ALGEBRAIC TORI AS NISNEVICH SHEAVES WITH TRANSFERS

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Abstract. We relate R-equivalence on tori with Voevodsky's theory of homotopy invariant Nisnevich sheaves with transfers and effective motivic complexes.

Contents

1. Introduction

Let k be a perfect field and let T be a k-torus. Then T has a natural structure of a *homotopy invariant étale sheaf with transfers* in the sense of Voevodsky [18, §3.3]. However, we shall be interested here in T as a *Nisnevich sheaf with transfers* [17].

Surprisingly, this point of view is related to R-equivalence on tori as studied by Colliot-Thélène and Sansuc in [3]. Let L be the group of cocharacters of T. Let us denote by HI the category of homotopy invariant Nisnevich sheaves with transfers over k. The main results of this note are:

Proposition 1. *There is a natural isomorphism* $T_{-1} \xrightarrow{\sim} L$ *in* HI.

(The definition of T_{-1} is recalled in the proof of Proposition 2).

Since $T_{-1} = \underline{\mathrm{Hom}}(\mathbb{G}_m, T)$ in HI (*e.g.* [5, Lemme 3.4.5] or [9, Prop. 4.3]), adjunction gives a morphism

$$L \otimes_{\mathrm{HI}} \mathbb{G}_m \to T \tag{1}$$

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where $\otimes_{\mathrm{HI}}$ denotes the tensor product in HI. This morphism becomes an isomorphism in the étale topology, but may not be so in the Nisnevich topology, which is the point here.

Theorem 1. *a)* (1) *is an isomorphism [in the Nisnevich topology] if T (i.e. L) is invertible.*
b) In general there is an exact sequence

$$0 \to S_1(k)/R \to (L \otimes_{\mathrm{HI}} \mathbb{G}_m)(k) \to T(k) \to T(k)/R \to 0$$

where S_1 is the kernel of a flasque resolution of T.

Corollary 1. *$S_1(k)/R$ only depends on T.* □

Of course, Corollary 1 is easy to prove directly: see end of Section 2.

Corollary 2. *The assignment $Sm(k) \ni X \mapsto \bigoplus_{x \in X^{(0)}} G(k(x))/R$ provides G/R with the structure of a homotopy invariant Nisnevich sheaf with transfers. In particular, any morphism $f : Y \to X$ of smooth connected schemes induces a morphism $f^* : G(k(X))/R \to G(k(Y))/R$.* □

In Section 3, we get a version of Theorem 1 in the category $\mathrm{DM}_-^{\mathrm{eff}}$ of [18], see Theorem 3 and Corollary 3. In Section 4, we relate this approach to questions of stable birationality as studied in [3] and [4]: this provides alternate proofs of some of their results (at least over a perfect field), *e.g.* Corollaries 4 and 5, the main result being Theorem 4. Finally, we raise a few questions in Section 5.

Acknowledgements. Theorem 1 and Theorem 2 below were obtained in the course of discussions with Takao Yamazaki during his stay at the IMJ in October 2010: I would like to thank him for inspiring exchanges. I also thank Daniel Bertrand for a helpful discussion. Finally, I wish to acknowledge inspiration from the work of Colliot-Thélène and Sansuc on tori [3, 4], which will be obvious throughout this paper.

2. Proofs

We shall actually prove a more general version of Proposition 1 and Theorem 1, which may be useful for other purposes. Let G be a semi-abelian variety over k, which is an extension of an abelian variety A by a torus T.

Lemma 1. *The exact sequence*

$$0 \to T(k) \to G(k) \to A(k)$$

induces an exact sequence

$$0 \to T(k)/R \xrightarrow{i} G(k)/R \to A(k).$$

Proof. Let $f : \mathbf{P}^1 \dashrightarrow G$ be a k-rational map defined at 0 and 1. Its composition with the projection $G \to A$ is constant: thus the image of f lies in a T-coset of G defined by a rational point. This implies the injectivity of i, and the rest is clear. □

Let L be the cocharacter group of T.

Proposition 2. *There is a natural isomorphism $G_{-1} \xrightarrow{\sim} L$ in* HI.

Proof. Recall [17, p. 96] that if $\mathcal{F}$ is a presheaf [with transfers] on smooth k-schemes, the presheaf [with transfers] $\mathcal{F}^p_{-1}$ is defined by

$$U \mapsto \operatorname{Coker}(\mathcal{F}(U \times \mathbf{A}^1) \to \mathcal{F}(U \times \mathbb{G}_m)).$$

If $\mathcal{F}$ is homotopy invariant, we may replace $U \times \mathbf{A}^1$ by U and the rational point $1 \in \mathbb{G}_m$ realises $\mathcal{F}^p_{-1}(U)$ as a functorial direct summand of $\mathcal{F}(U \times \mathbb{G}_m)$.

If $\mathcal{F}$ is a Nisnevich sheaf [with transfers], $\mathcal{F}_{-1}$ is defined as the sheaf associated to $\mathcal{F}^p_{-1}$.

