On the operator space $OUMD$ property for the column Hilbert space C

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Abstract

The operator space $OUMD$ property was introduced by Pisier in the context of vector-valued noncommutative L_p-spaces. It is still unknown whether the property is independent of p in this setting. In this paper, we prove that the column Hilbert space C is $OUMD_p$ for all $1 < p < \infty$, this answers positively a question asked by Ruan.

1 Introduction

In Banach space valued martingale theory, the UMD property plays an important role. Let us recall briefly the definition of the UMD property. Let $1 < p < \infty$, a Banach space B is UMD_p, if there exists a positive constant which depends on p and the Banach space B (the best one is usually denoted by $C_p(B)$), such that for all positive integers n, all sequences $\varepsilon = (\varepsilon_k)_{k=1}^n$ of numbers in $\{-1, 1\}$ and all B-valued martingale difference sequences $dx = (dx_k)_{k=1}^n$, we have

$$\|\sum_{k=1}^n \varepsilon_k dx_k\|_{L_p([0,1];B)} \leq C_p(B)\|\sum_{k=1}^n dx_k\|_{L_p([0,1];B)}.$$

The UMD property has very deep connections with the boundedness of certain singular integral operators such as the Hilbert transform. Burkholder and McConnell [Bur83] proved that if a Banach space B is UMD_p, then the

Hilbert transform is bounded on the Bochner space $L_p(\mathbb{T}, m; B)$. Bourgain [Bou83] showed that if the Hilbert transform is bounded on $L_p(\mathbb{T}, m; B)$, then B is UMD_p. Pisier proved that the finiteness of $C_p(B)$ for some $1 < p < \infty$ implies its finiteness for all $1 < p < \infty$. Thus we can say the UMD property without mentioning p. Examples of UMD spaces include all the finite dimensional Banach spaces, the Schatten p-classes S_p and more generally the noncommutative L_p-spaces associated to a von Neumann algebra M, for all $1 < p < \infty$. The readers are referred to Burkholder [Bur86, Bur01] for information on UMD spaces.

In his monograph [Pis98], Pisier developed a theory of vector-valued noncommutative L_p spaces $L_p(\tau; E)$ associated with a hyperfinite von Neumann algebra M equipped with an normal, semifinite, faithful trace τ, and E is equipped with an operator space structure, see [Pis03] for the details on operator space theory. Noncommutative conditional expectations and martingales arise naturally in this setting. Following Pisier, we say that an operator space E is $OUMD_p$ for some $1 < p < \infty$, if there exists a constant (as before, the best one is usually denoted by $C_p^{os}(E)$, which depends on p and the operator space structure on E) such that any martingale (f_n) in $L_p(\tau; E)$ satisfies

$$\forall n \geq 1 \qquad \varepsilon_k = \pm 1 \qquad \|f_0 + \sum_{k=1}^{n} \varepsilon_k (f_k - f_{k-1})\|_{L_p(\tau;E)} \leq C_p^{os}(E) \|f_n\|_{L_p(\tau;E)}.$$

By a main result of Pisier and Xu in their papers [PX96] and [PX97], the one dimensional operator space $\mathbb{C}$ and all the non-commutative L_p-spaces are $OUMD_p$. In particular, the Schatten p-class S_p is $OUMD_p$. Later, in her thesis, Musat [Mus06] studied the properties $OUMD_p$ and proved that for all $1 < p, q < \infty$, the Schatten p-class S_p is $OUMD_q$. More generally, she proved that for all $1 < u, v < \infty$, the spaces $S_u[S_v]$ are $OUMD_p$ for all $1 < p < \infty$. In the end of her paper, she stated explicitly Ruan's question and left it open.

The main theorem of this paper is:

Theorem 1 *Let $1 < p < \infty$, then $S_p[C]$ is UMD as a Banach space.*

Using an unpublished result of Musat, we solve completely the problem proposed by Ruan, we state it as follows

Theorem 2 *The column Hilbert space C is $OUMD_p$ for all $1 < p < \infty$.*

Our proof relies on properties of the Haagerup tensor product and complex interpolation. In section 2, we briefly recall some necessary definitions and collect some well known results we will use later. In section 3, we prove a slightly more general result and theorem 1 follows as a corollary. In the last section, we investigate some equivalent conditions for an operator space E to be $OUMD_p$.

