CONTACT STRUCTURES ON PRINCIPAL CIRCLE BUNDLES

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ABSTRACT. We describe a necessary and sufficient condition for a principal circle bundle over an even-dimensional manifold to carry an invariant contact structure. As a corollary it is shown that all circle bundles over a given base manifold carry an invariant contact structure, only provided the trivial bundle does. In particular, all circle bundles over 4-manifolds admit invariant contact structures.

1. Introduction

The study of invariant contact structures on principal S^1-bundles can be traced back to the work of Boothby and Wang [1], cf. [5, Section 7.2]. They observed that if the Euler class of the bundle can be represented by a symplectic form on the base, a suitable connection 1-form will be an invariant contact form. Their main result was that any contact form whose Reeb vector field is regular essentially arises in this way. These contact structures are transverse to the S^1-fibres.

A general classification of invariant contact structures on S^1-bundles over surfaces was obtained by Lutz [8]. In the present paper we extend his results to higher dimensions. We derive a necessary (Proposition 3) and sufficient (Theorem 4) condition for an S^1-bundle to admit an invariant contact structure. Our explicit way of building an invariant contact structure from certain data on the base manifold is inspired by the work of Giroux [7] on convex hypersurfaces (corresponding to $\mathbb{R}$-invariant contact structures), and it extends a construction by Stipsicz and the second author [6] from trivial to nontrivial bundles. In fact, as a consequence of our existence criterion we can show that if the trivial S^1-bundle over a given base manifold admits an invariant contact structure, then the same is true for all nontrivial S^1-bundles (Corollary 7). Combining this with the main result from [6] we conclude that all S^1-bundles over 4-manifolds carry invariant contact structures.

2. Conventions

This section merely serves to fix our normalisation conventions regarding principal connections. Let $\pi\colon M \to B$ be a principal S^1-bundle. Write ∂_θ for the vector field on M generating the S^1-action. By a connection 1-form ψ we mean an S^1-invariant form on M, i.e. $L_{\partial_\theta}\psi \equiv 0$, normalised by $\psi(\partial_\theta) \equiv 1$. Up to a factor 2π this ψ is what Bott–Tu [2] call the global angular form.

The 2-form $\mathrm{d}\psi$ is then S^1-invariant and horizontal, where the latter means that $i_{\partial_\theta}\mathrm{d}\psi \equiv 0$. It follows that $\mathrm{d}\psi$ induces a closed 2-form ω on B, that is, $\mathrm{d}\psi = \pi^*\omega$.

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This 2-form ω is called the curvature form of the connection ψ. The Euler class of the S^1-bundle is given by $e = -[\omega/2\pi] \in H^2_{\mathrm{dR}}(B)$, where H^*_{dR} denotes de Rham cohomology.

Given a further closed 2-form ω' on B with $[\omega'] = [\omega] \in H^2_{\mathrm{dR}}(B)$, one can find a connection 1-form ψ' with $\mathrm{d}\psi' = \pi^*\omega'$. Indeed, we have $\omega' = \omega + \mathrm{d}\gamma$ for some 1-form γ on B, and we may then set $\psi' = \psi + \pi^*\gamma$. The difference between two connection 1-forms with the same curvature form is a closed horizontal 1-form.

By slight abuse of notation we shall not usually distinguish between an S^1-invariant and horizontal differential form on M and the induced form on B.

3. Symplectic decompositions

Assume now that B is a closed, connected, oriented $2n$-manifold and $\pi\colon\ M \to B$ a principal S^1-bundle as before, with the corresponding orientation on M. Suppose that M admits an S^1-invariant cooriented contact structure $\xi = \ker\alpha$ such that $\alpha \wedge (d\alpha)^n$ is a (positive) volume form for this orientation of M. The contact form α may likewise be taken to be S^1-invariant, for we can always pass to the averaged contact form

$$\int_{\theta\in S^1} \theta^*\alpha.$$

Then $u := \alpha(\partial_\theta)$ defines a smooth S^1-invariant function on M. With ψ a connection 1-form on M, define a 1-form β on M by

$$\alpha = \beta + u\psi.$$

This β is S^1-invariant and horizontal. Thus, both the function u and the 1-form β descend to B.

