POSITIVITY OF INTEGRATED RANDOM WALKS

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Abstract. Take a centered random walk S_n and consider the sequence of its partial sums $A_n := \sum_{i=1}^n S_i$. Suppose S_1 is in the domain of normal attraction of an α-stable law with $1 < \alpha \le 2$. Assuming that S_1 is either right-exponential (that is $\mathbb{P}\{S > x | S > 0\} = e^{-ax}$ for some $a > 0$ and all $x > 0$) or right-continuous (skip free), we prove that

$$p_N := \mathbb{P}\Big\{\min_{1\le k\le N} A_k > 0\Big\} \sim c_\alpha N^{\frac{1}{2\alpha}-\frac{1}{2}}$$

as $N \to \infty$, where $c_\alpha > 0$ depends on the distribution of the walk. We also condition on $S_N = 0$ and study positivity of integrated discrete bridges.

1. Introduction

1.1. **The problem.** Consider a non-degenerate sequence of centered random variables. What is the probability that the sequence stays positive for a long time? Surprising little is known on this quesion. Only one situation is well understood besides the trivial case that the variables are independent: For a random walk S_n, the classical Sparre-Andersen theorem expresses the generating function of

$$q_n := \mathbb{P}\Big\{\min_{1\le k\le n} S_k > 0\Big\}$$

in terms of the probabilities $\mathbb{P}(S_n > 0)$. A Tauberian theorem then implies $n^{1/2} q_n \to c > 0$ in the typical case that $\mathbb{E}S_1 = 0, Var(S_1) < \infty$, and moreover, if $\mathbb{P}(S_n > 0) \to \gamma \in (0, 1)$, then $n^{1-\gamma} q_n$ is slowly varying at infinity.

Consider the sequence $A_n := \sum_{i=1}^n S_i$, which we call an *integrated random walk*. We are interested in the asymptotics of

$$p_N := \mathbb{P}\Big\{\min_{1\le k\le N} A_k > 0\Big\}$$

as $N \to \infty$. One may also refer to similar type of questions as to asymptotics of the tail of one-sided exit times, unilateral small deviation probabilities, or *persistence* if adopting the terminology from physics.

This problem was introduced in the seminal paper Sinai [21], which considered the specific case that S_n is a simple random walk. Sinai studied the question in connection with

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behavior of solutions of the Burgers equation with random initial data. The author's initial motivation comes from his study [24] of sticky particles systems with gravitational attraction. The asymptotics of p_N is directly related to behavior of such systems with random initial data at the critical moment of the total gravitational collapse. The probabilities p_N also arise in the wetting model of random polymers with Laplacian interaction considered in Caravenna and Deuschel [5].

Although continuous-time versions of our question drew more attention, there are not many results in this direction, most of them listed in Aurzada and Dereich [1]. The recent breakthrough here, [1] shows universality of the asymptotics in the *one-sided exit problem* for general integrated Lévy processes. We also mention that in addition to a great theoretical interest, persistence probabilities that a certain function does not change its sign over a large time scale, appear in many physical models, see Majumbar [17].

1.2. **The background.** The first result on the subject is due to Sinai [21] which explained that $p_N \asymp N^{-1/4}$ for a simple random walk. As the continuous-time analog with an integrated Wiener process $A(t) := \int_0^t W(s)ds$ shows the same asymptotics, namely

$$\mathbb{P}\{\inf_{0\le t\le N} A(t) \ge -1\} \asymp N^{-1/4}$$

(Isozaki and Watanabe [14]), [5, 24] conjectured that $p_N \asymp N^{-1/4}$ for any walk S_n with $\mathbb{E}S_1 = 0$ and $Var(S_1) < \infty$. At the present there are three different approaches to this open question which are briefly explained below.

Sinai's method relies on the observation that if S_n is a simple random walk, then all the local extrema of A_n occur at the times when S_n returns to zero, and such times form a renewal sequence. This property is based on the very specific structure of increments of the walk and does not hold for different distributions. However, the main message here is to partition the trajectory of S_n with a suitable sequence of regeneration times. Vysotsky [25] explored this idea and showed that $p_N \lesssim N^{-1/4}$ for integer-valued walks (we write $a_n \lesssim b_n$ for two non-negative sequences a_n and b_n if a_n/b_n stays bounded while $a_n \asymp b_n$ means $a_n \lesssim b_n$ and $b_n \lesssim a_n$). Further development method required to make restrictive assumptions on the positive increments of the walk. Due to technical difficulties, [25] also imposed analogous constraints on the negative increments of S_1 and proved $p_N \asymp N^{-1/4}$ for double-sided exponentials, symmetric geometric, lazy simple, and two other "mixed" types of random walks.

The second approach is by Aurzada and Dereich [1] which used strong approximation by a Wiener process assuming $\mathbb{E}e^{a|S_1|} < \infty$ for some $a > 0$. This powerful method allowed to prove universality of the asymptotics for general integrated Levy processes but strong approximation does not work well for small values of time resulting in extra factors in the estimates: $N^{-1/4}(\log N)^{-4} \lesssim p_N \lesssim N^{-1/4}(\log N)^4$.

The third method by Dembo and Gao [7] is based on decomposition of the sequence A_n at its maximum. [7] proved the desired $p_N \lesssim N^{-1/4}$ for any walks with $\mathbb{E}S_1 = 0$ and $Var(S_1) < \infty$ but still imposed assumptions on positive increments of the walk to prove the sharp lower bound. [7] showed that $N^{-1/4} \lesssim p_N$ when the positive tail of S_1 is either sub-exponential or behaves exponentially at infinity. Thus the results of [7] cover those of [25] and, essentially, of [1].

Hence the conjecture $p_N \asymp N^{-1/4}$ under $\mathbb{E}S_1 = 0$ and $Var(S_1) < \infty$ is not yet proved to the full extent.

1.3. **Results and organization of the paper.** The goal of this paper is to prove the sharp asymptotics for p_N. We follow our approach developed in [25] but only keep the assumptions on positive increments of S_n. It seems there is no way to get the sharp asymptotics using the other methods.

Let us state our assumptions. A random walk S_n is *right-exponential* if $\mathrm{Law}(S_1|S_1 > 0)$ is an exponential distribution. An integer-valued walk S_n is *right-continuous* (skip free) if $\mathbb{P}\{S_1 = 1|S_1 > 0\} = 1$; the name comes from analogy with spectrally negative integrable Lévy processes, which do not have positive jumps and take all values before reaching any positive horizontal level. The introduced distributions are well know in the renewal theory and have the characteristic property that all overshoots of S_n over any fixed level have the same common distribution.

Suppose that S_1 belongs to the domain of normal attraction (to be denoted as $S_1 \in \mathcal{DN}(\alpha)$) of a strictly stable law with the index $1 < \alpha \le 2$. Clearly, such law is spectrally negative, and by the stable central limit theorem and Eq. (2.2.30) in Zolotarev [26] for the positivity parameter, it holds that $\mathbb{P}\{S_n > 0\} \to 1/\alpha$. For $1 < \alpha \le 2$, define

$$\begin{aligned}\mathcal{R}_\alpha \ := \ \Big\{ & S_1 : S_n \text{ is either right-exponential or right-continuous}, \mathbb{E}S_1 = 0,\\ & S_1 \in \mathcal{DN}(\alpha), \text{ and } \sum_{n=1}^{\infty} \frac{1}{n}\Big(\mathbb{P}\{S_n > 0\} - \frac{1}{\alpha}\Big) \text{ converges}\Big\}.\end{aligned}$$

Recall that (Theorem 2.6.6 in Ibragimov and Linnik [13]) $S_1 \in \mathcal{DN}(2)$ is equivalent to $Var(S_1) < \infty$, which ensures convergence of the series (Feller [11, Ch. XVIII.5]. Due to Egorov [10], for $1 < \alpha < 2$ a sufficient condition for the convergence is $\int_0^\infty x^\alpha |d(F(-x) - G_\alpha(-x)| < \infty$, where $F(x)$ and $G_\alpha(x)$ are the distribution functions of S_1 and the limit stable law, respectively.

We now state the main result of the paper.

Theorem 1. *Let $S_1 \in \mathcal{R}_\alpha$ for some $1 < \alpha \le 2$. Then there exists a constant $c = c(\alpha, Law(S_1)) > 0$ such that*

$$\lim_{N\to\infty} N^{\frac{1}{2}-\frac{1}{2\alpha}} p_N = c.$$

Remark. *Computable bounds for c when $\alpha = 2$ and S_1 is upper-exponential are given below in* (7)*.*

It seems that our proof works if we drop convergence of the series in the definition of $\mathcal{R}_\alpha$, and then $N^{\frac{1}{2}-\frac{1}{2\alpha}} p_N$ just becomes slowly varying at infinity. We also point that [7] showed $p_N \asymp N^{\frac{1}{2\alpha}-\frac{1}{2}}$ for the class of distributions that includes $\mathcal{R}_\alpha$.

Our method could be also applied for a modified version of the problem with integrated walks replaced by integrated discrete bridges. For an integer-valued walk S_n, put

$$p^*_N := \mathbb{P}\Big\{\min_{1\le k\le N} A_k > 0 \Big| S_N = 0\Big\}$$

for $N \in \mathcal{D}_{S_1} := \{n : \mathbb{P}(S_n = 0) > 0\}$ where this expression is well-defined.

Proposition 1. *Let S_n be an integer-valued random walk with $\mathbb{E}S_1 = 0$ and $Var(S_1) < \infty$. Then $p^*_N \lesssim N^{-1/4}$ and moreover, $p^*_N \asymp N^{-1/4}$ if S_1 is right-continuous, as $N \to \infty$ along $\mathcal{D}_{S_1}$.*

Although this statement covers a narrow class of distributions, it is the first result of such type. An open and very challenging problem that draw a recent attention is to find the asymptotics of

$$\mathbb{P}\Big\{\min_{1\le k\le N} A_k > 0 \big| A_N = 0\Big\} \quad \text{and} \quad \mathbb{P}\Big\{\min_{1\le k\le N} A_k > 0 \big| S_N = 0, A_N = 0\Big\}.$$

These probabilities are related to polymer models similar to the one of Caravenna and Deuschel [5].