In the Banach space setting, the UMD property is independent of p, and it will be very interesting to know whether $OUMD_p$ is independent of p or not.

2 Preliminaries

We refer to Pisier [Pis03] for details on operator spaces. By an operator space we mean a closed subspace of $B(H)$ for some complex Hilbert space H. When $E \subset B(H)$ is an operator space, we denote by $M_n(E)$ the space of all $n \times n$ matrices with entries in E, equipped with the norm induced by the space $B(\ell_2^n \otimes_2 H)$. Let e_{ij} be the element of $B(\ell_2)$ corresponding to the matrix coefficients equal to one at the (i, j) entry and zero elsewhere. The column Hilbert space C is defined as

$$C = \overline{\text{span}}\{e_{i1} | i \geq 1\}$$

and the row Hilbert space R is defined as

$$R = \overline{\text{span}}\{e_{1j} | j \geq 1\}.$$

Ruan [Rua88] gave an abstract characterization of operator spaces in terms of matrix norms. An abstract operator space is a vector space E equipped with matrix norms $\| \cdot \|_m$ on $M_m(E)$ for each positive integer m, satisfying the axioms: for all $x \in M_m(E), y \in M_n(E)$ and $\alpha, \beta \in M_m(\mathbb{C})$, we have

$$\left\| \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}, \qquad \|\alpha x \beta\|_m \leq \|\alpha\| \|x\|_m \|\beta\|.$$

This abstract characterization allows us to define many important constructions of new operator spaces from the given ones. Among these are the projective tensor product, the quotient, the dual for operator spaces. The other two operations we will use later in our proof are the Haagerup tensor product and complex interpolation for operator spaces. Let us recall them briefly.

Let E, F be two operator spaces, the Haagerup tensor product $E \otimes_h F$ of E and F is defined as the completion of $E \otimes F$ with respect to the matrix norms

$$\|u\|_{h,m} = \inf\{\|v\|\|w\| : u = v \odot w, v \in M_{m,r}(E), w \in M_{r,m}(F), r \in \mathbb{N}\},$$

where the element $v \odot w \in M_m(E \otimes F)$ is defined by $(v \odot w)_{ij} = \sum_{k=1}^{m} v_{ik} \otimes w_{kj}$, for all $1 \leq i, j \leq m$.

We refer to [BL76] for details about interpolation spaces of Banach spaces. Now let E_0, E_1 be two operator space, such that (E_0, E_1) is a compatible couple in the sense of [BL76], following Pisier, we endow the interpolation space $E_\theta = (E_0, E_1)_\theta$ with a canonical operator space structure by defining for all positive integers m,

$$M_m(E_\theta) = (M_m(E_0), M_m(E_1))_\theta.$$

The Haagerup tensor product is injective, projective, self-dual in the finite dimensional case, however, it is not commutative, that is we do not have $E \otimes_h F = F \otimes_h E$ in general. Moreover the Haagerup tensor product behaves nicely with respect to the complex interpolation, see e.g. [Pis96].

Theorem 3 *(Kouba) Let (E_0, E_1) and (F_0, F_1) be two compatible couples of operator spaces. Then $(E_0 \otimes_h F_0, E_1 \otimes_h F_1)$ is a compatible couple, and for all $0 < \theta < 1$ we have a complete isometry*

$$(E_0 \otimes_h F_0, E_1 \otimes_h F_1)_\theta = (E_0, E_1)_\theta \otimes_h (F_0, F_1)_\theta.$$