Write ω for the curvature form of ψ as in the previous section.

Lemma 1. *The $2n$-form*

$$\Omega := (\mathrm{d}\beta + u\omega)^{n-1} \wedge [n\beta \wedge \mathrm{d}u + u(\mathrm{d}\beta + u\omega)]$$

is a volume form on B.

Proof. A straightforward computation gives $\alpha \wedge (\mathrm{d}\alpha)^n = \psi \wedge \Omega$. □

The terminology in the following definition is chosen because of the obvious analogy with the theory of $\mathbb{R}$-invariant contact structures near convex hypersurfaces in the sense of Giroux [7].

Definition. The **dividing set** of B induced by the contact structure ξ is the set

$$\Gamma := \{p \in B\colon\ u(p) = 0\}.$$

We write

$$B_\pm := \{p \in B\colon\ \pm u(p) \geq 0\},$$

so that $B \setminus \Gamma = \mathrm{Int}(B_+) \sqcup \mathrm{Int}(B_-)$.

Lemma 2. *The dividing set Γ is a (possibly empty) closed codimension 1 submanifold of B. The 1-form $\beta_0 := \beta|_{T\Gamma}$ is a contact form on Γ, inducing the orientation of Γ as the boundary of B_+.*

Proof. From the preceding lemma it follows that

$$-\mathrm{d}u \wedge \beta \wedge (\mathrm{d}\beta)^{n-1} > 0$$

along Γ, and $-\mathrm{d}u$ evaluates positively on vectors pointing out of B_+. □

We now want to show that there are symplectic forms on $B_\pm$ representing the Euler class of the bundle and compatible with the contact structure $\ker\beta_0$ on the boundary in the sense of the following definition.

Definition. Let (W,ω) be a compact symplectic manifold of dimension $2n$, oriented by the symplectic form ω, and $\eta=\ker\beta_0$ a cooriented (and hence oriented) contact structure on ∂W inducing the boundary orientation. We say that (W,ω) is a **weak filling** of $(\partial W,\eta)$ if

$$(\mathrm{w}_1)\qquad (\mathrm{d}\beta_0)^k\wedge\omega^{n-1-k}|_\eta>0\ \text{ for all }\ k=0,\ldots,n-1.$$

Observe that this condition does not depend on the choice of contact form for η (among forms inducing the given coorientation). This definition of a weak filling is essentially the one proposed by Massot et al. [9] (in contrast with earlier definitions, where only $\omega^{n-1}|_\eta>0$ had been required). In fact, they demand

$$(\mathrm{w}_2)\qquad (f\,\mathrm{d}\beta_0+\omega)^{n-1}|_\eta>0\ \text{ for all smooth functions }\ f\colon\ \partial W\to\mathbb{R}_0^+,$$

which on the face of it is weaker than (w_1). Lemma 5 below remains true with this weaker requirement, which implies that by a modification in a collar one can pass from (w_2) to (w_1). So for our purposes (w_1) and (w_2) can be used interchangeably.

Proposition 3. *Given a principal S^1-bundle $\pi\colon\ M\to B$ of Euler class e with an S^1-invariant contact structure ξ, then with notation as above the following holds. There are symplectic forms $\omega_\pm$ on $\pm B_\pm$, where $-B_-$ denotes B_- with reversed orientation, such that*

(i) $\mp[\omega_\pm/2\pi]=e|_{B_\pm}$;
(ii) *if Γ is non-empty, then $(\pm B_\pm,\omega_\pm)$ are weak fillings of* $(\Gamma,\ker\beta_0)$.

