The hierarchy of ω_1-Borel sets

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Abstract

We consider the ω_1-Borel subsets of the reals in models of ZFC. This is the smallest family of sets containing the open subsets of the 2^ω and closed under ω_1 intersections and ω_1 unions. We show that Martin's Axiom implies that the hierarchy of ω_1-Borel sets has length ω_2. We prove that in the Cohen real model the length of this hierarchy is at least ω_1 but no more than $\omega_1 + 1$.

Some authors have considered ω_1-Borel sets in other spaces, $\omega_1^{\omega_1}$ Mekler and Vaananen [10] and or completely metrizable spaces of uncountable density, Willmott [21]. But in this paper we only consider the space 2^ω.

Define the levels of the ω_1-Borel hierarchy of subsets of 2^ω as follows:

1. $\mathbf{\Sigma}^*_0 = \mathbf{\Pi}^*_0 =$ clopen subsets of 2^ω
2. $\mathbf{\Sigma}^*_\alpha = \{\bigcup_{\beta<\omega_1} A_\beta \ : \ (A_\beta : \beta < \omega_1) \in (\mathbf{\Pi}^*_{<\alpha})^{\omega_1}\}$
3. $\mathbf{\Pi}^*_\alpha = \{\bigcup_{\beta<\omega_1} A_\beta \ : \ (A_\beta : \beta < \omega_1) \in (\mathbf{\Sigma}^*_{<\alpha})^{\omega_1}\}$
4. $\mathbf{\Pi}^*_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Pi}^*_\beta \qquad \mathbf{\Sigma}^*_{<\alpha} = \bigcup_{\beta<\alpha} \mathbf{\Sigma}^*_\beta$

The length of this hierarchy is the smallest $\alpha \geq 1$ such that

$$\mathbf{\Pi}^*_\alpha = \mathbf{\Sigma}^*_\alpha.$$

It is easy to show that if $\alpha < \omega_2$ and every ω_1-Borel set is $\mathbf{\Pi}^*_{<\alpha}$, then $\mathbf{\Pi}^*_\beta = \mathbf{\Sigma}^*_\beta$ for some $\beta < \alpha$, i.e., bounded hierarchies must have a top class (see Miller [11] Proposition 4 p.235).

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The classes $\mathbf{\Pi}^*_1$ and $\mathbf{\Sigma}^*_1$ are the ordinary closed sets and open sets, respectively, so the length of the hierarchy of ω_1-Borel sets is at least 2.

Assuming the continuum hypothesis, $\mathbf{\Pi}^*_2 = \mathbf{\Sigma}^*_2 = \mathcal{P}(2^\omega)$, so CH implies the order of the hierarchy is 2. It also known to be consistent that

$$\mathbf{\Pi}^*_3 = \mathbf{\Sigma}^*_3 = \mathcal{P}(2^\omega) \text{ and } \mathbf{\Pi}^*_2 \neq \mathbf{\Sigma}^*_2$$

see Steprans [20]. In Stepran's model, the continuum is $\aleph_{\omega_1}$. Carlson [5] showed that if subset of 2^ω is ω_1-Borel, then the cofinality of the continuum must be ω_1. Stepran's model was used earlier by Bukovsky [3] and latter by Miller-Prikry [13].

The following is an open question from Brendle, Larson, and Todorcevic [4].

Question 1 *Is it consistent with the negation of the continuum hypothesis that $\mathbf{\Pi}^*_2 = \mathbf{\Sigma}^*_2$?*

Steprans noted that it would be too much to ask for

$$\neg CH \quad + \quad \mathbf{\Pi}^*_2 = \mathbf{\Sigma}^*_2 = \mathcal{P}(2^\omega)$$

since a $\mathbf{\Sigma}^*_2$ set, i.e., an ω_1 union of closed sets, of size greater than ω_1 would have to contain a perfect subset, hence $\neg CH$ implies a Bernstein set cannot be $\mathbf{\Sigma}^*_2$. It is also known that $\mathbf{\Pi}^*_2 \neq \mathbf{\Sigma}^*_2$ in the iterated Sacks model, see Ciesielski and Pawlikowski [6].