Let S_∞ be the space of compact operators on ℓ_2, then the embedding $S_\infty \subset B(\ell_2)$ gives S_∞ a natural operator space structure. We know the trace class S_1 is the dual space of S_∞^*, thus we can equip S_1 with the dual operator space structure. Let E be an operator space, following Pisier, the noncommutative vector-valued L_p-spaces in the discrete case are defined by

$$S_\infty^m[E] = S_\infty^m \otimes_{min} E, \qquad S_\infty[E] = S_\infty \otimes_{min} E,$$

$$S_1^m[E] = S_1^m \otimes^\wedge E, \qquad S_1[E] = S_1 \otimes^\wedge E.$$

It turns out that $(S_\infty[E], S_1[E])$ is a compatible couple, and for $1 < p < \infty$, we define

$$S_p^m[E] = (S_\infty^m[E], S_1^m[E])_{\frac{1}{p}}, \qquad S_p[E] = (S_\infty[E], S_1[E])_{\frac{1}{p}}.$$

For $1 \le p \le \infty$, denote by C_p and R_p the column subspace and the row subspace of S_p, we endow S_p with the canonical operator space structure given by the complex interpolation $S_p = (S_\infty, S_1)_{\frac{1}{p}}$, and endow C_p and R_p with the induced operator space structure. It is easy to see that the natural identification between C_p and $R_{p'}$ is a complete isometry, where p' is the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. We will need the following equality from [Pis98]:

$$C_p = (C_\infty, C_1)_{\frac{1}{p}},$$

and more generally, if $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, then $C_{p_\theta} = (C_{p_0}, C_{p_1})_\theta$. With these notations, we have

$$S_p[E] = C_p \otimes_h E \otimes_h R_p.$$

Let us end this section by stating some well known results, we refer to Musat [Mus06] for the details.

Proposition 4 *Let $1 < p < \infty$, and E be an operator space. If E is $OUMD_p$, then $S_p[E]$ is UMD (as a Banach space).*

Theorem 5 ([Mus06]) *If $1 < p, q, u < \infty$, then $S_q[S_u]$ is $OUMD_p$.*

Combining these two statements, we have

Theorem 6 *If $1 < p, q, u < \infty$, then $S_p[S_q[S_u]]$ is UMD as a Banach space.*

3 Main results

Lemma 7 *Let $1 < p_1, p_2, p_3 < \infty$, then $C_{p_1} \otimes_h C_{p_2} \otimes_h C_{p_3}$ is UMD as a Banach space.*

Proof. We have the following embeding

$$C_{p_1} \otimes_h C_{p_2} \otimes_h C_{p_3} \subset C_{p_1} \otimes_h C_{p_2} \otimes_h C_{p_3} \otimes_h R_{p_3} \otimes_h R_{p_2} \otimes_h R_{p_1} = S_{p_1}[S_{p_2}[S_{p_3}]],$$

The space $S_{p_1}[S_{p_2}[S_{p_3}]]$ is UMD as a Banach space, hence $C_{p_1} \otimes_h C_{p_2} \otimes_h C_{p_3}$ is also UMD. $\square$

Now we can state and prove our main result.

Theorem 8 *Let either* $1 < p_1, p_2, p_3 \leq \infty$ *or* $1 \leq p_1, p_2, p_3 < \infty$*, then* $C_{p_1} \otimes_h C_{p_2} \otimes_h C_{p_3}$ *is* UMD *as a Banach space.*

Proof. Assume that $1 < p_1, p_2, p_3 \leq \infty$. Since $p_i > 1$ for all $i = 1, 2, 3$, we can choose $0 < \theta < 1$ small enough so that $\tilde{p}_i = (1 - \theta)p_i > 1$. In other words, we have

$$\frac{1}{p_i} = \frac{1 - \theta}{\tilde{p}_i} + \frac{\theta}{\infty},$$

by complex interpolation, we have $C_{p_i} = (C_{\tilde{p}_i}, C_\infty)_\theta$. It follows from the multilinear version of theorem 3,

$$C_{p_1} \otimes_h C_{p_2} \otimes_h C_{p_3} = (C_{\tilde{p}_1} \otimes_h C_{\tilde{p}_2} \otimes_h C_{\tilde{p}_3}, C_\infty \otimes_h C_\infty \otimes_h C_\infty)_\theta.$$