Proof. On $B\setminus\Gamma$ one can write the volume form Ω from Lemma 1 as

$$\Omega=u^{n+1}\big(\mathrm{d}(\beta/u)+\omega\big)^n,$$

so

$$\omega_\pm:=\pm\big(\mathrm{d}(\beta/u)+\omega\big)|_{\mathrm{Int}(B_\pm)}$$

are symplectic forms on $\mathrm{Int}(B_\pm)$, respectively, inducing the positive orientation on $\mathrm{Int}(B_+)$ and the negative orientation on $\mathrm{Int}(B_-)$. Moreover, we have

$$\mp[\omega_\pm/2\pi]=-[\omega/2\pi]|_{\mathrm{Int}(B_\pm)}=e|_{\mathrm{Int}(B_\pm)}.$$

If $\Gamma\neq\emptyset$, we may choose $\varepsilon>0$ so small that

$$B_\pm^\varepsilon:=\{p\in B:\ \pm u(p)\geq\varepsilon\}$$

is an isotopic copy of $B_\pm$ in B, and such that for each $s\in[-\varepsilon,\varepsilon]$ the set

$$\Gamma_s:=\{p\in B:\ u(p)=s\}$$

is an isotopic copy of Γ in B with $\beta_s:=\beta|_{T\Gamma_s}$ a contact form. In other words, $B_\pm^\varepsilon$ is obtained from $B_\pm$ by shrinking a collar neighbourhood of its boundary. By Gray stability [5, Theorem 2.2.2], the contact manifolds $(\Gamma,\ker\beta_0)$ and $(\Gamma_{\pm\varepsilon},\ker\beta_{\pm\varepsilon})$ are diffeomorphic.

On Γ_ε we have for all $k=0,\ldots,n-1$, possibly after choosing a smaller $\varepsilon>0$,

$$\beta_\varepsilon\wedge(\mathrm{d}\beta_\varepsilon)^k\wedge\omega_+^{n-1-k}|_{T\Gamma_\varepsilon}=\beta\wedge(\mathrm{d}\beta)^k\wedge(\mathrm{d}\beta/\varepsilon+\omega)^{n-1-k}|_{T\Gamma_\varepsilon}>0,$$

so $(B_+^\varepsilon,\omega_+|_{B_+^\varepsilon})$ is a weak filling of $(\Gamma_\varepsilon,\ker\beta_\varepsilon)\cong(\Gamma,\ker\beta_0)$. Condition (i) holds under the obvious diffeomorphism between B_+^ε and B_+.

Similarly, on $\Gamma_{-\varepsilon}$ we have

$$\beta_{-\varepsilon}\wedge(\mathrm{d}\beta_{-\varepsilon})^k\wedge\omega_-^{n-1-k}|_{T\Gamma_{-\varepsilon}}=\beta\wedge(\mathrm{d}\beta)^k\wedge(\mathrm{d}\beta/\varepsilon-\omega)^{n-1-k}|_{T\Gamma_{-\varepsilon}}>0,$$

so $(-B_-^{\varepsilon},\omega_-|_{B_-^{\varepsilon}})$ is a weak filling of $(\Gamma_{-\varepsilon},\ker\beta_{-\varepsilon})\cong(\Gamma,\ker\beta_0)$. □

4. Constructing an invariant contact structure

We are now going to show that the conditions listed in Proposition 3 are in fact also sufficient for the existence of an S^1-invariant contact structure on M.

Theorem 4. *Let $\pi\colon M\to B$ be a principal S^1-bundle of Euler class e over a closed, connected, oriented manifold B of dimension $2n$. Suppose that B admits a splitting $B=B_+\cup_\Gamma B_-$ along a (possibly empty) codimension 1 submanifold Γ such that there are symplectic forms $\omega_\pm$ on $\pm B_\pm$ and a cooriented contact structure $\ker\beta$ on Γ satisfying conditions* (i) *and* (ii) *of Proposition 3. Then M admits an S^1-invariant contact structure with dividing set Γ.*

Proof of Theorem 4 for $\Gamma=\emptyset$. If ω_+ is a symplectic form on B with $-[\omega_+/2\pi]=e$, take α to be a connection 1-form ψ with curvature form ω_+. Then we have $\alpha\wedge(\mathrm{d}\alpha)^n=\psi\wedge\pi^*\omega_+^n>0$. If $-B$ admits a symplectic form ω_- with $[\omega_-/2\pi]=e$, set $\alpha=-\psi$, where ψ is a connection 1-form with curvature form $-\omega_-$. Then $\alpha\wedge(\mathrm{d}\alpha)^n=-\psi\wedge\pi^*\omega_-^n>0$. □