The paper is organized as follows. In Sec. 2 we explain in detail our approach of partitioning the trajectory of S_n into independent parts (so-called cycles) by appropriate moments of regeneration. The pivotal result of the section is Proposition 2 on bivariate random walks staying in the right half-plane. This is, very roughly speaking, a bivariate version of the famous Sparre-Andersen theorem that q_n do not depend on the distribution of the walk if S_1 is symmetric and continuous. In addition to its independent interest, Proposition 2 is the keystone for a simple and very intuitive way to prove $p_N \asymp N^{\frac{1}{2\alpha}-\frac{1}{2}}$ for $S_1 \in \mathcal{R}_\alpha$; once may regard it as a rigorous version of the heuristic arguments from [25, Sec. 2.1]. We give the proof here as we believe this makes the paper more readable and shows the advantage of the used technique. In Sec. 2.4 we apply Proposition 2 to get our results on positivity of integrated bridges. Theorem 1 is proved in Sec. 4. The necessary ingredients are prepared in Sec. 3, where we study joint tails of areas and lengths of Brownian and stable excursions, cycles and meanders. There we also discuss conditional limit theorems for bivariate random walks.

2. Partitioning into cycles and non-sharp asymptotics of p_N

2.1. **Partitioning by regenerating times.** The main idea of our approach is to partition trajectory of the random walk S_n into appropriate independent parts. Define the moments of *crossing* the zero level *from below* as

$$\Theta_0 := \min\{n \ge 0 : S_{n+1} > 0\}, \quad \Theta_{k+1} := \min\{n > \Theta_k : S_n \le 0, S_{n+1} > 0\}$$

for $k \ge 0$. A non-degenerate centered random walk is recurrent hence the r.v.'s defined above are proper. We stress that Θ_k are not stopping times as opposed to $\Theta_k + 1$. The trajectory of S_n is thus partitioned into cycles, and each cycle (possibly excluding the first one) starts with a positive excursion followed by a negative excursion. For $k \ge 1$, let $\theta_k := \Theta_k - \Theta_{k-1}$ be durations of cycles and $\psi_k := A_{\Theta_k} - A_{\Theta_{k-1}}$ be their areas; also, set $\Psi_k := A_{\Theta_k}$ for the total area of the first k cycles so $\psi_k = \Psi_k - \Psi_{k-1}$.

Define $\widetilde{\mathbb{P}}(\cdot) := \mathbb{P}(\cdot|S_1 > 0)$ as it is more convenient to think that S_n starts with a positive excursion and so $\Theta_0 = 0$ $\widetilde{\mathbb{P}}$-a.s.; also put $\sigma^2 := Var(S_1)$. The next observation from Vysotsky [25] (Lemmae 1, 2 and Proposition 1) plays the crucial role for our method.

Lemma 1. *Let S_n be a centered random walk that is either right-exponential or right-continuous. Then $(\theta_n, \psi_n)_{n\geq 1}$ are i.i.d. and $(\theta_1, \psi_1) \stackrel{\mathcal{D}}{=} (\theta_1, -\psi_1)$. If $S_1 \in \mathcal{R}_2$, then $\theta_1 \in \mathcal{DN}(1/2)$, and, moreover, $\lim_{n\to\infty} n^{1/2}\mathbb{P}\{\theta_1 \geq n\} = \sqrt{\frac{8}{\pi}} \frac{\sigma}{\mathbb{E}|S_1|}$ if S_1 is right-exponential.*

[25] actually gives a proof only for right-exponential walks but the right-continuous case should be considered in exactly the same way. Here is a brief explanation for the result. The i.i.d. property follows as each cycle starts with the overshoot S_{Θ_k+1} which is independent of the preceding part $S_1, \dots, S_{\Theta_k}$ of the trajectory. The symmetry holds by

$$\big(S_1, \dots, S_{\hat{\theta}_1}, \hat{\theta}_1\big) \stackrel{\mathcal{D}}{=} \big(-S_{\hat{\theta}_1}, \dots, -S_1, \hat{\theta}_1\big) \quad \text{under } \widetilde{\mathbb{P}}, \tag{1}$$

where $\hat{\theta}_1 := \min\{n > \Theta_0 : S_n < 0, S_{n+1} \geq 0\}$. This relation follows from the duality principle for random walks with the use that under $\widetilde{\mathbb{P}}$, S_1 either identically equals 1 or is distributed exponentially.

For the tail of θ_1, the Sparre-Andersen theorem implies that length of the first positive excursion $\theta_+ \in \mathcal{DN}(1-1/\alpha)$ if $S_1 \in \mathcal{R}_\alpha$ with $1 < \alpha \leq 2$, and by (1), the same holds for the first negative excursion. Thus one also expects $\theta_1 \in \mathcal{DN}(1-1/\alpha)$ as stated in Lemma 1 for $\alpha = 2$. We prove this statement for $1 < \alpha < 2$ in the next section in Lemma 2 but we will use this fact already in this section in the proof of Theorem 3. Note that there is a significant difference in the shape of long cycles: for $\alpha = 2$ the walk essentially does not change its sign while for $1 < \alpha < 2$ it stays on both half-axes for a nonzero fraction of time.

2.2. **Bivariate walks staying in the right half-plane.** Thus under the conditions of Lemma 1, (Θ_k, Ψ_k) is a bivariate random walk with the symmetric second component, and the walk starts at $(\Theta_0, \Psi_0) = (0, 0)$ under $\widetilde{\mathbb{P}}$. It turns that such walks enjoy a useful property stated below in Proposition 2, which is a slight improvement of the Sparre-Andersen theorem. The later implies, in particular, that the probabilities q_n that a symmetric continuous random walk stays positive does not depend on the distribution of increments. Proposition 2 is inspired by Lemma 3 of Sinai [21]; we also refer to Feller [11, Ch. XII] for appropriate definitions and general ideas. It is worth mentioning that Sinai's lemma is a special case of the results from Greenwood and Shaked [12], which gives a half-plane Wiener-Hopf type factorization for bivariate distributions. This reference was pointed to the author by Vitaliy Wachtel.

Proposition 2. *Let $(S_n^{(1)}, S_n^{(2)})$ be a bivariate random walk such that $(S_1^{(1)}, S_1^{(2)}) \stackrel{\mathcal{D}}{=} (-S_1^{(1)}, S_1^{(2)})$. Then for any $n \geq 1$ and $y \in \mathbb{R}$,*

$$\mathbb{P}\Big\{\min_{1\leq i\leq n} S_i^{(1)} \geq 0, S_n^{(2)} \in dy\Big\} \geq \mathbb{P}\Big\{\min_{1\leq i\leq n} S_i^{(1)} > 0\Big\}\mathbb{P}\Big\{S_n^{(2)} \in dy\Big\}. \tag{2}$$

and

$$\mathbb{P}\Big\{\min_{1\leq i\leq n} S_i^{(1)} > 0, S_n^{(2)} \in dy\Big\} \leq \mathbb{P}\Big\{\min_{1\leq i\leq n} S_i^{(1)} \geq 0\Big\}\mathbb{P}\Big\{S_n^{(2)} \in dy\Big\} \tag{3}$$

Corollary. *If in addition the distribution of $S_1^{(1)}$ is continuous, then the events $\big\{\min_{1\leq i\leq n} S_i^{(1)} > 0\big\}$ and $\big\{S_n^{(2)} \in dy\big\}$ are independent.*

Proof. We start noting that the characteristic function $\chi(s,t)$ of the left-hand side of (2) satisfies

$$1+\chi(s,t)=\frac{1}{1-\chi_{T_1,U_1}(s,t)},$$

where T_1 is the first weak ascending ladder epoch of the walk $S^{(1)}$ and $U_1 := S^{(2)}_{T_1}$. This can be verified by standard arguments once we consider the dual random walk to obtain

$$\mathbb{P}\Big\{\min_{1\le i\le n} S_i^{(1)} \ge 0, S_n^{(2)}\in dy\Big\}=\mathbb{P}\Big\{\max_{1\le i\le n-1} S_i^{(1)} \le S_n^{(1)}, S_n^{(2)}\in dy\Big\},$$

which means that n is a ladder index of $S^{(1)}$.