Now $A(U \times \mathbf{A}^1) \xrightarrow{\sim} A(U \times \mathbb{G}_m)$ since A is an abelian variety, hence $A^p_{-1} = 0$. We therefore have an isomorphism of presheaves $T^p_{-1} \xrightarrow{\sim} G^p_{-1}$, and *a fortiori* an isomorphism of sheaves $T_{-1} \xrightarrow{\sim} G_{-1}$. If U is affine, the sequence $T(U) \to T(U \times \mathbb{G}_m) \to L(U) \to 0$ is exact by a direct computation, which concludes the proof. □

Proposition 2 and adjunction now give a morphism

$$L \otimes_{\mathrm{HI}} \mathbb{G}_m \to G. \tag{2}$$

Theorem 2. *a)* (2) *is an isomorphism if G is an invertible torus.*
b) In general there is an exact sequence

$$0 \to S_1(k)/R \to (L \otimes_{\mathrm{HI}} \mathbb{G}_m)(k) \to G(k) \to G(k)/R \to 0$$

where S_1 is the kernel of a flasque resolution of T.

Proof. a) We reduce to the case $T = R_{E/k}\mathbb{G}_m$, where E is a finite extension of k. Let us write more precisely $\mathrm{HI}(k)$ and $\mathrm{HI}(E)$. There is a pair of adjoint functors

$$\mathrm{HI}(k) \xrightarrow{f^*} \mathrm{HI}(E), \quad \mathrm{HI}(E) \xrightarrow{f_*} \mathrm{HI}(k)$$

where $f : \operatorname{Spec} E \to \operatorname{Spec} k$ is the projection. Clearly,

$$f_*\mathbf{Z} = \mathbf{Z}_{\mathrm{tr}}(\operatorname{Spec} E), \quad f_*\mathbb{G}_m = T$$

where $\mathbf{Z}_{\mathrm{tr}}(\operatorname{Spec} E)$ is the (homotopy invariant) Nisnevich sheaf with transfers represented by $\operatorname{Spec} E$. Since $\mathbf{Z}_{\mathrm{tr}}(\operatorname{Spec} E) = L$, this proves the claim.

b) A diagram chase based on Lemma 1 reduces us to the case $G = T$. Choose a flasque resolution $0 \to S_1 \to T_0 \xrightarrow{p} T \to 0$ of T as in [3, §5]. Recall that, if

$$0 \to Q_1 \to L_0 \to L \to 0 \tag{3}$$

is the corresponding sequence of cocharacter groups, L_0 is an invertible lattice (i.e. a direct summand of a permutation lattice) chosen so that $L_0(E) \to L(E)$ is surjective for any extension E/k. In particular, (3) is exact as a sequence of Nisnevich sheaves.

Since $\otimes_{\mathrm{HI}}$ is right exact, the sequence

$$Q_1 \otimes_{\mathrm{HI}} \mathbb{G}_m \to L_0 \otimes_{\mathrm{HI}} \mathbb{G}_m \to L \otimes_{\mathrm{HI}} \mathbb{G}_m \to 0$$

is exact, which implies that in the commutative diagram

$$\begin{array}{ccccccccc} & & (Q_1 \otimes_{\mathrm{HI}} \mathbb{G}_m)(k) & \to & (L_0 \otimes_{\mathrm{HI}} \mathbb{G}_m)(k) & \to & (L \otimes_{\mathrm{HI}} \mathbb{G}_m)(k) & \to & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0}\,\wr & & \downarrow{\scriptstyle \alpha} & & \\ 0 \to & & S_1(k) & \to & T_0(k) & \to & T(k) & \to & T(k)/R \to 0 \end{array}$$

the top sequence is exact, while the bottom one is exact by [3, p. 199, Th. 2]. This gives isomorphisms $\operatorname{Coker} \alpha \simeq T(k)/R$ and $\operatorname{Ker} \alpha \simeq \operatorname{Coker} \alpha_1$. □

Direct proof of Corollary 1. Let $0 \to S_1' \to T_0' \xrightarrow{p'} T \to 0$ be another flasque resolution of T. Consider the third flasque resolution

$$0 \to S_1'' \to T_0 \times T_0' \xrightarrow{(p,p')} T \to 0$$

where S_0'' is defined as the kernel of (p, p'). We have exact sequences

$$0 \to S_1 \to S_1'' \to T_0' \to 0, \quad 0 \to S_1' \to S_1'' \to T_0 \to 0$$

which are split by [3, Lemme 1 (viii)] (cf. *loc. cit.*, proof of Lemme 5). Thus we have

$$S_1''(k)/R \simeq S_1(k)/R \times T_0'(k)/R \simeq S_1'(k)/R \times T_0(k)/R.$$

But $T_0(k)/R = T_0'(k)/R = \{1\}$ because T_0 are T_0' are open subsets of affine spaces (cf. [3, proof of Th. 2]). □

3. Extension to DM

Let $\mathrm{DM}_-^{\mathrm{eff}}$ be the category of effective motivic complexes introduced in [18]: it has a t-structure with heart HI. It also has a tensor structure that we shall just denote by $\otimes$. This tensor structure is right t-exact; its relationship with $\otimes_{\mathrm{HI}}$ is given by the formula

$$\mathcal{F} \otimes_{\mathrm{HI}} \mathcal{G} = H_0(\mathcal{F}[0] \otimes \mathcal{G}[0])$$

for $\mathcal{F},\mathcal{G}\in \mathrm{HI}$.