From now on it will be assumed that B decomposes as $B=B_+\cup_\Gamma B_-$ with $\Gamma\neq\emptyset$. We begin by considering B_+ and B_- separately. Our first aim is to modify $\omega_\pm$ in a neighbourhood of the boundary such that the new symplectic manifolds resemble strong fillings of $(\Gamma,\ker\beta)$. The next lemma mildly generalises an idea of Eliashberg [3], cf. [4].

Lemma 5. *The symplectic forms $\omega_\pm$ can be modified in a collar neighbourhood of $\Gamma=\partial(\pm B_\pm)$ in $\pm B_\pm$ such that in a smaller collar neighbourhood $(-\varepsilon,0]\times\Gamma$ we can write*

$$\omega_\pm=\pm\omega_\pm^\Gamma+\mathrm{d}(\mathrm{e}^s\beta),$$

possibly after replacing β by $K\beta$ for some large $K\in\mathbb{R}^+$. Here $\omega_\pm^\Gamma$ are 2-forms on Γ, pulled back to $(-\varepsilon,0]\times\Gamma$ under the projection map to Γ.

Proof. For ease of notation we first consider (B_+,ω_+). Consider a tubular neighbourhood $[0,1]\times\Gamma$ of the boundary, where $\{1\}\times\Gamma\equiv\Gamma=\partial B_+$. Define the 2-form ω_+^Γ on $[0,1]\times\Gamma$ by first restricting ω_+ to $T(\{0\}\times\Gamma)$ (i.e. pulling back under the inclusion $\{0\}\times\Gamma\subset[0,1]\times\Gamma$) and then pulling back again to $[0,1]\times\Gamma$. Then both forms ω_+ and ω_+^Γ represent the cohomology class $-2\pi e|_{[0,1]\times\Gamma}$, so there is a 1-form γ on $[0,1]\times\Gamma$ such that

$$\omega_+=\omega_+^\Gamma+\mathrm{d}\gamma.$$

We continue to write β for the 1-form on $[0,1]\times\Gamma$ obtained by pulling back the original β from $\Gamma=\{1\}\times\Gamma$. Since (B_+,ω_+) is a weak filling of $(\Gamma,\ker\beta)$, we may assume that the collar $[0,1]\times\Gamma$ had been chosen so small that for all $k=0,\ldots,n-1$, $t\in[0,1]$ and $c(t)\in[0,1]$ we have

$$\beta\wedge(\mathrm{d}\beta)^k\wedge(\omega_+^\Gamma+c(t)\,\mathrm{d}\gamma)^{n-1-k}>0\ \text{ on }\ T(\{t\}\times\Gamma).$$

Now set

$$\tilde\omega_+=\omega_+^\Gamma+\mathrm{d}(c\gamma)+\mathrm{d}(b\beta)$$

on $[0,1]\times\Gamma$, where the smooth functions $b(t)$ and $c(t)$, $t\in[0,1]$, are chosen as follows: Fix a small $\varepsilon>0$. Choose $b\colon [0,1]\to\mathbb{R}_0^+$ monotonically increasing, identically 0 near $t=0$ and with $b'(t)>0$ for $t>\varepsilon/2$. Choose $c\colon [0,1]\to[0,1]$ identically 1 on $[0,\varepsilon]$ and identically 0 near $t=1$.