Next, Lemma 3 of Sinai [21], which is a little improvement of the Sparre-Andersen theorem, states

$$\log\frac{1}{1-\chi_{T_1,U_1}(s,t)}=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{s^n}{n}e^{ity}\mathbb{P}\Big\{S_n^{(1)}\ge 0, S_n^{(2)}\in dy\Big\},$$

hence with the symmetry of $S_n^{(1)}$,

$$\log\frac{1}{1-\chi_{T_1,U_1}(s,t)}=\sum_{n=1}^{\infty}\frac{s^n}{2n}\chi^n_{S_1^{(2)}}(t)+\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{s^n}{2n}e^{ity}\mathbb{P}\Big\{S_n^{(1)}=0, S_n^{(2)}\in dy\Big\}.$$

The second term in the left-hand side could be transformed as above and we get

$$\log\frac{1}{1-\chi_{T_1,U_1}(s,t)}=\frac{1}{2}\log\frac{1}{1-s\chi_{S_1^{(2)}}(t)}+\frac{1}{2}\log\frac{1}{1-\chi_{T_1^*,U_1^*}(s,t)},$$

where $\chi_{T_1^*,U_1^*}(s,t)$ is the characteristic function of the non-probability measure

$$Q^*(k,dy):=\mathbb{P}\big\{S_1^{(1)}<0,\dots,S_{k-1}^{(1)}<0, S_k^{(1)}=0, S_k^{(2)}\in dy\big\}.$$

In a certain sense T_1^* is the first moment when $S_n^{(1)}$ hits to zero from below and $U_1^*=S^{(2)}_{T_1^*}$. Then

$$1+\chi(s,t)=\sqrt{\frac{1}{\big(1-s\chi_{S_1^{(2)}}(t)\big)\big(1-\chi_{T_1^*,U_1^*}(s,t)\big)}},\tag{4}$$

and similarly, for the characteristic function $\chi^+(s,t)$ of the left-hand side of (3),

$$1+\chi^+(s,t)=\sqrt{\frac{1-\chi_{T_1^*,U_1^*}(s,t)}{1-s\chi_{S_1^{(2)}}(t)}}.\tag{5}$$

Finally, the characteristic function of the right-hand side of (2) is

$$\tilde\chi(s,t)=\sum_{n=1}^{\infty}s^n\chi^n_{S_1^{(2)}}(t)\mathbb{P}\Big\{\min_{1\le i\le n}S_i^{(1)}>0\Big\}=\chi_{T_1^+}\big(s\chi_{S_1^{(2)}}(t)\big),$$

where T_1^+ is the first strong ascending ladder epoch of $S^{(1)}$, and by (4), (5), and $\chi_{T_1^+}(s) = \chi^+(s,0)$ we get

$$1+\chi(s,t) = \big(1+\tilde\chi(s,t)\big)\sqrt{\frac{1}{\big(1-\chi_{T_1^*}(s\chi_{S_1^{(2)}}(t))\big)\big(1-\chi_{T_1^*,U_1^*}(s,t)\big)}}. \tag{6}$$

Since all the coefficients of the Maclaurin series of $(1-x)^{-1/2}$ are positive, the square root factor in the right-hand side of (6) is the characteristic function of a certain *measure* $Q(k,dy)$ satisfying $Q(0,dy) = \delta_0(y)$. Then (2) follows as product of characteristic functions corresponds to convolution of measures. The same argument of course applies for (3). □

Note that (4) and (5) actually follow from Eq. (4) in Greenwood and Shaked [12] with τ and ν from their Example (a) on p. 568, but we have chosen to start from the Sinai lemma to go along the original lines of our proof.

2.3. **Weak asymptotics of p_N.** The following result is not new and of course is weaker than Theorem 1. As we explained in the introduction, we give the proof here to show a simple and very intuitive way to understand the asymptotics of p_N and demonstrate advantage of the technique.

Proposition 3. *If $S_1 \in \mathcal{R}_\alpha$ for some $1<\alpha\le 2$, then $p_N \asymp N^{\frac{1}{2\alpha}-\frac12}$.*

Remark. *If S_1 is right-exponential and $S_1\in\mathcal{R}_2$, then*

$$\Big[\varliminf_{N\to\infty} p_N N^{\frac12-\frac{1}{2\alpha}}, \varlimsup_{N\to\infty} p_N N^{\frac12-\frac{1}{2\alpha}}\Big] \subset \frac{2^{1/4}}{\pi}\Gamma\Big(\frac14\Big)\sqrt{\frac{\sigma}{\mathbb{E}|S_1|}}\mathbb{P}\{S_1>0\}\times\Big[\frac12,1\Big]. \tag{7}$$

Proof. The key observation is that

$$\mathbb{P}\Big\{\min_{1\le k\le N} A_k>0\Big\} = \mathbb{P}\Big\{\min_{1\le k\le \eta(N)} A_{\Theta_k}>0,\, A_1>0, A_N>0\Big\}, \tag{8}$$

where

$$\eta(N) := \max\{n\ge 0 : \Theta_n \le N\};$$

under $\widetilde{\mathbb{P}}$, this is just the number of up-crossing of the zero level by the walk S_n by the time N. Then

$$\widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)+1}\Psi_k>0\Big\} \le \frac{p_N}{\mathbb{P}\{S_1>0\}} \le \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)}\Psi_k>0\Big\}.$$

By the conditional symmetry of ψ_i (Lemma 2), we estimate the lower bound flipping the last cycle to make sure it has a positive area: condition on $\eta(N)$ and $\Theta_{\eta(N)}$ and get

$$\begin{aligned} &\widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k > 0, \psi_{\eta(N)+1} \ge 0\Big\} \\ = \ & \sum_n \sum_{i\le N} \widetilde{\mathbb{P}}\Big\{\Theta_n = i, \theta_{n+1} > N-i, \min_{1\le k\le n} \Psi_k > 0, \psi_{n+1} \ge 0\Big\} \qquad (9) \\ = \ & \sum_n \sum_{i\le N} \widetilde{\mathbb{P}}\Big\{\Theta_n = i, \min_{1\le k\le n} \Psi_k > 0\Big\} \widetilde{\mathbb{P}}\Big\{\theta_{n+1} > N-i, \psi_{n+1} \ge 0\Big\} \\ \ge \ & \frac{1}{2} \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k > 0\Big\}. \end{aligned}$$

Hence

$$\frac{1}{2} \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k > 0\Big\} \le \frac{p_N}{\mathbb{P}\{S_1 > 0\}} \le \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k > 0\Big\}.$$

Now assume that S_1 is right-exponential. By conditioning on $\eta(N)$ and using Proposition 2, we proceed as above in (9) and get the most important relation

$$\widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k > 0\Big\} = \sum_{n=0}^{\infty} \widetilde{\mathbb{P}}\big\{\eta(N) = n\big\} \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le n} \Psi_k > 0\Big\}. \qquad (10)$$

As the distribution of Φ_k is symmetric, Sparre-Andersen's theorem implies existence of a positive limit

$$c_1 := \lim_{n\to\infty} n^{1/2} \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le n} \Psi_k > 0\Big\},$$

hence

$$\begin{aligned} \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k > 0\Big\} &= \sum_{k=0}^{\infty} \widetilde{\mathbb{P}}\Big\{\eta(N) = n\Big\} \frac{c_1 + o(1)}{\sqrt{n+1}} \\ &= (c_1 + o(1)) \widetilde{\mathbb{E}} \frac{1}{\sqrt{\eta(N)+1}} + O\Big(\widetilde{\mathbb{P}}\big\{\eta(N) < \ln N\big\}\Big) \\ &= \frac{c_1 + o(1)}{N^{\frac{1}{2} - \frac{1}{2\alpha}}} \widetilde{\mathbb{E}} \sqrt{\frac{N^{1-1/\alpha}}{\eta(N)+1}} + O\Big(\widetilde{\mathbb{P}}\big\{\eta(N) < \ln N\big\}\Big). \qquad (11) \end{aligned}$$

We estimate the remainder as

$$\begin{aligned} &\widetilde{\mathbb{P}}\big\{\eta(N) < \ln N\big\} = \widetilde{\mathbb{P}}\big\{\Theta_{\ln N} > N\big\} \le \ln N \, \widetilde{\mathbb{P}}\big\{\theta_1 > N/\ln N\big\} \\ = \ & O(N^{-\alpha} (\ln N)^{1+\alpha}) = o(N^{-1}). \qquad (12) \end{aligned}$$

Further, it follows from Feller [11, Ch. XI.5] that the number of renewal epochs $\eta(N)$ satisfies

$$\frac{\eta(N)}{N^{1-1/\alpha}} \xrightarrow{\mathcal{D}} c_2^{-1} \tau_{1-1/\alpha}^{1/\alpha - 1} \quad \text{under } \widetilde{\mathbb{P}}, \qquad (13)$$

where τ_γ is a spectrally positive γ-stable r.v. such that

$$\lim_{n\to\infty} n^\gamma \mathbb{P}\{\tau_\gamma > n\} = 1, \text{ and } c_2 := \lim_{n\to\infty} n^{1-1/\alpha} \mathbb{P}\{\theta_1 > n\}.$$

Since $\mathbb{E}\tau_{1-1/\alpha}^{(1-1/\alpha)/2} < \infty$, the theorem will be proved once we check uniform integrability of $\sqrt{\frac{N^{1-1/\alpha}}{\eta(N)+1}}$ in order to get convergence of the expectations. Estimate the tail as follows: for any $0 < x \le N^{\frac{1}{2}-\frac{1}{2\alpha}}$,

$$\begin{aligned}\widetilde{\mathbb{P}}\left\{\sqrt{\frac{N^{1-1/\alpha}}{\eta(N)+1}} \ge x\right\} &= \widetilde{\mathbb{P}}\Big\{\eta(N) \le [x^{-2}N^{1-1/\alpha}] - 1\Big\}\\ = \ \ \widetilde{\mathbb{P}}\Big\{\Theta_{[x^{-2}N^{1-1/\alpha}]} > N\Big\} &\le \widetilde{\mathbb{P}}\Big\{\Theta_k > (x^2k)^{\frac{1}{1-1/\alpha}}\Big\},\end{aligned}$$

where $k := [x^{-2}N^{1-1/\alpha}] \ge 1$. The last probability could be estimated by the following analog of the Chebyshev inequality attributed by Nagaev [18] to Tkachuk (1977): if X_n are i.i.d. r.v.'s and $X_1 \in \mathcal{DN}(\gamma)$ for $0 < \gamma < 1$, then there exist $c, K > 0$ such that

$$\mathbb{P}\{X_1 + \dots X_n > Rn^{1/\gamma}\} \le cR^{-\gamma}$$

for all n and $R \ge K$. Thus uniform integrability follows as x^{-2} is integrable at infinity.