We know $C_\infty \otimes_h C_\infty \otimes_h C_\infty$ is a Hilbertian space, and the same for $C_1 \otimes_h C_1 \otimes_h C_1$. Thus, we have an isometry equality for the underlying Banach spaces

$$C_{p_1} \otimes_h C_{p_2} \otimes_h C_{p_3} = (C_{\tilde{p}_1} \otimes_h C_{\tilde{p}_2} \otimes_h C_{\tilde{p}_3}, C_1 \otimes_h C_1 \otimes_h C_1)_\theta.$$

Applying once more the multilinear version of theorem 3, we have equality of operator spaces

$$(C_{\tilde{p}_1} \otimes_h C_{\tilde{p}_2} \otimes_h C_{\tilde{p}_3}, C_1 \otimes_h C_1 \otimes_h C_1)_\theta = C_{q_1} \otimes_h C_{q_2} \otimes_h C_{q_3},$$

where $\frac{1}{q_i} = \frac{1-\theta}{\tilde{p}_i} + \frac{\theta}{1}$, in particular $1 < q_i < \infty$ for $i = 1, 2, 3$. Now we are exactly in the situation of lemma 7, so the underlying Banach space of $C_{q_1} \otimes_h C_{q_2} \otimes_h C_{q_3}$ is UMD, and hence $C_{p_1} \otimes_h C_{p_2} \otimes_h C_{q_3}$ is UMD as a Banach space. By a duality argument, $C_{p_1} \otimes_h C_{p_2} \otimes_h C_{q_3}$ is also UMD if we assume $1 \leq p_1, p_2, p_3 < \infty$. □

Now we can prove theorem 1

Proof. Let $1 < p < \infty$, $S_p[C] = C_p \otimes_h C \otimes_h R_p = C_p \otimes_h C_\infty \otimes_h C_{p'}$, applying theorem 3, we get that $S_p[C]$ is UMD. □

Let us end this section by mentioning an unpublished result of Musat, she proved that the converse of proposition 4 is also true,

Theorem 9 *(Musat) Let $1 < p < \infty$, and E be an operator space. Then E is $OUMD_p$ if and only if $S_p[E]$ is UMD as a Banach space.*

Thus our theorem 2 follows directly from theorem 1.

Remark 10 *If one compares with Banach space theory, our main result is slightly surprising. Indeed, it is known that for any UMD Banach space X there is a function $n \mapsto F(n)$ that is $o(\sqrt{n})$ such that for any n and any n-dimensional subspace $E \subset X$, the Banach-Mazur distance $d(E, \ell_2^n)$ is $\le F(n)$. This follows from a result due to Milman and Wolfson [MmW78] (and the fact that UMD implies that X does not contain ℓ_1^n's uniformly). In sharp contrast, it is known (see [Pis03] p. 219) that if we denote by R_n and C_n the n-dimensional versions of R and C, we have $d_{cb}(C_n, OH_n) = d_{cb}(R_n, OH_n) = \sqrt{n}$.*

4 Further results

In this section, we give some necessary and sufficient conditions for the space $S_p[E]$ to be UMD. We give the equivalence between the UMD property and the boundedness of the triangular projection on $S_p[E]$. Then we apply this equivalence to prove that E is $OUMD_p$ if and only if E is $OUMD_p$ with respect to the so-called canonical filtration of matrix algebras.

We first give the following simple propostion.

Proposition 11 *Let $1 < p < \infty$, if we denote by $\mathcal{R}$ the Riesz projection $\mathcal{R} : L_p(\mathbb{T}, m) \to L_p(\mathbb{T}, m)$ defined by*

$$\sum_{finite} x_n z^n \mapsto \sum_{n \ge 0} x_n z^n.$$

Then $S_p[E]$ is UMD if and only if

$$\mathcal{R}_E := Id_E \otimes \mathcal{R} : L_p(\mathbb{T}, m; E) \to L_p(\mathbb{T}, m; E)$$

is completely bounded.