The category $\mathrm{DM}_-^{\mathrm{eff}}$ also enjoys a (partially defined) internal Hom. Coming back the the semi-abelian variety G of the previous section, we have an isomorphism

$$L[0]=G_{-1}[0]\simeq \underline{\mathrm{Hom}}_{\mathrm{DM}_-^{\mathrm{eff}}}(\mathbb{G}_m[0],G[0])$$

[9, Rk. 4.4], hence (2) extends to a morphism in $\mathrm{DM}_-^{\mathrm{eff}}$

$$L[0]\otimes\mathbb{G}_m[0]\to G. \tag{4}$$

By [10, Lemma 6.3] or [7, §2], the cone of (4) is the *birational motivic complex* $\nu_{\le 0}G[0]$ associated to G. We are going to compute its homology sheaves.

Let NST denote the category of Nisnevich sheaves with transfers. Recall that $\mathrm{DM}_-^{\mathrm{eff}}$ may be viewed as a localisation of $D^-(\mathrm{NST})$, and that its tensor structure is a descent of the tensor structure on the latter category [18, Prop. 3.2.3].

Lemma 2. *If G is an invertible torus, there is a canonical isomorphism in $D^-(\mathrm{NST})$*

$$L[0]\otimes\mathbb{G}_m\xrightarrow{\sim}G[0]$$

which descends to (4)*. In particular, $\nu_{\le 0}G[0]=0$.*

Proof. Same as for Theorem 2 a) (reduction to $G=\mathbb{G}_m$). □

Consider a coflasque resolution (3) of L. Taking a coflasque resolution of Q_1 and iterating, we get a resolution of L by invertible lattices:

$$\cdots\to L_n\to\cdots\to L_0\to L\to 0. \tag{5}$$

Note that this is a version of Voevodsky's "canonical resolutions" of [18, §3.2] p. 202; in particular, $L[0]$ is isomorphic in $D^-(\mathrm{NST})$ to the complex

$$L_\cdot=\cdots\to L_n\to\cdots\to L_0\to 0.$$

Lemma 3. *Let T_n denote the torus with cocharacter group L_n. Then the object $\nu_{\le 0}G[0]$ of $\mathrm{DM}_-^{\mathrm{eff}}$ is isomorphic to the complex*

$$\cdots\to T_n\to\cdots\to T_0\to G\to 0.$$

Proof. By Lemma 2, $L_n[0]\otimes\mathbb{G}_m[0]\simeq T_n[0]$ is homologically concentrated in degree 0 for all n. It follows that the commplex

$$T_\cdot=\cdots\to T_n\to\cdots\to T_0\to 0$$

is isomorphic to $L[0]\otimes\mathbb{G}_m[0]$ in $D^-(\mathrm{NST})$, hence *a fortiori* in $\mathrm{DM}_-^{\mathrm{eff}}$. □

Let $Q_1 = \operatorname{Ker}(L_0 \to L)$ as in (3), and for $n > 0$ let $Q_{n+1} = \operatorname{Ker}(L_n \to L_{n-1})$. By construction of $L.$, Q_n is coflasque for all $n > 0$ and

$$0 \to Q_{n+1} \to L_n \to Q_n \to 0$$

is a coflasque resolution of Q_n.

Theorem 3. *Let S_n be the torus with cocharacter group Q_n. For any connected smooth k-scheme X with function field K, one has*

$$H_n(\nu_{\leq 0}G[0])(X) = \begin{cases} 0 & \text{if } n < 0 \\ G(K)/R & \text{if } n = 0 \\ S_n(K)/R & \text{if } n > 0. \end{cases}$$

Proof. For any nonempty open subscheme $U \subseteq X$ we have isomorphisms

$$H_n(\nu_{\leq 0}G[0])(X) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(U) \xrightarrow{\sim} H_n(\nu_{\leq 0}G[0])(K) \tag{6}$$

(e.g. [7, p. 912]). We now conclude via Lemma 3 and Theorem 2. □

Corollary 3. *a) If k is finitely generated, the n-th homology sheaf of $\nu_{\leq 0}G[0]$ takes values in finitely generated abelian groups, and even in finite groups if $n > 0$ or G is a torus.*
b) If G is a torus, then (4) is an isomorphism (equivalently $\nu_{\leq 0}G[0] = 0$) if G is split by a Galois extension E/k whose Galois group has cyclic Sylow subgroups.