Let us compute the constant for $\alpha = 2$. It is well known that $\tau_{1/2}$ could be represented as the first moment when a standard Brownian motion W hits a certain level, and we use the reflection principle to get $\mathbb{P}\{\tau_{1/2} > x\} = \hat{\Phi}\big(\sqrt{\frac{\pi}{2x}}\big)$, where $\hat{\Phi}(x)$ is the distribution function of $|W(1)|$. Then

$$\lim_{N\to\infty} \widetilde{\mathbb{E}}\sqrt{\frac{N^{1-\frac{1}{\alpha}}}{\eta(N)+1}} = c_2^{1/2}\widetilde{\mathbb{E}}\tau_{1/2}^{1/4} = c_2^{1/2}\int_0^\infty 2^{-1}x^{-5/4}e^{-\frac{\pi}{4x}}dx = \sqrt{\frac{c_2}{2\sqrt{\pi}}}\Gamma\Big(\frac{1}{4}\Big),$$

and it remains to recall that $c_2 = \sqrt{\frac{8}{\pi}\frac{\sigma}{\mathbb{E}|S_1|}}$ and $c_1 = \sqrt{\frac{1}{\pi}}$ (as Ψ_n is continuous and symmetric).

Now assume S_1 is right-continuous. Denote

$$r_N := \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k > 0\Big\}, \quad \bar{r}_N := \widetilde{\mathbb{P}}\Big\{\min_{1\le k\le \eta(N)} \Psi_k \ge 0\Big\}$$

and replace (10) by the appropriate inequality; then get an analog of (11) with "=" and c_1 replaced by "$\le$" and $\bar{c_1}$, respectively, and by the uniform integrability conclude with $r_N \lesssim N^{\frac{1}{2}-\frac{1}{2\alpha}}$. The same argument implies $N^{\frac{1}{2}-\frac{1}{2\alpha}} \lesssim \bar{r}_N$, and since

$$\bar{r}_N \ge r_N \ge \mathbb{P}\{\psi_1 > 0\}\bar{r}_N,$$

we obtain $\bar{r}_N \asymp r_N \asymp N^{\frac{1}{2}-\frac{1}{2\alpha}}$. □

2.4. **Positivity of integrated bridges.** Let us show how Proposition 2 could be used to obtain asymptotics of

$$p_N^* = \mathbb{P}\Big\{\min_{1\le k\le N} A_k > 0 \big| S_N = 0\Big\}$$

as $N \to \infty$ along $\mathcal{D}_{S_1}$. Recall we assumed that S_n is centered integer-valued. Let d be the maximal positive integer such that $\mathbb{P}\{S_1 \in d\mathbb{N}\} = 1$, and let h be the maximal step of S_1/d, that is the maximal positive integer such that there exists an $0 \le a \le h-1$ satisfying $\mathbb{P}\{S_1 \in d(a + h\mathbb{N})\} = 1$. Then $\mathcal{D}_{S_1} \subset h\mathbb{N}$ and $h\mathbb{N} \setminus \mathcal{D}_{S_1}$ is finite.

We need to modify the definition of the regeneration moments considered in Sec. 2.1. Define the moments of *leaving* zero as $\Theta_0^* := \min\{n \geq 0 : S_{n+1} \neq 0\}$ and $\Theta_{k+1}^* := \min\{n > \Theta_k : S_n = 0, S_{n+1} \neq 0\}$ for $k \geq 0$, and introduce $\theta_k^*, \psi_k^*, \Psi_k^*, \eta^*(N)$ accordingly. Put $\mathbb{P}^*(\cdot) := \mathbb{P}(\cdot | S_1 \neq 0)$. The following result is completely analogous to Lemma 1 and is essentially proved in [25] while the local asymptotics is by Kesten [16].

Lemma 1′. *Let S_n be a centered random walk. Then $(\theta_n^*, \psi_n^*)_{n\geq 1}$ are i.i.d. and $(\theta_1^*, \psi_1^*) \overset{\mathcal{D}}{=} (\theta_1^*, -\psi_1^*)$. If $Var(S_1) =: \sigma^2 < \infty$, then $\mathbb{P}^*\{\theta_1^* = hn\} \sim \sqrt{\frac{h}{2\pi}} \frac{\sigma}{d} n^{-3/2}$ as $n \to \infty$.*

Recall the result we introduced in Sec. 1.

Proposition 1. *Let S_n be an integer-valued random walk with $\mathbb{E}S_1 = 0$ and $Var(S_1) < \infty$. Then $p_{hN}^* \lesssim N^{-1/4}$ and moreover, $p_{hN}^* \asymp N^{-1/4}$ if S_1 is right-continuous.*

Proof. Similarly to (8), write

$$\mathbb{P}\Big\{\min_{1\leq k\leq hN} A_k > 0, S_{hN} = 0\Big\} \leq \mathbb{P}\Big\{\min_{1\leq k\leq \eta^*(hN)} A_{\Theta_k^*} > 0, S_{hN} = 0\Big\},$$

which becomes an equality when S_1 is right-continuous, and then condition on the number of returns to zero $\eta^*(hN)$ as in (9) to get

$$\mathbb{P}\Big\{\min_{1\leq k\leq hN} A_k > 0, S_{hN} = 0\Big\} \leq \sum_{n=1}^{\infty} \mathbb{P}^*\Big\{\min_{1\leq k\leq n} \Psi_k^* > 0, \Theta_n^* = hN\Big\} =: r_N.$$

By Proposition 2 and (12) we get the following analog of (11):

$$r_N \leq (\bar{c}_1 + o(1)) \sum_{n=1}^{\infty} n^{-1/2} \mathbb{P}^*\big\{\Theta_n^* = hN\big\} + o(N^{-1}).$$

With Lemma 1′, we use a result by Doney [9] on local large deviation probabilities that $\mathbb{P}^*\big\{\Theta_n^* = hN\big\} \sim n\mathbb{P}^*\big\{\theta_1^* = hN\big\}$ as $N \to \infty$ uniformly in $n = o(\sqrt{N})$. Then by Lemma 1′, the contribution of the terms with $n = o(\sqrt{N})$ is $o(N^{-3/4})$, implying

$$r_N \leq (\bar{c}_1 + o(1)) \varlimsup_{\varepsilon\to 0+} \sum_{n=\varepsilon\sqrt{N}}^{\infty} n^{-1/2} \mathbb{P}^*\big\{\Theta_n^* = hN\big\} + o(N^{-3/4}).$$

Once we bounded $\sqrt{N}/n$ away from zero, the local limit theorem gives

$$\begin{aligned} \lim_{\varepsilon\to 0+} \lim_{N\to\infty} N^{3/4} \sum_{n=\varepsilon\sqrt{N}}^{\infty} n^{-1/2} \mathbb{P}^*\big\{\Theta_n^* = hN\big\} &= \lim_{\varepsilon\to 0+} \frac{1}{\sqrt{N}} \sum_{n=\varepsilon\sqrt{N}}^{\infty} \Big(\frac{n}{\sqrt{N}}\Big)^{-5/2} n^2 \mathbb{P}^*\big\{\Theta_n^* = hN\big\} \\ &= \lim_{\varepsilon\to 0+} \frac{1}{\sqrt{N}} \sum_{n=\varepsilon\sqrt{N}}^{\infty} \Big(\frac{n}{\sqrt{N}}\Big)^{-5/2} hg\Big(\frac{hN}{n^2}\Big) \\ &= h^{1/4} \mathbb{E}\tau^{-5/4}, \end{aligned}$$

where g is the density of a strictly 1/2-stable r.v. τ that appears as a weak limit of Θ_n^*/n^2.

Thus $r_N \lesssim N^{-3/4}$ and by Gnedenko's local limit theorem, $p^*_{hN} \lesssim N^{-1/4}$. For the estimate in the other direction when S_1 is right-continuous, condition on the $(hN+1)$st step of the walk to get $p^*_{hN} \ge (\mathbb{P}\{S_1 = 1\})^2 \bar{p}^*_{hN}$, where $\bar{p}^*_n$ is defined as p^*_n with "$>$" replaced by "$\ge$". Arguing as above we get $\bar{p}^*_{hN} \gtrsim N^{-1/4}$ that implies $p^*_{hN} \asymp \bar{p}^*_{hN} \asymp N^{-1/4}$. □

3. Areas and lengths of excursions of asymptotically stable random walks

3.1. **Conditional limit theorems for random walks.** Let S_n be a random walk such that the first descending ladder moment $T = \min\{k \ge 1 : S_1 < 0\} < \infty$ a.s. Bolthausen [4] showed that if $\mathbb{E}S_1 = 0$ and $Var(S_1) = \sigma^2 < \infty$, then

$$\mathrm{Law}\Big(\frac{S_{[n\cdot]}}{n^{1/2}}\Big| T \ge n\Big) \xrightarrow{\mathcal{D}} \mathrm{Law}\big(\sigma W_+(\cdot)\big) \tag{14}$$

in the Skorokhod space $D[0,1]$ as $n \to \infty$, where W_+ is a Brownian meander on $[0,1]$ defined below in terms of a standard Brownian motion W.

The proof of [4] is based on the following insightfully simple observation. For any $f : [0,\infty) \to \mathbb{R}$ define $\tau_f := \inf\{t \ge 0 : f(s+t) \ge f(t) \text{ for } 0 \le s \le 1\}$, where $\inf_\varnothing := \infty$, and $\Gamma(f)(\cdot) := f(\cdot + \tau_f) - f(\tau_f)$ if $\tau_f < \infty$ and $\Gamma(f) :\equiv 0$ if otherwise. Then

$$\mathrm{Law}\big(S_{[n\cdot]}\big| T \ge n\big) = \mathrm{Law}\big(\Gamma(S_{[n\cdot]})\big). \tag{15}$$

Bolthausen [4] essentially showed that $\mathbb{P}\{\tau_W < \infty\} = 1$ and Γ considered as a mapping $C[0,\infty) \to C[0,1]$ is measurable and continuous $\mathbb{P}\{W \in \cdot\}$-a.s. (Wiener measure). With a linear smoothing, convergence (14) with $W_+ = \Gamma(W)$ immediately follows from the invariance principle in $C[0,\infty)$ and the continuous mapping theorem, see [3, Sec. 2].