Proof. As is well-known, $S_p[E]$ is UMD if and only if the corresponding Riesz projection $Id_{S_p[E]} \otimes \mathcal{R} : L_p(\mathbb{T}, m; S_p[E]) \to L_p(\mathbb{T}, m; S_p[E])$ is bounded. By the non-commutative Fubini theorem, the natural identification between

$L_p(\mathbb{T}, m; S_p[E])$ and $S_p[L_p(\mathbb{T}, m; E)]$ is a complete isometric isomorphism. In this identification, $Id_{S_p[E]} \otimes \mathcal{R}$ becomes

$$Id_{S_p} \otimes \mathcal{R}_E : S_p[L_p(\mathbb{T}, m; E)] \to S_p[L_p(\mathbb{T}, m; E)].$$

A very useful result in [Pis98] tell us that $\|\mathcal{R}_E\|_{cb} = \|Id_{S_p} \otimes \mathcal{R}_E\|$, this ends our proof. □

The next theorem can be viewed as a special case of one result in [NR]

Theorem 12 *Let T_E be the triangular projection on $S_p[E]$ defined by*

$$(x_{ij}) \mapsto (x_{ij} 1_{j \geq i}).$$

Then $\|T_E\|_{cb} = \|T_E\| = \|\mathcal{R}_E\|_{cb}$.

We refer to [JX05] and [JX08] for details of the canonical matrix filtration. As usual, we regard M_n as a non-unital subalgebra of $M_\infty = B(\ell_2)$ by viewing an $n \times n$ matrix as an infinite one whose left upper corner of size $n \times n$ is the given $n \times n$ matrix, and all other entries are zero. The unit of M_n is the projection $e_n \in M_\infty$ which projects a sequence in ℓ_2 into its first n coordinates. The canonical matrix filtration is the increasing filtration $(M_n)_{n \geq 1}$ of subalgebras of M_∞. We denote by $E_n : M_\infty \to M_n$ the corresponding conditional expectation, clearly, we have

$$E_n(a) = e_n a e_n = \sum_{\max(i,j) \leq n} a_{ij} \otimes e_{ij}, \qquad a = (a_{ij}) \in M_\infty.$$

Note that E_n is not faithful.

We can define the $OUMD_p$ property with this canonical matrix filtration. Let $x \in S_p[E]$. We have $d_1 x = E_1(x)$ and $d_n x = E_n(x) - E_{n-1}(x)$ for $n \geq 2$. Then E is said to be $OUMD_p$ with respect to the canonical matrix filtration, if there exists a constant K_p, such that for all positive integer N and all choice of signs $\varepsilon_n = \pm 1$, we have

$$\| \sum_{n=1}^{N} \varepsilon_n d_n x \|_{S_p[E]} \leq K_p \|x\|_{S_p[E]}.$$

Let us denote the best such constant by $K_p(E)$.

Every choice of signs ε generates a transformation T_ε defined by $T_\varepsilon(x) = \sum_n \varepsilon_n d_n x$. We will say an element $x \in S_p[E]$ is of finite support if the support of x defined as the subset $\mathtt{supp}(x) = \{(i,j) \in \mathbb{N}^2 : x_{ij} \neq 0\}$ is finite. Note that T_ε is always well-defined on the subspace of finite supported elements.

An operator space E is $OUMD_p$ with respect to the canonical matrix filtration if for every choice of signs ε, we have

$$\|T_\varepsilon(x)\|_{S_p[E]} \leq K_p(E)\|x\|_{S_p[E]}, \quad |\mathtt{supp}(x)| < \infty.$$

Remark 13 *The transformation T_ε is a Schur multiplication associated with the function $f_\varepsilon(i,j) = \varepsilon_{\max(i,j)}$. Indeed, pick up an arbitrary element $x = (x_{ij}) \in S_p^N[E]$, we have*

$$d_n x = \sum_{\max(i,j)\leq n} x_{ij} \otimes e_{ij} - \sum_{\max(i,j)\leq n-1} x_{ij} \otimes e_{ij} = \sum_{\max(i,j)=n} x_{ij} \otimes e_{ij},$$

thus

$$T_\varepsilon(x) = \sum_{n=1}^{N} \varepsilon_n d_n x = \sum_{n=1}^{N} \varepsilon_n \sum_{\max(i,j)=n} x_{ij} \otimes e_{ij} = (\varepsilon_{\max(i,j)} x_{ij}).$$

.