We compute

$$\begin{aligned}\tilde{\omega}_+^n &= nc'\,\mathrm{d}t\wedge\gamma\wedge\sum_{k=0}^{n-1}\binom{n-1}{k}(b\,\mathrm{d}\beta)^k(\omega_+^\Gamma+c\,\mathrm{d}\gamma)^{n-1-k}\\ &\quad + nb'\,\mathrm{d}t\wedge\beta\wedge\sum_{k=0}^{n-1}\binom{n-1}{k}(b\,\mathrm{d}\beta)^k(\omega_+^\Gamma+c\,\mathrm{d}\gamma)^{n-1-k}\\ &\quad + \sum_{k=0}^{n-1}\binom{n}{k}(b\,\mathrm{d}\beta)^k(\omega_+^\Gamma+c\,\mathrm{d}\gamma)^{n-k}.\end{aligned}$$

The terms in the second line are volume forms on $[0,1]\times\Gamma$ up to a non-negative factor $b'b^k$, so is the restriction to $[0,\varepsilon]\times\Gamma$ of the term in the last line with $k=0$. By choosing b small on $[0,\varepsilon]$ and b' large compared with $\max\{1,|c'|\}$ on $[\varepsilon,1]$, one can ensure that these positive terms dominate the remaining terms over which we have no control. Then $\tilde{\omega}_+$ is a symplectic form on $[0,1]\times\Gamma$ and, in terms of the coordinate $s:=\log b(t)-\log b(1)$, this symplectic form looks like $\omega_+^\Gamma+\mathrm{d}(\mathrm{e}^s b(1)\beta)$ near $\{1\}\times\Gamma$.

For $(-B_-,\omega_-)$ the argument is completely analogous, except that we take ω_-^Γ to be the restriction of $-\omega_-$ to $T(\{0\}\times\Gamma)$. The value $b(1)$ may be chosen the same for both ω_+ and ω_-. □

Remark. Our choice of sign in the preceding lemma implies that when we regard the 2-forms $\omega_\pm^\Gamma$ as forms on Γ, we have $-[\omega_\pm^\Gamma/2\pi]=e|_\Gamma$, so both forms $\omega_\pm^\Gamma$ are curvature forms for the restriction of the S^1-bundle to Γ.

The following is a generalisation of the argument used for proving [6, Theorem 1].

Proof of Theorem 4 for $\Gamma\neq\emptyset$. By Lemma 5 we find a collar neighbourhood

$$(-1-\varepsilon,-1]\times\Gamma$$

of $\{-1\}\times\Gamma\equiv\Gamma=\partial(B_+)$ in B_+ where

$$\omega_+=\omega_+^\Gamma+\mathrm{d}(\mathrm{e}^{t+1}\beta);$$

the shift in the collar parameter is made for notational convenience below. Likewise, we have a collar neighbourhood

$$[1,1+\varepsilon)\times\Gamma$$

of $\{1\}\times\Gamma\equiv\Gamma=\partial(-B_-)$ in B_- where

$$\omega_-=-\omega_-^\Gamma+\mathrm{d}(\mathrm{e}^{-t+1}\beta);$$

Write the base B of the S^1-bundle as

$$B_+\cup_\Gamma([-1,1]\times\Gamma)\cup_\Gamma B_-.$$

Let $\psi_\pm$ be connection 1-forms of the restriction of the S^1-bundle to $B_\pm$ with curvature forms $\pm\omega_\pm$. Let $\psi_\pm^\Gamma$ be connection 1-forms of the S^1-bundle over Γ with curvature form $\omega_\pm^\Gamma$.

Lemma 6. *These choices can be made in such a way that over the two collars* $(-1-\varepsilon,-1]\times\Gamma$ *and* $[1,1+\varepsilon)\times\Gamma$ *we have*

$$\psi_\pm = \psi_\pm^\Gamma \pm \mathrm{e}^{\pm t+1}\beta,$$

respectively, perhaps at the cost of taking a slightly smaller $\varepsilon>0$.