Shimura [20] used the same method to extended (14) to convergence of excursions. For any $f : [0,\infty) \to \mathbb{R}$ define $\delta_f := \inf\{t \ge 0 : f(t) < 0\}$ and $\Lambda(f)(\cdot) := \big(f(\cdot \wedge \delta_f), \delta_f\big)$. [20] proved that $\mathbb{P}\{\delta_{W_+} < \infty\} = 1$ and $\Lambda\Gamma$ considered as a mapping $D[0,\infty) \to D[0,\infty) \times \mathbb{R}$ is measurable and continuous $\mathbb{P}\{W \in \cdot\}$-a.s.; Shimura actually checked continuity along step functions but they are dense in $D[0,\infty)$. Hence under assumptions $\mathbb{E}S_1 = 0$ and $Var(S_1) = \sigma^2 < \infty$, the continuous mapping theorem implies Shimura's main result

$$\mathrm{Law}\Big(\Big(\frac{S_{[n\cdot\wedge T]}}{n^{1/2}}, \frac{T}{n}\Big)\Big| T \ge n\Big) \xrightarrow{\mathcal{D}} \mathrm{Law}\big(\sigma W_+(\cdot \wedge \delta_{W_+}), \delta_{W_+}\big) \tag{16}$$

in $D[0,\infty) \times \mathbb{R}$. Now define rescalings $\widehat{\Lambda}_a(f)(\cdot) := \delta_f^{-1/a} f(\cdot\, \delta_f)$, then $\widehat{\Lambda}_a\Gamma : D[0,\infty) \to D[0,\infty)$ is continuous $\mathbb{P}\{W \in \cdot\}$-a.s. for any $a > 0$. As trajectories of W are continuous, $\widehat{\Lambda}_a\Gamma$ is also a.s. continuous as a mapping $D[0,\infty) \to D[0,1]$, and we restate (16) as

$$\mathrm{Law}\Big(\Big(\frac{S_{[T\cdot]}}{T^{1/2}}, \frac{T}{n}\Big)\Big| T \ge n\Big) \xrightarrow{\mathcal{D}} \mathrm{Law}\big(\sigma W_{ex}(\cdot), \delta_{W_+}\big), \tag{17}$$

in $D[0,1] \times \mathbb{R}$, where $W_{ex} = \widehat{\Lambda}_2\Gamma(W)$ is a standard Brownian excursion on $[0,1]$. Note that W_{ex} is independent of the length δ_{W_+} of the excursion of $W_+ = \Gamma(W)$ while $\mathbb{P}\{\delta_{W_+} \ge x\} = x^{-1/2}$ for $x \ge 1$, see Bertoin [2, Ch. VIII.4].

A further refinement is due Doney [8], which essentially proved that $\mathbb{P}\{\delta_{S_+} < \infty\} = 1$ and $\Lambda\Gamma : D[0,\infty) \to D[0,\infty) \times \mathbb{R}$ is continuous $\mathbb{P}\{S \in \cdot\}$-a.s. for any strictly stable centered process S with the index $1 < \alpha \le 2$. The continuity was actually checked along step functions but they are dense in $D[0,\infty)$. Then $\Gamma : D[0,\infty) \to D[0,1]$ is continuous $\mathbb{P}\{S \in \cdot\}$-a.s. but the same does not immediately follow for $\widehat{\Lambda}_\alpha \Gamma$ since meanders $S_+ := \Gamma(S)$ are continuous at $t = 1$ but normalized excursions $S_{ex} := \widehat{\Lambda}_\alpha \Gamma(S)$ are discontinuous at $t = 1$ if $\alpha < 2$. One should first use that $\widehat{\Lambda}_\alpha \Gamma : D[0,\infty) \to D[0,2]$ is a.s. continuous, which follows from the a.s. continuity of $\widehat{\Lambda}_\alpha \Gamma : D[0,\infty) \to D[0,\infty)$ because S_{ex} is a constant for $t > 1$, and only then get the continuity from $D[0,\infty)$ to $D[0,1]$.

Now assuming $n^{-1/\alpha} l(n) S_n \xrightarrow{\mathcal{D}} S(1)$ for some slowly varying $l(n)$, we restate the result of Doney as we did above with (16) in the form

$$\mathrm{Law}\Big(\Big(\frac{S_{[T\cdot]}}{T^{1/\alpha} l(T)}, \frac{T}{n}\Big)\Big| T \geq n\Big) \xrightarrow{\mathcal{D}} \mathrm{Law}\big(S_{ex}(\cdot), \delta_{S_+}\big) \tag{18}$$

in $D[0,1] \times \mathbb{R}$. As before, S_{ex} is independent with δ_{S_+} and $\mathbb{P}\{\delta_{S_+} \geq x\} = x^{1/\alpha - 1}$ for $x \geq 1$, see Bertoin [2, Ch. VIII.4].

We stress that all the mentioned results follow from the functional central or stable limit theorem with the use of the continuous mapping theorem. Let us give a little straightening to (14). First extend the definitions of τ_f and Γ to higher dimensions: for $\mathbf{f} = (f^{(1)}, f^{(2)})$, put $\tau_{\mathbf{f}} := \tau_{f^{(1)}}$ and $\Gamma(\mathbf{f}) := \mathbf{f}(\cdot + \tau_{\mathbf{f}}) - \mathbf{f}(\tau_{\mathbf{f}})$.

Let $\mathbf{S}_n = (S_n^{(1)}, S_n^{(2)})$ be a bivariate random walk that satisfies

$$\Big(\frac{S^{(1)}_{[n\cdot]}}{n^{1/\alpha_1} l_1(n)}, \frac{S^{(2)}_{[n\cdot]}}{n^{1/\alpha_2} l_2(n)}\Big) \xrightarrow{\mathcal{D}} \mathbf{S}(\cdot) \tag{19}$$

in $D^2[0,\infty)$ for some bivariate Lévy process $\mathbf{S}$, slowly varying $l_1(n), l_2(n)$, and $0 < \alpha_1, \alpha_2 \leq 2$. By Resnick and Greenwood [19], (19) is equivalent to existence of the finite positive

$$\lim_{n\to\infty} n\mathbb{P}\big\{\epsilon_1 S_1^{(1)} > x n^{1/\alpha_1} l_1(n),\ \epsilon_2 S_1^{(2)} > y n^{1/\alpha_2} l_2(n)\big\} \tag{20}$$

for all $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ and $x, y \geq 0$ such that $x + y > 0$. [19] also shows that (19) is equivalent to the weak convergence of the one-dimensional distributions at $t = 1$. The limit random vector $\mathbf{S}(1)$ is sometimes called bivariate stable with indices α_1, α_2 as its independent copies $\mathbf{S}'(1), \mathbf{S}''(1)$ satisfy

$$a_1 \mathbf{S}'(1) + a_2 \mathbf{S}''(1) \stackrel{\mathcal{D}}{=} \big((a_1^{\alpha_1} + a_2^{\alpha_1})^{1/\alpha_1} S^{(1)}(1),\ (a_1^{\alpha_2} + a_2^{\alpha_2})^{1/\alpha_2} S^{(2)}(1)\big)$$

for any $a_1, a_2 > 0$. [19] gave a complete characterization of such bivariate distributions.

By the $\mathbb{P}\{\mathbf{S} \in \cdot\}$-a.s. continuity of $\Gamma : D^2[0,\infty) \to D^2[0,1]$ we get

$$\mathrm{Law}\Big(\Big(\frac{S^{(1)}_{[n\cdot]}}{n^{1/\alpha_1} l_1(n)}, \frac{S^{(2)}_{[n\cdot]}}{n^{1/\alpha_2} l_2(n)}\Big)\Big| T^{(1)} \geq n\Big) \xrightarrow{\mathcal{D}} \mathrm{Law}\big(\mathbf{S}_+(\cdot)\big) \tag{21}$$

in $D^2[0,1]$, where $T^{(1)}$ is the first ladder moment of $S^{(1)}$ and $\mathbf{S}_+ := \Gamma(\mathbf{S})$. A simple consideration of (21) shows that it also holds true if $T^{(1)}$ is replaced by the first *strict* ladder moment.

3.2. **Areas of cycles.** The first statement of this section complements analogous Proposition 1 from [25] which covers $S_1 \in \mathcal{R}_2$. We stress that long cycles of a random walk from $\mathcal{R}_\alpha$ behave very differently for $\alpha = 2$ and $1 < \alpha < 2$, and the same could be said about excursions. For $\alpha < 2$, a typical positive excursion looks like a meander and then it takes only one step to drop down to the negative half-axis, see (23).

Lemma 2. *Let S_n be a random walk such that $S_1 \in \mathcal{R}_\alpha$ for some $1 < \alpha < 2$. Then*
1) $\theta_1 \in \mathcal{DN}(1 - 1/\alpha)$;
2) For any $\epsilon \in \{-1, 1\}$ and $s, t \geq 0$ such that $s + t > 0$ there exists a finite positive

$$F^\epsilon(s,t) := \lim_{n\to\infty} n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\{\theta_1 > sn, \epsilon\psi_1 > tn^{1+\frac{1}{\alpha}}\}.$$

Proof. $S_1 \in \mathcal{R}_\alpha$ implies $T \in \mathcal{DN}(1 - 1/\alpha)$ and we denote

$$c_3 := \lim_{n\to\infty} n^{1-1/\alpha} \widetilde{\mathbb{P}}\{T > n\}, \quad c_4 := \lim_{n\to\infty} n^\alpha \mathbb{P}\{S_1 < -n\}.$$

We first claim that

$$\lim_{\varepsilon\to 0+} \overline{\lim_{n\to\infty}}\, n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\big\{\theta_1 > n, T < \varepsilon n\big\} = 0 \tag{22}$$

describing shape of long cycles.