Proof. a) This follows via Theorem 3 and Lemma 1 from [3, p. 200, Cor. 2] and the Mordell-Weil-Néron theorem. b) We may choose the L_n and hence the S_n split by E/k. The conclusion then follows from Theorem 3 and [3, p. 200, Cor. 3]. □

Remark 1. In characteristic $p > 0$, all finitely generated perfect fields are finite and tori over such fields are invertible. To give some contents to Corollary 3 a) in this case, one may pass to the perfect closure k of a finitely generated field k_0. If G is a semi-abelian variety on k, it is defined over some finite extension k_1 of k_0. If k_2/k_1 is a finite (purely inseparable) subextension of k/k_1, then the composition

$$G(k_2) \xrightarrow{N_{k_2/k_1}} G(k_1) \to G(k_2)$$

equals multiplication by $[k_2 : k_1]$. Hence Corollary 3 a) remains true at least after inverting p.

4. Stable birationality

If X is a smooth variety over a field k, we write $\mathrm{Alb}(X)$ for its generalised Albanese variety in the sense of Serre [15]: it is a semi-abelian variety, and a rational point $x_0 \in X$ determines a morphism $X \to \mathrm{Alb}(X)$ which is universal for morphisms from X to semi-abelian varieties sending x_0 to 0.

We also write $\mathrm{NS}(X)$ for the group of cycles of codimension 1 on X modulo algebraic equivalence. This group is finitely generated if k is algebraically closed [8, Th. 3].

4.1. **Well-known lemmas.** I include proofs for lack of reference.

Lemma 4. *a) Let G, G' be two semi-abelian k-varieties. Then any k-morphism $f : G \to G'$ can be written uniquely $f = f(0) + f'$, where f' is a homomorphism.*
b) For any semi-abelian k-variety G, the canonical map $G \to \mathrm{Alb}(G)$ sending 0 to 0 is an isomorphism.

Proof. a) amounts to showing that if $f(0) = 0$, then f is a homomorphism. By an adjunction game, this is equivalent to b). Let us give two proofs: one of a) and one of b).

Proof of a). We may assume k to be a universal domain. This is classical for abelian varieties [13, p. 41, Cor. 1] and an easy computation for tori. In the general case, let T, T' be the toric parts of G and G' and $A; A$ be their abelian parts. Let $g \in G(k)$. As any morphism from T to A' is constant, the k-morphism

$$\varphi_g : T \ni t \mapsto f(g + t) - f(g) \in G'$$

(which sends 0 to 0) lands in T', hence is a homomorphism. Therefore it only depends on the image of g in $A(k)$. This defines a morphism $\varphi : A \to \underline{\mathrm{Hom}}(T, T')$, which must be constant with value $\varphi_0 = f$. It follows that

$$(g, h) \mapsto f(g + h) - f(g) - f(h)$$

induces a morphism $A \times A \to T'$. Such a morphism is constant, of value 0.

Proof of b). This is true if G is abelian, by rigidity and the equivalence between a) and b). In general, any morphism from G to an abelian variety is trivial on T. This shows that the abelian part of $\mathrm{Alb}(G)$ is A. Let $T' = \mathrm{Ker}(\mathrm{Alb}(G) \to A)$. We also have the counit morphism $\mathrm{Alb}(G) \to G$, and the composition $G \to \mathrm{Alb}(G) \to G$ is the identity. Thus T is a direct summand of T'. It suffices to show that $\dim T' = \dim T$. Going to the algebraic closure, we may reduce to $T = \mathbb{G}_m$.

Then consider the line bundle completion $\bar G \to A$ of the $\mathbb{G}_m$-bundle $G \to A$. It is sufficient to show that the kernel of

$$\mathrm{Alb}(G) \to \mathrm{Alb}(\bar G) = A$$

is 1-dimensional. This follows for example from [1, Cor. 10.5.1]. □

Lemma 5. *Let G be a semi-abelian variety over an algebraically closed field k. Let A be the abelian quotient of A. Then the map*

$$\mathrm{NS}(A) \to \mathrm{NS}(G) \tag{7}$$

is an isomorphism.

Proof. Let $T = \mathrm{Ker}(G \to A)$ and $X(T)$ its character group. Choosing a basis (e_i) of $X(T)$, we may complete the $\mathbb{G}_m^n$-torsor G into a product of line bundles $\bar G \to A$. The surjection

$$\mathrm{Pic}(A) \xrightarrow{\sim} \mathrm{Pic}(\bar G) \twoheadrightarrow \mathrm{Pic}(G)$$

show the surjectivity of (7). Its kernel is generated by the classes of the irreducible components D_i of the divisor with normal crossings $\bar G - G$. These components correspond to the basis elements e_i. Since the corresponding $\mathbb{G}_m$-bundle is a group extension of A by $\mathbb{G}_m$, the class of the 0 section of its line bundle completion lies in $\mathrm{Pic}^0(A)$, hence goes to 0 in $\mathrm{NS}(\bar G)$. □

Lemma 6. *Let X be a smooth k-variety, and let $U \subseteq X$ be a dense open subset. Then there is an exact sequence of semi-abelian varieties*

$$0 \to T \to \mathrm{Alb}(U) \to \mathrm{Alb}(X) \to 0$$

with T a torus. If $\mathrm{NS}(\bar U) = 0$ (this happens if U is small enough), there is an exact sequence of character groups

$$0 \to X(T) \to \bigoplus_{x \in X^{(1)} - U^{(1)}} \mathbf{Z} \to \mathrm{NS}(\bar X) \to 0.$$