Proof. We only deal with ψ_+; the argument for ψ_- is completely analogous. Over $(-1-\varepsilon,-1]\times\Gamma$ the connection forms ψ_+ and $\psi_+^\Gamma+\mathrm{e}^{t+1}\beta$ have the same curvature form ω_+. It follows that

$$\psi_+ = \psi_+^\Gamma + \mathrm{e}^{t+1}\beta + \gamma$$

with γ a closed horizontal 1-form. Choose a closed 1-form γ^Γ on Γ, which we also interpret as a 1-form on $(-1-\varepsilon,-1]\times\Gamma$, representing the same class as γ in $H^1_{\mathrm{dR}}((-1-\varepsilon,-1]\times\Gamma)$. Then we can write

$$\gamma = \gamma^\Gamma + \mathrm{d}h$$

for some smooth function on $(-1-\varepsilon,-1]\times\Gamma$. Replace ψ_+^Γ by $\psi_+^\Gamma+\gamma^\Gamma$, and ψ_+ by $\psi_+-\mathrm{d}(\chi h)$, where $\chi\colon\,(-1-\varepsilon,-1]\to[0,1]$ interpolates smoothly between 0 near $-1-\varepsilon$ and 1 near -1. Then the new ψ_+ still extends as before over B_+, and near $\{-1\}\times\Gamma$ we have the equality claimed in the lemma. □

We continue with the proof of Theorem 4. Let ψ_t^Γ, $t\in[-1,1]$, be a smooth family of connection 1-forms on the S^1-bundle over Γ with $\psi_t^\Gamma=\psi_\pm^\Gamma$ for t near ∓ 1.

Now choose two smooth functions f and g on the interval $(-1-\varepsilon,1+\varepsilon)$ subject to the following conditions (see Figure 1):

- f is an even and nowhere zero function with $f(t)=\mathrm{e}^{t+1}$ near $(-1-\varepsilon,-1]$,
- g is an odd function with $g(t)=1$ near $(-1-\varepsilon,-1]$ and a single zero at 0,
- $f'g-fg'>0$,
- $f\gg 1$ and $f'g-fg'\gg 1$ where $g'\neq 0$.

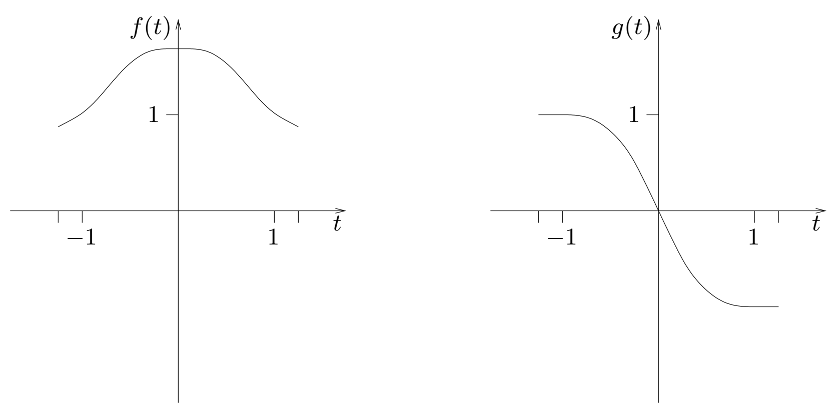

FIGURE 1. The functions f and g.

Define a smooth S^1-invariant 1-form α on the S^1-bundle over B by

$$\alpha = \begin{cases} \psi_+ & \text{over } B_+,\\ f\beta + g\psi_t^\Gamma & \text{over } [-1,1]\times\Gamma,\\ -\psi_- & \text{over } B_-.\end{cases}$$

Over $B_\pm$ this defines a contact form by the same computation as in the proof for the case $\Gamma = \emptyset$. Over $[-1,1]\times\Gamma$ we compute

$$\begin{aligned}\alpha\wedge(\mathrm{d}\alpha)^n &= (f\beta+g\psi_t^\Gamma)\wedge n\,(f\,\mathrm{d}\beta+g\,\mathrm{d}\psi_t^\Gamma)^{n-1}\wedge\mathrm{d}t\\ &\quad\wedge\big(f'\beta+g'\psi_t^\Gamma+g(\partial\psi_t^\Gamma/\partial t)\big)\\ &= n\psi_t^\Gamma\wedge\mathrm{d}t\wedge\big((f'g-fg')\beta+g^2(\partial\psi_t^\Gamma/\partial t)\big)\\ &\quad\wedge(f\,\mathrm{d}\beta+g\,\mathrm{d}\psi_t^\Gamma)^{n-1}.\end{aligned}$$

Notice that $\partial\psi_t^\Gamma/\partial t$ is a horizontal 1-form, so the term where we wedge this with β rather than ψ_t^Γ from the first factor yields a horizontal $(2n+1)$-form, i.e. zero.