Theorem 2 *(MA_{ω_1}) $\mathbf{\Pi}^*_\alpha \neq \mathbf{\Sigma}^*_\alpha$ for every $\alpha < \omega_2$.*

We prove this using the following two lemmas. A well-known consequence of MA_{ω_1} is that every subset $Q \subseteq 2^\omega$ of size ω_1 is a Q-set, i.e., for every subset $X \subseteq Q$ there is a G_δ set $G \subseteq 2^\omega$ with $G \cap Q = X$ (see Fleissner and Miller [7]).

Lemma 3 *Suppose there exists a Q-set of size ω_1. Then there exists an onto map $F : 2^\omega \to 2^{\omega_1}$ such for every subbasic clopen set $C \subseteq 2^{\omega_1}$ the set $F^{-1}(C)$ is either G_δ or F_σ.*

Proof

Fix $Q = \{u_\alpha \in 2^\omega \ : \ \alpha < \omega_1\}$ a Q-set. Let $G \subseteq 2^\omega \times 2^\omega$ be a universal G_δ set, i.e., G is G_δ and for every G_δ set $H \subseteq 2^\omega$ there exists $x \in 2^\omega$ with $G_x = H$. Define F as follows, given $x \in 2^\omega$ let

$$F(x)(\alpha) = 1 \text{ iff } u_\alpha \in G_x$$

If C is a subbasic clopen set, then for some α and $i = 0$ or $i = 1$

$$C_{\alpha,i} = \{p \in 2^{\omega_1} \ : \ p(\alpha) = i\}.$$

Then for $i = 1$

$$F^{-1}(C_{\alpha,1}) = \{x \ : \ u_\alpha \in G_x\}$$

which is a G_δ set. Since $C_{\alpha,0}$ is the complement of $C_{\alpha,1}$ we have that $F^{-1}(C_{\alpha,0})$ is an F_σ-set

Finally, we note that since Q is a Q-set, i.e., every subset is a relative G_δ, it follows that F is onto.

QED

The next Lemma is true without any additional assumptions beyond ZFC. Its proof is a generalization of Lebesgue's 1905 proof (see Kechris [9] p.168) for the standard Borel hierarchy.

Lemma 4 *For any α with $0 < \alpha < \omega_2$ there exists a $\mathbf{\Sigma}^*_\alpha$ set $U \subseteq 2^{\omega_1} \times 2^\omega$ which is universal for $\mathbf{\Sigma}^*_\alpha$ subsets of 2^ω, i.e., for any $Q \subseteq 2^\omega$ which is $\mathbf{\Sigma}^*_\alpha$ there exists $p \in 2^{\omega_1}$ with $U_p = Q$. Similarly, there is a universal $\mathbf{\Pi}^*_\alpha$ set.*

Proof

The proof is by induction on α. Note that the complement of a universal $\mathbf{\Sigma}^*_\alpha$ set is a universal $\mathbf{\Pi}^*_\alpha$-set.

For $\alpha = 1$, $\mathbf{\Sigma}^*_\alpha$ is just the open sets. There is a universal open set $V \subseteq 2^\omega \times 2^\omega$. Put

$$U = \{(p, x) \in 2^{\omega_1} \times 2^\omega \ : \ (p \restriction \omega, x) \in V\}$$

For α such that $2 \leq \alpha < \omega_2$ proceed as follows. Let $(\delta_\beta < \alpha : \beta < \omega_1)$ have the property that for every $\gamma < \alpha$ there are ω_1 many $\delta_\beta \geq \gamma$. It follows that for every $\mathbf{\Sigma}^*_\alpha$ set $Q \subseteq 2^\omega$ there is $(Q_\beta \in \mathbf{\Pi}^*_{\delta_\beta} \ : \ \beta < \omega_1)$ with

$$Q = \bigcup_{\beta < \omega_1} Q_\beta.$$

By induction, there are $U_\beta \subseteq 2^{\omega_1} \times 2^\omega$ universal $\mathbf{\Pi}^*_{\delta_\beta}$ sets. Let $a : \omega_1 \times \omega_1 \to \omega_1$ be a bijection. For each β define

$$\pi_\beta : 2^{\omega_1} \times 2^\omega \to 2^{\omega_1} \times 2^\omega, \ \ (p, x) \mapsto (q, x)$$