Remark 14 *Let $D_\varepsilon = diag\{\varepsilon_1, \cdots, \varepsilon_n, \cdots\}$. Then $T_\varepsilon(x)$ multiplied on the left by the scalar matrix D_ε, we get $D_\varepsilon T_\varepsilon(x) = (\varepsilon_i \varepsilon_{\max(i,j)} x_{ij})$. After taking the average according to independent uniformly distributed choices of signs, we get the lower triangular projection of x, i.e, we have*

$$\int D_\varepsilon T_\varepsilon(x)\, d\varepsilon = \int (\varepsilon_i \varepsilon_{\max(i,j)} x_{ij})\, d\varepsilon = (x_{ij} 1_{i\geq j}).$$

The following result is inspired by [JX05] and [JX08]

Theorem 15 *Let $1 < p < \infty$, then E is $OUMD_p$ if and only if it is $OUMD_p$ with respect to the canonical matrix filtration. Moreover, we have:*

$$\frac{1}{2}(K_p(E) - 1) \leq \|T_E\| \leq K_p(E) + 2.$$

Proof. Assume that E is $OUMD_p$, then $S_p[E]$ is UMD and the triangular projection T_E is bounded. Let T_E^- be the triangular projection defined by $(x_{ij}) \mapsto (x_{ij} 1_{j \le i})$, it is clear that $\|T_E\| = \|T_E^-\|$. We have

$$d_n x = d_n T_E x + d_n T_E^- - D_n x,$$

where $D_n x = e_{nn} x e_{nn}$. Thus

$$\begin{aligned} \|\sum \varepsilon_n d_n x\|_{S_p[E]} &\le \|\sum \varepsilon_n d_n T_E x\|_{S_p[E]} + \|\sum \varepsilon_n d_n T_E^- x\|_{S_p[E]} \\ &\quad + \|\sum \varepsilon_n D_n x\|_{S_p[E]}. \end{aligned}$$

Since $d_n T_E x$ is the n-th column of $T_E x$, it is easy to see

$$\|\sum \varepsilon_n d_n T_E x\|_{S_p[E]} = \|\sum d_n T_E x\|_{S_p[E]} = \|T_E x\|_{S_p[E]} \le \|T_E\| \|x\|_{S_p[E]}.$$

The same reason shows that

$$\|\sum \varepsilon_n d_n T_E^- x\|_{S_p[E]} = \|\sum d_n T_E^- x\|_{S_p[E]} = \|T_E^- x\|_{S_p[E]} \le \|T_E^-\| \|x\|_{S_p[E]}.$$

For the third term, we have obviously that

$$\|\sum \varepsilon_n D_n x\|_{S_p[E]} = \|\sum D_n x\|_{S_p[E]} \le \|x\|_{S_p[E]}.$$

Combining these inequalities, we have

$$\|\sum \varepsilon_n d_n x\|_{S_p[E]} \le (\|T_E\| + \|T_E^-\| + 1)\|x\|_{S_p[E]} = (2\|T_E\| + 1)\|x\|_{S_p[E]}.$$

So E is $OUMD_p$ with respect to the canonical matrix filtration with $K_p(E) \le 2\|T_E\| + 1$.

Conversely, assume that E is $OUMD_p$ with respect to the canonical matrix filtration. We shall show that E is $OUMD_p$. It suffices to show that the triangular projection T_E is bounded. According to the remark 14, we have

$$\|(x_{ij} 1_{i \ge j})\|_{S_p[E]} \le \int \|D_\varepsilon T_\varepsilon(x)\|_{S_p[E]} \mathrm{d}\varepsilon \le K_p(E) \|x\|_{S_p[E]}.$$

Then it is rather easy to deduce that $\|(x_{ij} 1_{j \ge i})\|_{S_p[E]} \le (2 + K_p(E))\|x\|_{S_p[E]}$, so the upper triangular projection on $S_p[E]$ is bounded and

$$\|T_E\| \le 2 + K_p(E).$$

□

Remark 16 *We have a sligtly better estimation for* $\|T_E\|$ *and* $K_p(E)$*, i.e, we can prove that*

$$\frac{1}{2}(K_p(E)-1) \leq \|T_E\| \leq \frac{1}{2}(K_p(E)+1).$$

We omit the proof here.

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