Note that the weak limit of $\widetilde{\mathbb{P}}\big\{\frac{S_T}{T^{1/\alpha}} \in -dz \big| T \geq n\big\}$, which exists by (18), does not have an atom at $z = 0$. It actually has density, as follows from

$$\begin{aligned}
\lim_{n\to\infty} \widetilde{\mathbb{P}}\Big\{\frac{S_T}{T^{1/\alpha}} \leq -z \Big| T \geq n\Big\} &= \lim_{n\to\infty} \sum_{k=n}^{\infty} \int_0^\infty \widetilde{\mathbb{P}}\Big\{\frac{S_{k-1}}{k^{1/\alpha}} \in dx \Big| S_1 \geq 0, \dots, S_{k-1} \geq 0\Big\} \\
&\quad \times \frac{\widetilde{\mathbb{P}}\{T \geq k-1\} \mathbb{P}\{S_1 \leq -(x+z)k^{1/\alpha}\}}{\widetilde{\mathbb{P}}\{T \geq n\}} \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=n}^{\infty} \Big(\frac{k}{n}\Big)^{\frac{1}{\alpha}-2} \int_0^\infty p_+(x)(x+z)^{-\alpha} dx \\
&= \frac{\alpha c_4}{\alpha - 1} \mathbb{E}(z + S_+)^{-\alpha},
\end{aligned} \tag{23}$$

where we use the conditional local limit theorems from Vatutin and Wachtel [23] on convergence to the meander S_+ whose density $p_+(x)$ satisfies $p_+(x) \sim c'x$ as $x \to 0+$.

Let $\theta_+ := T - 1$ and $\theta_- := \theta_1 - \theta_+$ be durations of the first positive and first negative excursions of S_n, respectively. From (1) it follows that

$$\theta_- \stackrel{\mathcal{D}}{=} \xi + \theta_+ \text{ under } \widetilde{\mathbb{P}},$$

where $\xi := \theta_1 - \hat{\theta}_1$ is independent of θ_+ and distributed geometrically with parameter $\mathbb{P}\{S_1 \neq 0\}$ if S_1 is right-continuous and $\xi \equiv 0$ if S_1 is right-exponential. Therefore θ_+ and θ_- have the same tails, and applying (1) again,

$$\begin{aligned}\lim_{\varepsilon\to 0+}\overline{\lim_{n\to\infty}}\, n^{1-\frac{1}{\alpha}}\widetilde{\mathbb{P}}\big\{\theta_1>n, T<\varepsilon n\big\} &\le \lim_{\varepsilon\to 0+}\overline{\lim_{n\to\infty}}\, n^{1-\frac{1}{\alpha}}\widetilde{\mathbb{P}}\big\{\theta_->(1-\varepsilon)n, \theta_+<\varepsilon n\big\}\\ &= \lim_{\varepsilon\to 0+}\overline{\lim_{n\to\infty}}\, n^{1-\frac{1}{\alpha}}\widetilde{\mathbb{P}}\big\{T>(1-\varepsilon)n, S_T\le -(\varepsilon^{1/2}n)^{1/\alpha}, \theta_-<\varepsilon n\big\}\\ &= c_3\lim_{\varepsilon\to 0+}\overline{\lim_{n\to\infty}}\,\mathbb{P}\big\{T'((\varepsilon^{1/2}n)^{1/\alpha})<\varepsilon n\big\}\\ &= c_3\lim_{\varepsilon\to 0+}\mathbb{P}\big\{T''(\varepsilon^{1/(2\alpha)})<\varepsilon\big\}=0,\end{aligned}$$

where we denoted $T'(u):=\min\{k\ge 0: S_k>u\}$ and $T''(u):=\inf\{r\ge 0: S(r)>u\}$ and used that $T''(u)\overset{\mathcal{D}}{=}u^\alpha T''(1)$ by self-similarity of S. Thus (22) is proved.

We are now ready to prove the statements of Lemma 2. The Part 1 of course follows from Part 2. By the symmetry given in (1), we may consider only $\epsilon=1$. We start with $s\neq 0$ and by the obvious change of variables it suffices to consider $s=1$. Fix an $\varepsilon\in(0,1/2)$, then condition on the parameters of the first positive excursion and write

$$\begin{aligned}&\lim_{n\to\infty}\widetilde{\mathbb{P}}\big\{\theta_1>n, \psi_1>tn^{1+\frac{1}{\alpha}}\big| T\ge\varepsilon n\big\}\\ =\ &\lim_{n\to\infty}\int_1^\infty\int_0^\infty\int_0^\infty f_n^+\big((\varepsilon x)^{1/\alpha}z, 1-\varepsilon x, t-(\varepsilon x)^{1+1/\alpha}y\big)P_n^{(\varepsilon)}(dx,dy,dz),\end{aligned}\tag{24}$$

where

$$P_n^{(\varepsilon)}(dx,dy,dz):=\widetilde{\mathbb{P}}\Big\{\frac{T}{\varepsilon n}\in dx, \frac{A_T}{T^{1+\frac{1}{\alpha}}}\in dy, \frac{S_T}{T^{\frac{1}{\alpha}}}\in -dz\Big| T\ge\varepsilon n\Big\}$$

and

$$f_n^+(u,v,w):=\mathbb{P}\Big\{T'(un^{1/\alpha})-1>nv, \sum_{i=1}^{T'(un^{1/\alpha})-1}(S_i-un^{1/\alpha})>wn^{1+1/\alpha}\Big\}.$$

It clear that for any $u>0$ and v,w it holds that

$$\lim_{n\to\infty}f_n^+(u,v,w)=f^+(u,v,w):=\mathbb{P}\Big\{T''(u)\ge v, \int_0^{T''(u)}(S(r)-u)dr\ge w\Big\},$$

and our claim is that this convergence is uniform in $(u,v,w)\in[\delta,\infty)\times\mathbb{R}^2$ for any $\delta>0$. As $f_n^+(u,v,w)=f_{nu^\alpha}^+(1,u^{-\alpha}v,u^{-\alpha-1}w)$ and $f^+(u,v,w)=f^+(1,u^{-\alpha}v,u^{-\alpha-1}w)$ by self-similarity of S, we should check that the convergence is uniform in $(v,w)\in\mathbb{R}^2$ for $u=1$. This statement just a little improvement of the standard idea that the distribution functions of weakly convergent r.v.'s converge uniformly if the limit distribution is continuous. We prove the later if show that the r.v.'s $T''(1)$ and $\int_0^{T''(1)}(S(r)-u)dr$ are continuous. The first clearly is, say, since $T''(u)$ is a stable subordinator with index $1/\alpha$ as S does not have positive jumps. For the second, use that $\int_0^x(S(r)-u)dr$ and $S(x)$ are jointly stable and, consequently, have a joint density for any $x>0$.

Hence the integrands in (24) converge uniformly in $(x,y,z)\in[1,\infty)\times[0,\infty)\times[\delta,\infty)$. Further, (18) ensures $P_n^{(\varepsilon)}\overset{\mathcal{D}}{\to}P$ in $[1,\infty)\times\mathbb{R}_+^2$ for any fixed ε, where

$$P(dx,dy,dz):=d(-x^{\frac{1}{\alpha}-1})\mathbb{P}\Big\{\int_0^1 S_{ex}(s)ds\in dy, S_{ex}(1)\in -dz\Big\}.$$

As we seen above, $\mathbb{P}(S_{ex}(1)=0)=0$ so (22) and (24) imply

$$\begin{aligned} F^+(1,t) &= \lim_{n\to\infty} n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\big\{\theta_1 > n, \psi_1 > tn^{1+\frac{1}{\alpha}}\big\} \\ &= c_3 \lim_{\varepsilon\to 0+} \varepsilon^{\frac{1}{\alpha}-1} \int_1^\infty \int_0^\infty \int_0^\infty f^+\big((\varepsilon x)^{1/\alpha} z, 1-\varepsilon x, t-(\varepsilon x)^{1+1/\alpha} y\big) P(dx,dy,dz) \\ &= c_3 \iiint\limits_{\mathbb{R}^3_+} f^+\big(x^{1/\alpha} z, 1-x, t-x^{1+1/\alpha} y\big) P(dx,dy,dz). \end{aligned} \tag{25}$$

The last expression is finite as by (1) it does not exceed

$$\overline{\lim_{n\to\infty}}\, n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\big\{\theta_+ + \theta_+ > n\big\} \le 2 \lim_{n\to\infty} n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\big\{\theta_+ > n/2\big\} = 2^{2-\frac{1}{\alpha}} c_3.$$

It remains to consider $s=0$ and $t>0$, and it suffices to take $t=1$. Since

$$\lim_{\varepsilon\to 0+} \lim_{n\to\infty} n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\{T < \varepsilon n, \psi_1 > n^{1+\frac{1}{\alpha}}\} \le \lim_{\varepsilon\to 0+} \lim_{n\to\infty} n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\{T<\varepsilon n, A_T > n^{1+\frac{1}{\alpha}}\} = 0,$$

we argue as above in (24) and (25) to get

$$F^+(0,1) = c_3 \iiint\limits_{\mathbb{R}^3_+} f^+\big(x^{1/\alpha} z, 0, 1-x^{1+1/\alpha} y\big) P(dx,dy,dz).$$

With $\psi_1 \le A_T$ in mind, the last expression is finite as $A_T \in \mathcal{DN}(\frac{\alpha-1}{\alpha+1})$ by Corollary 2 and Example 6 in Doney [8].

□

3.3. **Areas of incomplete cycles.** The same technique allows to get the following result.