Proof. This follows for example from [1, Cor. 10.5.1]. □

Lemma 7. *Let k be an infinite field and let $f : G \dashrightarrow G'$ be a rational map between semi-abelian k-varieties, with G a torus. Then there exists an extension $\tilde G$ of G by a permutation torus and a homomorphism $\tilde f : \tilde G \to G'$ which lifts f up to translation in the following sense: there exists a rational section $s : G \dashrightarrow \tilde G$ of the projection $\pi : \tilde G \to G$ and a rational point $g' \in G'(k)$ such that $f = \tilde f s + g'$. If f is defined at 0_G and sends it to $0_{G'}$, then $g' = 0$.*

Proof. Let U be an open subset of G where f is defined. We define $\tilde{G} = \mathrm{Alb}(U)$. Applying Lemmas 6 and 4 b) and using $\mathrm{NS}(\bar{G}) = 0$, we get an extension

$$0 \to P \to \tilde{G} \to G \to 0$$

where P is a permutation torus, as well as a morphism $\tilde{f} = \mathrm{Alb}(f) : \tilde{G} \to G'$.

This much does not use that k is infinite. With this assumption, $U(k) \neq \emptyset$ because G is unirational. A rational point $g \in U$ defines an Albanese map $U \to \tilde{G}$ sending g to $0_{\tilde{G}}$. Since P is a permutation torus, $g \in G(k)$ lifts to $\tilde{g} \in \tilde{G}(k)$ (Hilbert 90) and we may replace s by a morphism sending g to $\tilde{g}$. Then s is a rational section of π. Moreover, $f = \tilde{f}s + g'$ with $g' = f(g) - \tilde{f}(\tilde{g})$. The last assertion follows. □

Lemma 8. *Let G be a finite group, and let A be a finitely generated G-module. Then*
a) There exists a short exact sequence of G-modules $0 \to P \to F \to A \to 0$, with F torsion-free and flasque, and P permutation.
b) Let B be another finitely generated G-module, and let $0 \to P' \to E \to B \to 0$ be an exact sequence with P' an invertible module. Then any G-morphism $f : A \to B$ lifts to $\tilde{f} : F \to E$.

Proof. a) is the contents of [4, Lemma 0.6, (0.6.2)]. b) The obstruction to lifting f lies in $\mathrm{Ext}^1_G(F, P')$. This group is isomorphic to $\mathrm{Ext}^1_G(P'^*, F^*)$ ($\mathbf{Z}$-duals), which is 0 since P'^* is invertible and F^* is coflasque. □

Lemma 9. *Let G, G' be as in Lemma 4, with toric parts T, T', and let $0 \to S_1 \to T_0 \to T$, $0 \to S_1' \to T_0' \to T'$ be flasque resolutions of T and T'. Then any homomorphism $f : G \to G'$ lifts to a homomorphism $f_0 : T_0 \to T_0'$.*

Proof. Reasoning as in the proof of Lemma 4, we first see that f induces a homomorphism $f' : T \to T'$. The obstruction to lifting f' to f_0 lies in $\mathrm{Ext}^1_k(T_0, S_1')$.

This group is 0: indeed, it sits in a short exact sequence

$$0 \to H^1(k, \underline{\mathrm{Hom}}(T_0, S_1')) \to \mathrm{Ext}^1_k(T_0, S_1') \to H^0(k, \underline{\mathrm{Ext}}^1(T_0, S_1'))$$

and it suffices to show that both sides are 0. The étale sheaf $\underline{\mathrm{Ext}}^1(T_0, S_1')$ is associated to the presheaf $K \mapsto \mathrm{Ext}^1_K(T_0, S_1')$. It is 0: for K/k large enough, T_0 and S_1' are split over K and we reduce to computing $\mathrm{Ext}^1_K(\mathbb{G}_m, \mathbb{G}_m)$. Any extension of $\mathbb{G}_m$ by $\mathbb{G}_m$ defines a $\mathbb{G}_m$-torsor with base $\mathbb{G}_m$, which is trivial since $\mathrm{Pic}(\mathbb{G}_m) = 0$.

It remains to show that $H^1(K/k, \mathrm{Hom}_K(T_0, S_1')) = 0$ for any finite Galois extension K/k splitting T_0 and S_1. Let $G = Gal(K/k)$. If L_0, Q_1'

are the cocharacter groups of L_0 and S_1, the G-module $\operatorname{Hom}_K(T_0, S_1')$ is isomorphic to $\operatorname{Hom}_K(L_0, Q_1')$, which is coflasque since L_0 is invertible and Q_1' is coflasque. □

4.2. **Less standard lemmas.**

Lemma 10. *Let*

$$0 \to P \to G \to H \to 0 \tag{8}$$

be an exact sequence of semi-abelian varieties, with P an invertible torus. Then $\nu_{\leq 0}G[0] \xrightarrow{\sim} \nu_{\leq 0}H[0]$.