where $q(\alpha) = p(a(\beta, \alpha))$. Put

$$U = \bigcup_{\beta<\omega_1} \pi_\beta^{-1}(U_\beta)$$

then U will be a universal $\mathbf{\Sigma}^*_\alpha$ set.
QED

Now we prove Theorem 2. Suppose for contradiction, that every ω_1-Borel set is $\mathbf{\Sigma}^*_\alpha$ for some fixed $\alpha < \omega_2$. Let $U \subseteq 2^{\omega_1} \times 2^\omega$ be a universal $\mathbf{\Sigma}^*_\alpha$ and define

$$V = \{(x, y) \in 2^\omega \times 2^\omega \ : \ (F(x), y) \in U\}.$$

Then V is an ω_1-Borel set (although not necessarily at the $\mathbf{\Sigma}^*_\alpha$) because the preimage of any clopen box $C \times D$ is ω_1-Borel by Lemma 3. Define

$$D = \{x : (x, x) \notin V\}.$$

But then D is ω_1-Borel but not $\mathbf{\Sigma}^*_\alpha$. We see this by the usual diagonal argument that if $D = U_p$, then since F is onto there would be $x \in 2^\omega$ such that $F(x) = p$ but then

$$x \in D \text{ iff } (F(x), x) \notin U \text{ iff } x \notin U_p \text{ iff } x \notin D.$$

QED

Remark 5 *Note that in the proof $V \subseteq 2^\omega \times 2^\omega$ is a $\mathbf{\Sigma}^*_{2+\alpha}$-set, since the preimage of a clopen set under F is $\mathbf{\Delta}^0_3$. Hence for levels $\alpha \geq \omega$ the set V is a $\mathbf{\Sigma}^*_\alpha$ set which is universal for $\mathbf{\Sigma}^*_\alpha$ sets.*

Remark 6 *Our result easily generalizes to show that MA implies that for any κ a cardinal with $\omega \leq \kappa < |2^\omega|$ the κ-Borel hierarchy has length κ^+. This implies that for any $\kappa_1 < \kappa_2$ there are κ_2-Borel sets which are not κ_1-Borel.*[2] *It is also true for the Cohen real model that for $\omega \leq \kappa_1 < \kappa_2 < |2^\omega|$ that there are κ_2-Borel sets which are not κ_1-Borel.*

[2] Since κ_2-Borel sets at level κ_1^+ or higher cannot be κ_1-Borel.

Question 7 *Suppose MA and the continuum, $\mathfrak{c} = |2^\omega|$, is a weakly inaccessible cardinal. What is the length*[3] *of the hierarchy of $(< \mathfrak{c})$-Borel sets?*

Theorem 8 *In the Cohen real model every ω_1-Borel set is $\mathbf{\Sigma}^*_{\omega_1+1}$ and there is a $\mathbf{\Sigma}^*_{\omega_1}$ set which is not in $\mathbf{\Sigma}^*_{<\omega_1}$.*

Proof

We state the lower bound separately as Theorem 9.

We will use Steel forcing with tagged trees (Steel [18]) similarly to its use in Stern [19]. Stern proved that assuming MA_{ω_1} an ω_1 union of $\mathbf{\Sigma}^0_\alpha$ sets which is Borel, must be $\mathbf{\Sigma}^0_\alpha$. Since Steel forcing is countable, he only really needed MA(ctble). Similar results are proved in Solecki [16] Cor 2.3 and Becker and Dougherty [2] Thm 2. These authors do not consider ω_1-Borel sets but are interested only in ω_1-unions of ordinary Borel sets.

$MA_{\omega_1}(ctble)$ stands for Martin's axiom for countable posets. It says that for any countable poset and ω_1-family of dense sets there is a filter meeting all the dense sets in the family. It is equivalent to saying that the real line cannot be covered by ω_1 nowhere dense sets, see for example, Bartoszynski and Judah [1] p. 138. It holds in any generic extension obtained with a finite support ccc iteration of cofinality at least ω_2.

