist "naturality" formulas that allow one to compute (some) GW–invariants of $\overline{M}_{0,n}$ through (some) GW–invariants of X_n.

For "naturality" results see, e. g., [LeLWa], [MauPa], [Hu1], [Hu2], [BrK] (this paper contains corrections to [Hu1]), [Mano1], [Mano2], etc. We will discuss the relevant classes of morphisms below.

5.4.1. Blowing $\overline{M}_{0,n}$ down. The following choices of morphisms seem promising for application of this strategy, at least for small values of n.

(i) $X_n = \mathbf{P}^{n-3}$, p_n = Kapranov's morphism, representing $\overline{M}_{0,n}$ as the result of consecutive blowing up $n-1$ points, preimages of lines connecting pairs of these points, preimages of planes, passing through triples of them etc., cf. [HaT]. It involves forgetting the n–th point, then fixing $p_1, \dots, p_{n-1} \in \mathbf{P}^{n-3}$.

(ii) $X_n = (\mathbf{P}^1)^{n-3}$, p_n is a similar morphism that was described explicitly by Tavakol.

(iii) $X_n = \overline{L}_{n-2}$, the Losev–Manin moduli space parametrizing stable chains of $\mathbf{P}^1$'s with marked points and a specific stability condition; p_n the respective stabilization morphism.

It makes sense not just to use $\overline{L}_{n-2}$ in order to help calculate GW–invariants of $\overline{M}_{0,n}$, but to treat these moduli spaces as replacements of $\overline{M}_{0,n}$ in their own right. In fact, one can define GW–invariants based upon $\overline{L}_{n-2}$, essentially, no information is lost thereby: see [BaMa2].

The spaces $\overline{L}_{n-2}$ are toric, and have the largest Chow ring of these three examples. These manifolds are not Fano for $n \geq 6$, but according to [Ir2], any toric manifold has generically semisimple quantum cohomology, therefore it can be more accessible.

(iv) Finally, combining two or more forgetful morphisms, one can birationally map $\overline{M}_{0,n}$ and $\overline{L}_{n-2}$ onto products of similar manifolds, thus opening a way to an inductive calculation of GW–invariants. Here is the simplest example: for $n \geq 5$, forgetting at first x_n, and then all points except for (x_1, x_2, x_3, x_n), we get a birational morphism

$$\overline{M}_{0,n} \to \overline{M}_{0,n-1} \times \overline{M}_{0,4}, \quad \overline{M}_{0,4} \cong \mathbf{P}^1.$$

GW–invariants of a product can be calculated via the general quantum Künneth formula whenever they are known for lesser values of n.

For our main preoccupation here, that of understanding motivic properties of quantum cohomology correspondences, versions of this last suggestion are most promising.

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