Proof. As P is coflasque, (8) is exact in NST hence defines an exact triangle

$$P[0] \to G[0] \to H[0] \xrightarrow{+1}$$

in $\mathrm{DM}_-^{\mathrm{eff}}$. The conclusion then follows from Lemma 2. □

Lemma 11. *Let G, G' be as in Lemma 4. Then the group* $\operatorname{Hom}_{\mathrm{NST}}(G, G')$ *is canonically isomorphic to* $\operatorname{Hom}(G, G')$ *(homomorphism of semi-abelian varieties).*

Proof. Any homomorphism of semi-abelian varieties defines a morphism of the associated Nisnevich sheaves with transfers. Conversely, let $f : G \to G'$ be a morphism in NST. We argue à la Yoneda: we get a map

$$f_G : G(G) \to G'(G).$$

Then $f_G(1_G)$ defines a k-morphism $G \to G'$, sending 0 to 0; by Lemma 4, this is a homomorphism. □

Proposition 3. *Let G, G' be two semi-abelian k-varieties, with G a torus. Then a rational map $f : G \dashrightarrow G'$ induces a morphism $f_* : \nu_{\leq 0}G[0] \to \nu_{\leq 0}G'[0]$.*

Proof. If k is finite, then G is invertible and $\nu_{\leq 0}G[0] = 0$ by Lemma 2. Hence we may assume k infinite. Applying Lemma 7, f lifts to a homomorphism $\tilde{G} \to G'$ where $\tilde{G}$ is an extension of G by a permutation torus. By Lemma 10, the induced morphism

$$\nu_{\leq 0}\tilde{G}[0] \to \nu_{\leq 0}G'[0]$$

factors through a morphism $f_* : \nu_{\leq 0}G[0] \to \nu_{\leq 0}G'[0]$. □

Remark 2. The proof shows that f_* only depends on f up to translation by an element of $G(k)$ or $G'(k)$.

Corollary 4. *If T and T' are birationally equivalent k-tori, then $\nu_{\leq 0}T[0] \simeq \nu_{\leq 0}T'[0]$. In particular, the groups $T(k)/R$ and $T'(k)/R$ are isomorphic.*

Proof. The proof of Proposition 3 shows that $f \mapsto f_*$ is functorial for composable rational maps between tori. Let $f : T \dashrightarrow T'$ be a birational isomorphism, and let $g : T' \dashrightarrow T$ be the inverse birational isomorphism. Then we have $g_* f_* = 1_{\nu_{\le 0} T[0]}$ and $f_* g_* = 1_{\nu_{\le 0} T'[0]}$. The last claim follows from Theorem 3. □

Remark 3. It is known that a birational isomorphism of tori $f : T \dashrightarrow T'$ induces a set-theoretic bijection $f_* : T(k)/R \xrightarrow{\sim} T'(k)/R$ [3, p. 197, Cor. to Prop. 11] and that the *group* $T(k)/R$ is abstractly a birational invariant of T (ibid., p. 200, Cor. 4). The proof above shows that the bijection $f_* : T(k)/R \simeq T'(k)/R$ is an isomorphism of groups if f respects the origins of T and T'. The proofs of Lemma 7 and Proposition 3 may be seen as dual to the proof of [3, p. 189, Prop. 5], and are directly inspired from it.

4.3. **A converse.**

Proposition 4. *Let $f : G \dashrightarrow G'$ be a rational map between semi-abelian varieties, with G a torus. Assume that the map $f_* : G(K)/R \to G'(K)/R$ from Proposition 3 is identically 0 when K runs through the finitely generated extensions of k. Then there exists a permutation torus P and a factorisation of f as*

$$G \overset{\tilde f}{\dashrightarrow} P \xrightarrow{g} G'$$

where $\tilde f$ is a rational map and g is a homomorphism.
Conversely, if there is such a factorisation, then $f_* : \nu_{\le 0} G[0] \to \nu_{\le 0} G'[0]$ is the 0 morphism.

Proof. As in the proof of Proposition 3, we may assume k infinite. By Lemma 7, we may reduce to the case where f is a morphism. We shall then get $\tilde f$ as a homomorphism. Let $K = k(G)$. By hypothesis, the image of the generic point $\eta_G \in G(K)$ is R-equivalent to 0 on $G'(K)$. By a lemma of Gille [6, Lemme II.1.1 b)], it is directly R-equivalent to 0: in other words, there exists a rational map $h : G \times \mathbf{A}^1 \dashrightarrow G'$, defined in the neighbourhood of 0 and 1, such that $h_{|G\times\{0\}} = 0$ and $h_{|G\times\{1\}} = f$.

Let $U \subseteq G \times \mathbf{A}^1$ be an open set of definition of h. The 0 and 1-sections of $G \times \mathbf{A}^1 \to G$ induce sections

$$s_0, s_1 : G \to \mathrm{Alb}(U)$$

of the projection $\pi : \mathrm{Alb}(U) \to \mathrm{Alb}(G \times \mathbf{A}^1) = G$ such that $\mathrm{Alb}(h) \circ s_0 = 0$ and $\mathrm{Alb}(h) \circ s_1 = f$. If $T_1 = \operatorname{Ker} \pi$, then $s_0 - s_1$ induces a homomorphism $\tilde f : G \to T_1$ such that the composition

$$G \xrightarrow{\tilde f} T_1 \to \mathrm{Alb}(U) \xrightarrow{\mathrm{Alb}(h)} G'$$

equals f. Finally, T_1 is a permutation torus by Lemma 6.