Theorem 9 *Suppose $MA_{\omega_1}(ctble)$ holds. Then for any $\alpha < \omega_1$ there is an ordinary Borel set which is not $\mathbf{\Sigma}^*_\alpha$.*

Proof

We use Steel forcing with tagged trees[4] similarly to the way it is described in Harrington [8].

For any countable ordinal α define $\mathcal{Q}(\alpha)$ to be the following countable poset. Elements of $\mathcal{Q}(\alpha)$ have the form (t, h) where t is a finite subtree of $\omega^{<\omega}$ and $h : t \to \alpha \cup \{\infty\}$ is called a tagging. The ordering on $\alpha \cup \{\infty\}$ is $\infty < \infty$ and $\beta < \infty$ for each ordinal β along with the usual ordering on pairs of ordinals from α. A tagging h is a rank function which means it satisfies: if $\sigma, \tau \in t$ and σ is a strict initial segment of τ, then $h(\sigma) > h(\tau)$.[5]

The ordering on $\mathcal{Q}(\alpha)$ is $p \leq q$ (p extends q) iff

1. $t_q \subseteq t_p$ and

[3] The argument of Lemma 9 shows that it is at least ω_1.
[4] Sami [15] gives a proof of Harrington's Theorem which does not use Steel forcing.
[5] We differ from [8] by not requiring that $h(\langle\rangle) = \infty$.

2. $h_q \subseteq h_p$.

Note that nodes tagged with ∞ can always be extended and tagged with ∞ or any element of α. [6]

Now suppose that G is $\mathcal{Q}(\alpha)$ generic over M. Define

1. $T = T_G = \{\sigma \ : \ \exists (t,h) \in G \ \ \sigma \in t\}$
2. $H = H_G : T_G \to \alpha \cup \{\infty\}$ by $H(\sigma) = h(\sigma)$ for any h such that there exists $(t,h) \in G$ with $\sigma \in t$.

It is easily seen by a density argument that H is a rank function on the tree T where the symbol ∞ gets attached to the nodes of T which can be extended to an infinite branch.

Define $p(\beta)$ for $\beta \leq \alpha$ and $p \in \mathcal{Q}(\alpha)$ by $p(\beta) = (t, h_\beta)$ where $p = (t,h)$ and

$$h_\beta(s) = \begin{cases} h(s) & \text{if } h(s) < \omega \cdot \beta \\ \infty & \text{otherwise} \end{cases}$$

Lemma 10 *(Retagging Lemma) Suppose $p_1, p_2 \in \mathcal{Q}(\alpha)$ and $\beta + 1 \leq \alpha$ and $p_1(\beta+1) = p_2(\beta+1)$. Then for every $q_1 \leq p_1$ there is $q_2 \leq p_2$ such that $q_1(\beta) = q_2(\beta)$.*

Proof
Let $p_i = (t, h_i)$ for $i = 1,2$ and suppose $q_1 = (t', f_1)$. We define $q_2 = (t', f_2)$ as follows. Put $f_2 \upharpoonright t = h_2$. Fix $N < \omega$ greater than the height of t'. For each $\sigma \in t' \backslash t$ let $\tau \subseteq \sigma$ be the longest initial segment of σ which is in t.

Case 1. If $h_1(\tau) < \omega(\beta+1)$, then by assumption, $h_2(\tau) = h_1(\tau)$ and we can define $f_2(\sigma) = f_1(\sigma)$.

Case 2. $h_1(\tau) \geq \omega(\beta+1)$, then by assumption, $h_2(\tau) \geq \omega(\beta+1)$.

(a) If $h_1(\sigma) < \omega\beta$, then we put $h_2(\sigma) = h_1(\sigma)$.

(b) Otherwise $\omega\beta \leq h_1(\sigma)$ and we put $h_2(\sigma) = \omega\beta + (N - |\sigma|)$. Note that in this case when we look at $q_i(\beta)$ these σ will be retagged with ∞.
QED

Fix $\alpha < \omega_1$ and let T be the usual $\mathcal{Q}(\alpha)$-name for the generic tree T_G:

$$T = \{(p, \check{s}) : s \in t_p \text{ where } p = (t_p, h_p) \in \mathcal{Q}(\alpha)\}.$$

The following is the main property of Steel forcing. We identify $\mathcal{P}(\omega^{<\omega})$ with 2^ω.