Lemma 3. *Let S_n be a random walk such that $S_1 \in \mathcal{R}_\alpha$ for some $1<\alpha<2$. Then for any $s,t>0$ there exists a finite positive*

$$F(s,t) := \lim_{n\to\infty} n^{1-\frac{1}{\alpha}} \widetilde{\mathbb{P}}\Big\{\theta_1 \ge sn, A_{sn} > -tn^{1+\frac{1}{\alpha}}\Big\},$$

and this convergence is uniform in $(s,t) \in [\varepsilon,\infty)\times[0,\infty)$ for any $\varepsilon>0$. Moreover, $F(s,t)$ is continuous on $\mathbb{R}^2_+$.

Proof. By the obvious change of variable, is suffices to consider $s=1$. Then literally repeat the proof of Lemma 2 to get

$$F(1,t) = c_3 \iiint\limits_{\mathbb{R}^3_+} f\big(x^{1/\alpha} z, 1-x, -t-x^{1+1/\alpha} y\big) P(dx,dy,dz)$$

with

$$f(u,v,w) := \mathbb{P}\Big\{T''(u) \ge v^+, \int_0^{v^+} (S(r)-u)dr \ge w\Big\}.$$

The continuity of $F(1,t)$ follows from the same of $f(u,v,w)$ and the theorem of majorated convergence. As $F(1,t)$ is bounded, continuous and monotone and the converging functions are uniformly bounded and monotone, the convergence is uniform in t.

□

Let us prove analogous result for $\alpha = 2$. The main difference is that (22) is no longer true.

Lemma 4. *Let S_n be a random walk such that $S_1 \in \mathcal{R}_2$. Then*

$$\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\Big\{\theta_1 \geq sn, A_{sn} > -tn^{3/2}\Big\} = F(s,t) = c_3 s^{-1/2}\big(1 + G(\sigma^{-1}ts^{-3/2})\big)$$

uniformly in $(s,t) \in [\varepsilon, \infty) \times [0,\infty)$ for any $\varepsilon > 0$, where $G(x)$ is a continuous function defined as $G(x) := \mathbb{P}\big\{\int_0^1 W_+(u)du < x\big\}$ and W_+ is a standard Brownian meander on $[0,1]$.

Proof. As is Lemma 3, it suffices to consider $s = 1$ and prove pointwise convergence for each $t \geq 0$. We start with

$$F(s,t) = \lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\big\{\theta_+ \geq n\big\} + \lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\Big\{\theta_+ < n, \theta_1 \geq n, A_n > -tn^{3/2}\Big\},$$

and our goal that to show that only the negative excursion contributes to the second term:

$$\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\Big\{\theta_+ < n, \theta_1 \geq n, A_n > -tn^{3/2}\Big\} = \lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\Big\{\theta_- \geq n, A_{n+\theta_+} - A_{\theta_+} > -tn^{3/2}\Big\}. \tag{26}$$

Let us first find the value of the right-hand side. By (1),

$$\begin{aligned}\widetilde{\mathbb{P}}\Big\{\theta_- \geq n, A_{n+\theta_+} - A_{\theta_+} > -tn^{3/2}\Big\} &= \widetilde{\mathbb{P}}\Big\{\xi + \theta_+ \geq n, A_{\xi+\theta_+} - A_{\xi+\theta_+-n} < tn^{3/2}\Big\} \\ &\sim \widetilde{\mathbb{P}}\Big\{\theta_+ \geq n, A_{\theta_+} - A_{\theta_+-n} < tn^{3/2}\Big\} \end{aligned} \tag{27}$$

as $n \to \infty$, reducing the problem to consideration of the first positive excursion of S_n. Then we apply (17) to get

$$\begin{aligned}\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\Big\{\theta_+ \geq n, A_{\theta_+} - A_{\theta_+-n} < tn^{3/2}\Big\} &= c_3\mathbb{P}\Big\{\sigma\delta_{W_+}^{3/2}\int_{1-\delta_{W_+}^{-1}}^1 W_{ex}(s)ds < t\Big\} \\ &= c_3\mathbb{P}\Big\{\int_0^1 \delta_{W_+}^{1/2} W_{ex}(s\delta_{W_+}^{-1})ds < \sigma^{-1}t\Big\} \\ &= c_3 G(\sigma^{-1}t), \end{aligned} \tag{28}$$

where we used $\delta_{W_+}^{1/2} W_{ex}(\cdot\delta_{W_+}^{-1}) \overset{\mathcal{D}}{=} W_+(\cdot)$, which follows from (14) and (17). Due to Janson [15], the area of Brownian meander has density so G is continuous.

It remains to prove (26). Fix a $\delta \in (0, 1/2)$ and write

$$\big\{\theta_+ < n, \theta_1 \geq n\big\} = D \cup E_1, \tag{29}$$

where $E_1 := \big\{(1-\delta)n \leq \theta_+ < n\big\} \cup \big\{(1-\delta)n \leq \theta_- < n\big\} \cup \big\{\theta_+ \geq \delta n, \theta_- \geq \delta n\big\}$ and $D := \big\{\theta_+ < \delta n, \theta_- \geq n\big\}$. Note that

$$\lim_{\delta\to 0+}\lim_{n\to\infty} n^{1/2}\big(\widetilde{\mathbb{P}}\{\theta_- \geq n\} - \widetilde{\mathbb{P}}(D)\big) = 0, \quad \lim_{\delta\to 0+}\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}(E_1) = 0$$

by continuity in s of $\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\big\{\theta_{+/-} \geq sn\big\}$ and the relation

$$\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}\big\{\theta_+ \geq n, \theta_- \geq n\big\} = 0,$$

see (17) in [25]. Thus D gives the main contribution in (29).

Further, from (29) we have

$$\{\theta_+ < n, \theta_1 \geq n, A_n > -tn^{3/2}\} \subset D \cap \{A_n - A_{\theta_+} > -tn^{3/2}\} \cup E_1 \cup E_2 \cup E_3, \tag{30}$$

where

$$E_2 := \{\theta_+ < \delta n, \theta_- \geq n, -(t+\delta)n^{3/2} \leq A_n - A_{\theta_+} \leq -tn^{3/2}\}, \quad E_3 := \{\theta_+ < \delta n, A_{\theta_+} > \delta n^{3/2}\}.$$

Here $\lim_{\delta\to 0+}\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}(E_3) = 0$ by Proposition 1 in [25] while for the same relation for E_2, write

$$E_2 \subset \Big\{\theta_- \geq n, A_{n+\theta_+} - A_{(1-\delta)n+\theta_+} - (t+\delta)n^{3/2} \leq A_{n+\theta_+} - A_{\theta_+} \leq -tn^{3/2}\Big\},$$

use the symmetry as in (27) and then argue as in (28) to get

$$\lim_{\delta\to 0+}\lim_{n\to\infty} n^{1/2}\widetilde{\mathbb{P}}(E_2) \leq c_3\mathbb{P}\Big\{\sigma^{-1}t < \int_0^1 W_+(s)ds < \int_{1-\delta}^1 W_+(s)ds + \sigma^{-1}(t+\delta)\Big\} = 0.$$

Finally, combine

$$\begin{aligned} & \{\theta_+ < \delta n, \theta_- \geq n, A_{n+\theta_+} - A_{\theta_+} > -tn^{3/2}\} \\ \subset\ & D \cap \{A_n - A_{\theta_+} > -tn^{3/2}\} \\ \subset\ & \{\theta_- \geq (1-\delta)n, A_{(1-\delta)n+\theta_+} - A_{\theta_+} > -tn^{3/2}\} \end{aligned}$$

with (28) and the continuity of G to get

$$\lim_{\delta\to 0+}\lim_{n\to\infty} n^{1/2}\Big(\widetilde{\mathbb{P}}\{\theta_- \geq n, A_{n+\theta_+} - A_{\theta_+} > -tn^{3/2}\} - \widetilde{\mathbb{P}}\big(D \cap \{A_n - A_{\theta_+} > -tn^{3/2}\}\big)\Big) = 0.$$

Together with (30) and the estimates above this concludes (26). ☐

4. Sharp asymptotics of p_N

We are ready to prove the main result. Condition on $\eta(N)$ in (8) to obtain

$$\begin{aligned} \frac{p_N}{\mathbb{P}\{S_1 > 0\}} &= \sum_{k=0}^{\infty} \widetilde{\mathbb{P}}\Big\{\eta(N) = k, \min_{1\leq i\leq k} \Psi_i > 0, A_N > 0\Big\} \\ &= \sum_{k=\varepsilon N^{1-1/\alpha}}^{\varepsilon^{-1}N^{1-1/\alpha}} \widetilde{\mathbb{P}}\Big\{\Theta_k \leq N, \theta_{k+1} > N - \Theta_k, \min_{1\leq i\leq k} \Psi_i > 0, A_N - A_{\Theta_k} > -\Psi_k\Big\} \\ &\quad + R_1(\varepsilon, N) + R_2(\varepsilon, N), \end{aligned}$$

where $\varepsilon \in (0, 1/2)$ is fixed while R_1 and R_2 corresponds to $\eta(N) < \varepsilon N^{1-1/\alpha}$ and $\eta(N) > \varepsilon^{-1}N^{1-1/\alpha}$, respectively. Let S'_n be an independent copy of S_n. A classical result of the renewal theory (Feller [11, Ch. XIV.3]) states that $\theta_{\eta(N)+1}$ has the order N, motivating us to write

$$\frac{p_N}{\mathbb{P}\{S_1>0\}}=\sum_{k=\varepsilon N^{1-1/\alpha}}^{\varepsilon^{-1}N^{1-1/\alpha}}\widetilde{\mathbb{P}}\Big\{\Theta_k\le(1-\varepsilon^2)N,\theta_{k+1}>N-\Theta_k,\min_{1\le i\le k}\Psi_i>0,A'_{N-\Theta_k}>-\Psi_k\Big\}+R(\varepsilon,N)$$

with $R(\varepsilon,N):=R_1+R_2+R_3$ and $R_3=R_3(\varepsilon,N)$ corresponding to $(1-\varepsilon^2)N<\Theta_k<N$.