The last claim of Proposition 4 follows from Lemma 2. □

Theorem 4. *Let G, G' be two semi-abelian varieties, with G a torus. Suppose given, for every function field K/k, a homomorphism $f_K : G(K)/R \to G'(K)/R$ such that f_K is natural with respect to the functoriality of Corollary 2. Then*
a) There exists an extension $\tilde{G}$ of G by a permutation torus, and a homomorphism $f : \tilde{G} \to G'$ inducing (f_K).
b) f_K is surjective for all K if and only if there exist extensions $\tilde{G}, \tilde{G}'$ of G and G' by permutation tori such that f_K is induced by a split surjective homomorphism $\tilde{G} \to \tilde{G}'$.

Proof. a) As in the proof of Propositions 3 and 4, we may assume k infinite. Take $K = k(G)$. The image of the generic point η_G by f_K lifts to a (non unique) rational map $f : G \dashrightarrow G'$. Using Lemma 7, we may lift f to a homomorphism

$$\tilde{f} : \tilde{G} \to G'$$

where $\tilde{G}$ is an extension of G by a permutation torus P. Since $\tilde{G}(K)/R \xrightarrow{\sim} G(K)/R$, we reduce to $\tilde{G} = G$ and $\tilde{f} = f$.

Let L/k be a fonction field, and let $g \in G(L)$. Then g arises from a morphism $g : X \to G$ for a suitable smooth model X of L. By assumption on $K \mapsto f_K$, the diagram

$$\begin{array}{ccc} G(K)/R & \xrightarrow{f_K} & G'(K)/R \\ {\scriptstyle g^*}\downarrow & & {\scriptstyle g^*}\downarrow \\ G(L)/R & \xrightarrow{f_L} & G'(L)/R \end{array}$$

commutes. Applying this to $\eta_K \in G(K)$, we find that $f_L([g]) = [g \circ f]$, which means that f_L is the map induced by f.

b) The hypothesis implies that $G'(E)/R = 0$ for any algebraically closed extension E/k, which in turn implies that G' is also a torus. Applying a), we may, and do, convert f into a true homomorphism by replacing G by a suitable extension by a permutation torus. Applying Lemma 8 a) to the cocharacter group of G, we get a resolution $0 \to P_1 \to Q \to G \to 0$ with Q coflasque and P_1 permutation. Hence we may further (and do) assume G coflasque.

Let $K = k(G')$ and choose some $g \in G(K)$ mapping modulo R-equivalence to the generic point of G'. Then g defines a rational map $g : G' \dashrightarrow G$ such that fg is R-equivalent to $1_{G'}$. It follows that the induced map

$$1 - fg : G'/R \to G'/R \tag{9}$$

is identically 0.

Reapplying Lemma 7, we may find an extension $\tilde{G}'$ of G' by a suitable permutation torus which converts g into a true homomorphism. Since G is coflasque, Lemma 8 b) shows that $f : G \to G'$ lifts to $\tilde{f} : G \to \tilde{G}'$. Then (9) is still identically 0 when replacing (G', f) by $(\tilde{G}', \tilde{f})$.

Summarising: we have replaced the initial G and G' by suitable extensions by permutation tori, such that f lifts to these extensions and there is a homomorphism $g : G' \to G$ such that (9) vanishes identically. Hence $1-fg$ factors through a permutation torus P thanks to [the proof of] Proposition 4.

Write $u : G' \to P$ and $v : P \to G'$ for homomorphisms such that $1 - fg = vu$. Let $G_1 = G \times P$ and consider the maps

$$f_1 = (f, v) : G_1 \to G', \qquad g_1 = \begin{pmatrix} g \\ u \end{pmatrix} : G' \to G_1.$$

Then $f_1 g_1 = 1$ and G' is a direct summand of G_1 as requested. □

Corollary 5. *a) Let G' be a semi-abelian k-variety such that $G'(K)/R = 0$ for any function field K/k. Then G' is an invertible torus.*
b) In Theorem 4 b), assume that f_K is bijective for all K/k. Then there exist extensions $\tilde{G}$, $\tilde{G}'$ of G and G' by invertible tori such that f_K is induced by an isomorphism $\tilde{G} \xrightarrow{\sim} \tilde{G}'$.

Proof. a) This is the special case $G = 0$ of Theorem 4 b).

b) By Theorem 4 b), we may replace G and G' by extensions by permutation tori such that f_K lifts to a split surjection $f : G \to G'$. Let $T = \operatorname{Ker} f$. Then $T/R = 0$ universally. By a), T is invertible. □

Remark 4. Corollary 5 a) is a version of [4, Prop. 7.4] (taking [3, p. 199, Th. 2] into account). Theorem 4 was inspired by the desire to understand its proof from a different viewpoint.