Near $t=-1$ we have $g\equiv 1$ and $\psi_t^\Gamma\equiv\psi_+^\Gamma$, hence $\mathrm{d}\psi_t^\Gamma\equiv\omega_+^\Gamma$. With condition (ii) from Proposition 3 this implies

$$\alpha\wedge(\mathrm{d}\alpha)^n = nf'\psi_+^\Gamma\wedge\mathrm{d}t\wedge\beta\wedge(f\,\mathrm{d}\beta+\omega_+^\Gamma)^{n-1}>0.$$

Near $t=1$ we have $g\equiv -1$ and $\psi_t^\Gamma\equiv\psi_-^\Gamma$, hence $d\psi_t^\Gamma\equiv\omega_-^\Gamma$. Recall that ω_-^Γ was defined as the restriction of $-\omega_-$, so we get

$$\alpha\wedge(\mathrm{d}\alpha)^n = -nf'\psi_-^\Gamma\wedge\mathrm{d}t\wedge\beta\wedge(f\,\mathrm{d}\beta-\omega_-^\Gamma)^{n-1}>0.$$

Finally, over the region where $g'\neq 0$, we have $f\gg 1$ and $f'g-fg'\gg 1$. It follows that the positive summand

$$nf^{n-1}(f'g-fg')\psi_t^\Gamma\wedge\mathrm{d}t\wedge\beta\wedge(\mathrm{d}\beta)^{n-1}$$

in the expression for $\alpha\wedge(\mathrm{d}\alpha)^n$ will dominate all other summands.

The dividing set of the contact structure $\ker\alpha$ coincides with the zero set $\{0\}\times\Gamma$ of g. This completes the proof of the theorem. □

5. Examples

Here is a simple corollary of our main theorem.

Corollary 7. *Let B be a closed, oriented manifold of dimension $2n$. If the trivial S^1-bundle over B admits an S^1-invariant contact structure with dividing set Γ, then so do all S^1-bundles over B.*

Proof. If the trivial S^1-bundle over B admits an S^1-invariant contact structure with dividing set Γ, then by Proposition 3 the base B has a splitting $B=B_+\cup_\Gamma B_-$ with a contact structure $\ker\beta$ on Γ and exact symplectic forms $\mathrm{d}\lambda_\pm$ on $\pm B_\pm$ such that $(\pm B_\pm,\mathrm{d}\lambda_\pm)$ are weak fillings of $(\Gamma,\ker\beta)$.

Given the S^1-bundle over B of Euler class $e\in H^2_{\mathrm{dR}}(B)$, choose 2-forms $\sigma_\pm$ on $B_\pm$ with $\mp[\sigma_\pm/2\pi]=e|_{B_\pm}$. For $K\in\mathbb{R}^+$ sufficiently large, the 2-forms $\omega_\pm:=\sigma_\pm+K\,\mathrm{d}\lambda_\pm$ are symplectic forms satisfying conditions (i) and (ii) of Proposition 3. Then the result follows from Theorem 4. □

In [6, Corollary 2] it was shown that the trivial S^1-bundle over any closed, oriented 4-manifold admits an S^1-invariant contact structure. So the next corollary is immediate.

Corollary 8. *Any S^1-bundle over any closed, oriented 4-manifold admits an S^1-invariant contact structure.* □

In [6] it was also shown that $\mathbb{C}\mathrm{P}^2 \times S^1$ admits a contact structure in every homotopy class of almost contact structures (i.e. reduction of the structure group to $\mathrm{U}(2) \times 1$). The same argument as in the proof of Corollary 7 allows us to extend this to nontrivial bundles: on any given S^1-bundle over $\mathbb{C}\mathrm{P}^2$, any homotopy class of S^1-invariant almost contact structures contains a contact structure.

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