For each k, condition on (Θ_k,Ψ_k) and rewrite the last formula as

$$\begin{aligned}&\frac{p_N}{\mathbb{P}\{S_1>0\}}\\ =\ &\sum_{k=\varepsilon N^{1-1/\alpha}}^{\varepsilon^{-1}N^{1-1/\alpha}}\widetilde{\mathbb{P}}\Big\{\min_{1\le i\le k}\Psi_i>0\Big\}\int_0^{(1-\varepsilon^2)N}\int_0^\infty\widetilde{\mathbb{P}}\Big\{\theta_{k+1}>N-x,A'_{N-x}>-y\Big\}\\ &\times\widetilde{\mathbb{P}}\Big\{\Theta_k\in dx,\Psi_k\in dy\Big|\min_{1\le i\le k}\Psi_i>0\Big\}+R(\varepsilon,N),\\ =\ &\frac{1}{N^{1-1/\alpha}}\sum_{k=\varepsilon N^{1-1/\alpha}}^{\varepsilon^{-1}N^{1-1/\alpha}}\widetilde{\mathbb{P}}\Big\{\min_{1\le i\le k}\Psi_i>0\Big\}\int_0^{\frac{(1-\varepsilon^2)N}{k^{\alpha/(\alpha-1)}}}\int_0^\infty F_N\Big(1-x\Big(\frac{k}{N^{1-1/\alpha}}\Big)^{\frac{\alpha}{\alpha-1}},y\Big(\frac{k}{N^{1-1/\alpha}}\Big)^{\frac{\alpha+1}{\alpha-1}}\Big)\\ &\times\widetilde{\mathbb{P}}\Big\{\frac{\Theta_k}{k^{\alpha/(\alpha-1)}}\in dx,\frac{\Psi_k}{k^{(\alpha+1)/(\alpha-1)}}\in dy\Big|\min_{1\le i\le k}\Psi_i>0\Big\}+R(\varepsilon,N)\end{aligned}$$

with

$$F_n(u,v):=n^{1-1/\alpha}\widetilde{\mathbb{P}}\Big\{\theta_1>un,A_{un}>-vn^{1+1/\alpha}\Big\}$$

corresponding to the last incomplete cycle. Let $Q_k(dy,dx)$ denote the conditional probability measure in the last integral (where we intentionally switched the coordinates). Thinking of the summation as of integration over the discretization of the Lebesgue measure λ, we introduce

$$U_n(dz):=n^{-1}\delta_0(\{zn\}),\quad P_n(dz,dy,dx):=Q_{zn^{1-1/\alpha}}(dy,dx)U_{n^{1-1/\alpha}}(dz)$$

and get

$$\begin{aligned}\frac{p_N}{\mathbb{P}\{S_1>0\}}\ =\ &\int_\varepsilon^{\frac{1}{\varepsilon}}\int_0^\infty\int_0^{\frac{1-\varepsilon^2}{z^{\alpha/(\alpha-1)}}}\widetilde{\mathbb{P}}\Big\{\min_{1\le i\le zN^{1-1/\alpha}}\Psi_i>0\Big\}F_N\big(1-xz^{\frac{\alpha}{\alpha-1}},yz^{\frac{\alpha+1}{\alpha-1}}\big)P_N(dz,dy,dx)\\ &+R(\varepsilon,N).\end{aligned}$$

By Lemmae 3 and 4,

$$\begin{aligned}\frac{p_N N^{\frac12-\frac{1}{2\alpha}}}{c_1\mathbb{P}\{S_1>0\}}\ =\ &\int_\varepsilon^{\frac{1}{\varepsilon}}\int_0^\infty\int_0^{\frac{1-\varepsilon^2}{z^{\alpha/(\alpha-1)}}}z^{-1/2}F\big(1-xz^{\frac{\alpha}{\alpha-1}},yz^{\frac{\alpha+1}{\alpha-1}}\big)P_N(dz,dy,dx)\\ &+o_\varepsilon(1)+R(\varepsilon,N)N^{\frac12-\frac{1}{2\alpha}}\end{aligned}\tag{31}$$

as $N\to\infty$. Further, Lemma 2 for $1<\alpha<2$ and Proposition 1 from [25] for $\alpha=2$ combined with (20) ensure

$$\Big(\frac{\Psi_n}{n^{(\alpha+1)/(\alpha-1)}},\frac{\Theta_n}{n^{\alpha/(\alpha-1)}}\Big)\xrightarrow{\mathcal{D}}\mathbf{S}(1)\text{ under }\widetilde{\mathbb{P}},$$

where $\mathbf{S}(1)$ is a bivariate stable r.v. (in the sense of Resnick and Greenwood [19]) with indices $\frac{\alpha-1}{\alpha+1}, \frac{\alpha-1}{\alpha}$. Thus (19) holds and by (21), we have $Q_n \xrightarrow{\mathcal{D}} \mathrm{Law}(\mathbf{S}_+(1))$ implying $P_N \xrightarrow{\mathcal{D}} \lambda|_{[\varepsilon,\varepsilon^{-1}]} \otimes \mathrm{Law}(\mathbf{S}_+(1))$ as we are concerned with $z \geq \varepsilon$. The integrand in (31) continuous a.s. with respect to the limit measure so

$$\lim_{N\to\infty} \frac{p_N N^{\frac{1}{2}-\frac{1}{2\alpha}}}{c_1\mathbb{P}\{S_1>0\}} = \int_0^\infty\int_0^\infty\int_0^{z^{-\frac{\alpha}{\alpha-1}}} z^{-1/2}F\big(1-xz^{\frac{\alpha}{\alpha-1}}, yz^{\frac{\alpha+1}{\alpha-1}}\big)dz\mathbb{P}\{\mathbf{S}_+(1)\in(dy,dx)\}$$

if we check that

$$\lim_{\varepsilon\to0+}\varlimsup_{N\to\infty} R(\varepsilon,N)N^{\frac{1}{2}-\frac{1}{2\alpha}} = 0. \tag{32}$$

We simply the formula for the constant using $F\big(1-xz^{\frac{\alpha}{\alpha-1}}, yz^{\frac{\alpha+1}{\alpha-1}}\big) = z^{-1}F\big(z^{-\frac{\alpha}{\alpha-1}}-x, y\big)$ and making the change in the integral:

$$\lim_{N\to\infty} \frac{p_N N^{\frac{1}{2}-\frac{1}{2\alpha}}}{2c_1\mathbb{P}\{S_1>0\}} = \int_0^\infty\int_0^\infty\int_0^u F(u-x,y)d(u^{\frac{\alpha-1}{2\alpha}})\mathbb{P}\{\mathbf{S}_+(1)\in(dy,dx)\}. \tag{33}$$

The right-hand side is finite by Theorem 3. Of course this can be checked directly using $F(u-x,y)\leq c_3(u-x)^{\frac{1}{\alpha}-1}$ and the observation that $\mathbf{S}_+^{(2)}(1) \overset{\mathcal{D}}{=} \mathbf{S}^{(2)}(1)$, which follows from Proposition 2.

It remains to check (32) to show that the contribution of $R = R_1+R_2+R_3$ is negligible. Combine (12) with (11) in the "continuous" case and the analogous inequality in the "discrete" case to get

$$\begin{aligned} R_1(\varepsilon,N)+R_2(\varepsilon,N) &\leq \widetilde{\mathbb{P}}\Big\{\min_{1\leq k\leq\eta(N)}\Psi_k>0, \frac{\eta(N)}{N^{1-1/\alpha}}\notin[\varepsilon,\varepsilon^{-1}]\Big\} \\ &\leq \frac{\bar{c}_1+o(1)}{N^{\frac{1}{2}-\frac{1}{2\alpha}}}\widetilde{\mathbb{E}}\,\mathbb{1}_{[\varepsilon,\varepsilon^{-1}]^c}\Bigg(\sqrt{\frac{N^{1-1/\alpha}}{\eta(N)}}\Bigg)\sqrt{\frac{N^{1-1/\alpha}}{\eta(N)+1}}+o(N^{-1}). \end{aligned}$$

Then the required estimate for R_1+R_2 follows from (13) and the uniform integrability of $\sqrt{\frac{N^{1-1/\alpha}}{\eta(N)+1}}$, which we checked when proved Theorem 3.

For the last term we proceed as above to obtain

$$\begin{aligned} R_3(\varepsilon,N) &\leq \widetilde{\mathbb{P}}\Big\{\min_{1\leq k\leq\eta(N)}\Psi_k>0, \varepsilon\leq\frac{\eta(N)}{N^{1-1/\alpha}}\leq\varepsilon^{-1}, 1-\varepsilon^2\leq\frac{\Theta_{\eta(N)}}{N}\leq1\Big\} \\ &\leq \sum_{k=\varepsilon N^{1-1/\alpha}}^{\varepsilon^{-1}N^{1-1/\alpha}}\widetilde{\mathbb{P}}\Big\{\min_{1\leq i\leq k}\Psi_i\geq0\Big\}\widetilde{\mathbb{P}}\Big\{(1-\varepsilon^2)N\leq\Theta_k\leq N, \theta_{k+1}>N-\Theta_k\Big\} \\ &\leq \frac{\bar{c}_1+o(1)}{\varepsilon^{\frac{1}{2}}N^{\frac{1}{2}-\frac{1}{2\alpha}}}\widetilde{\mathbb{P}}\Big\{\frac{N-\Theta_{\eta(N)}}{N}\leq\varepsilon^2\Big\}, \end{aligned}$$

and by Feller [11, Ch. XIV.3], the last probability converges to

$$\int_0^{\varepsilon^2}\frac{\sin(\pi(1-1/\alpha))dx}{\pi x^{1-1/\alpha}(1-x)^{1/\alpha}} < \varepsilon^{\frac{2}{\alpha}}.$$

Combine the estimates above to conclude (32) and the proof of the theorem.

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