Corollary 6. *Let $f : G \dashrightarrow G'$ be a rational map of semi-abelian varieties, with G a torus. Then the following conditions are equivalent:*

(i) *$f_* : \nu_{\leq 0} G[0] \to \nu_{\leq 0} G'[0]$ is an isomorphism (see Proposition 3).*
(ii) *$f_* : G(K)/R \to G'(K)/R$ is bijective for any function field K/k.*
(iii) *f is an isomorphism, up to extensions of G and G' by invertible tori and up to a translation.* □

5. Some open questions

Question 1. Are lemma 7 and Proposition 3 still true when G is not a torus?

This is far from clear in general, starting with the case where G is an abelian variety and G' a torus. Let me give a positive answer in the case of an elliptic curve.

Proposition 5. *The answer to Question 1 is yes if the abelian part A of G is an elliptic curve.*

Proof. Arguing as in the proof of Proposition 3, we get for an open subset $U \subseteq G$ of definition for f an exact sequence

$$0 \to \mathbb{G}_m \to P \to \operatorname{Alb}(U) \to G \to 0$$

where P is a permutation torus. Here we used that $\operatorname{NS}(\bar{G}) \simeq \mathbf{Z}$, which follows from Lemma 5.

The character group $X(P)$ has as a basis the geometric irreducible components of codimension 1 of $G - U$. Up to shrinking U, we may assume that $G - U$ contains the inverse image D of $0 \in A$. As the divisor class of 0 generates $\operatorname{NS}(\bar{A})$, D provides a Galois-equivariant splitting of the map $\mathbb{G}_m \to P$. Thus its cokernel is still a permutation torus, ans we conclude as before. □

Question 2. Can one formulate a version of Theorem 4 and Corollary 5 providing a description of $\operatorname{Hom}(\nu_{\leq 0}G[0], \nu_{\leq 0}G'[0])$ (at least in case G and G' are tori)?

The proof of Theorem 4 suggests the presence of a closed model structure on the category of tori (or lattices), which might provide an answer to this question.

For the last question, let G be a semi-abelian variety. Forgetting its group structure, it has a motive $M(G) \in \mathrm{DM}_-^{\mathrm{eff}}$. Recall the canonical morphism

$$M(G) \to G[0]$$

induced by the "sum" maps

$$c(X, G) \xrightarrow{\sigma} G(X) \tag{10}$$

for smooth varieties X ([16, (6), (7)], [1, §1.3]).

The morphism (10) has a canonical section

$$G(X) \xrightarrow{\gamma} c(X, G) \tag{11}$$

given by the graph of a morphism: this section is functorial in X but is not additive.

Consider now a smooth equivariant compactification $\bar{G}$ of G. It exists in all characteristics. For tori, this is written up in [2]. The general case reduces to this one by the following elegant argument I learned from M. Brion: if G is an extension of an abelian variety A by a torus T, take a smooth projective equivariant compactification Y of T. Then the bundle $G \times^T Y$ associated to the T-torsor $G \to A$ also exists: this is the desired compactification.

Then we have a diagram of birational motives

$$\begin{array}{ccc} \nu_{\leq 0} M(G) & \xrightarrow{\sim} & \nu_{\leq 0} M(\bar{G}) \\ {\scriptstyle \nu_{\leq 0}\sigma}\downarrow & & \\ \nu_{\leq 0} G[0]. & & \end{array} \tag{12}$$

By [10], we have $H_0(\nu_{\leq 0} M(\bar{G}))(X) = CH_0(\bar{G}_{k(X)})$ for any smooth connected X. Hence the above diagram induces a homomorphism

$$CH_0(\bar{G}_{k(X)}) \to G(k(X))/R \tag{13}$$

which is natural in X for eht action of finite correspondences (compare Corollary 2). One can probably check that this is the homomorphism of [11, (17) p. 78], reformulating [3, Proposition 12 p. 198]. Similarly, the set-theoretic map

$$G(k(X))/R \to CH_0(\bar{G}_{k(X)}) \tag{14}$$

of [3, p. 197] can presumably be recovered as a birational version of (11), using perhaps the homotopy category of schemes of Morel and Voevodsky.

In [11], Merkurjev shows that (13) is an isomorphism for G a torus of dimension at most 3. This suggests:

Question 3. Is the map $\nu_{\leq 0}\sigma$ of Diagram (12) an isomorphism when G is a torus of dimension ≤ 3?

In [12], Merkurjev gives examples of tori G for which (14) is not a homomorphism; hence its (additive) left inverse (13) cannot be an isomorphism. Merkurjev's examples are of the form $G = R^1_{K/k}\mathbb{G}_m \times R^1_{L/k}\mathbb{G}_m$, where K and L are distinct biquadratic extensions of k. This suggests:

Question 4. Can one study Merkurjev's examples from the above viewpoint? More generally, what is the nature of the map $\nu_{\leq 0}\sigma$ of Diagram (12)?

We leave all these questions to the interested reader.

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