[6] Harrington [8] makes the additional requirement that the top node, $\langle\rangle$, be tagged with ∞, but this is unnecessary and makes our proof clumsy, as in Miller [14] Lemma 4.4.

Lemma 11 *Suppose $p, q \in \mathcal{Q}(\alpha)$, $1+\beta \leq \alpha$, $p(1+\beta) = q(1+\beta)$, and $B \subseteq \mathcal{P}(\omega^{<\omega})$ is $\mathbf{\Pi}^*_\beta$ set coded in the ground model.*[7] *Then*

$$p \Vdash T \in B \text{ iff } q \Vdash T \in B.$$

Proof
This is proved by induction on β.

For $\beta = 0$ we take for $\mathbf{\Pi}^*_0$ basic clopen subsets of $\mathcal{P}(\omega^{<\omega})$. This means that for some pair F_0, F_1 of disjoint finite subsets of $\omega^{<\omega}$ that

$$B = \{X \subseteq \omega^{<\omega} \ : \ F_0 \subseteq X \text{ and } F_1 \cap X = \emptyset\}.$$

So the statement $X \in B$ is a finite conjunction of statements of the form $\sigma \in X$ or $\sigma \notin X$. But note that:

1. $p \Vdash \check{\sigma} \in T$ iff $\sigma = \langle\rangle$, $\sigma \in t_p$, or $\tau \in t_p$ and $h_p(\tau) > 0$ where τ is the initial segment of σ of length exactly one less than σ.
2. $p \Vdash \check{\sigma} \notin T$ iff there exists $\tau \subseteq \sigma$ with $\tau \in t_p$ and $h_p(\tau) < |\sigma| - |\tau|$.

Both of these are preserved when we look at $p(1)$. Hence if $p(1) = q(1)$ then

$$p \Vdash T \in B \text{ iff } q \Vdash T \in B.$$

For $\beta > 0$ suppose that B is $\mathbf{\Pi}^*_{\beta+1}$ and coded in the ground model. Working in the ground model let $B = \bigcap_{\alpha<\omega_1} \sim B_\alpha$ where[8] each B_α is $\mathbf{\Pi}^*_{<1+\beta}$. And suppose for contradiction that $p_2 \Vdash T \in B$ but p_1 does not force this. Then there exists a $q_1 \leq p_1$ and $\alpha < \omega_1$ such that

$$q_1 \Vdash T \in B_\alpha.$$

And suppose that B_α is $\mathbf{\Pi}^*_{1+\gamma}$ where $\gamma < \beta$. Since $1+\gamma+1 \leq 1+\beta$, by the retagging lemma we may find $q_2 \leq p_2$ with $q_1(1+\gamma) = q_2(1+\gamma)$. By inductive hypothesis

$$q_2 \Vdash T \in B_\alpha$$

[7] There are many ways to code Borel (or more generally κ-Borel) sets. Solovay [17] p.25 gives a clear definition of coding and absoluteness which is similar to what we use in the proof of Lemma 4. Harrington [8] Definition 2.5 and Steel [18] code using infinitary propositional logic. We like to use well-founded trees as in Lemma 12.

[8] We use $\sim B$ to denote the complement of B.

which contradicts that

$$p_2 \Vdash T \in B \supseteq \sim B_\alpha.$$

QED

Suppose for contradiction that in the Cohen real model there is an $\alpha_0 < \omega_1$ such that every ω_1-Borel set is $\mathbf{\Pi}^*_{\alpha_0}$. It well-known that for every countable ordinal α the set

$$WF_\alpha = \{T \subseteq \omega^{<\omega} : T \text{ is a well-founded tree of rank } \alpha\}$$

is an (ordinary) Borel set.[9] Consequently it must be a $\mathbf{\Pi}^*_{\alpha_0}$-set. Fix a countable $\alpha > \alpha_0 \cdot \omega$. Take a sufficiently large[10] regular cardinal κ and let H_κ be the sets whose transitive closure has cardinality less than κ. Take N to be an elementary substructure of V_κ of cardinality ω_1 which contains $\alpha + 1$. Then N will contain a code for B the $\mathbf{\Pi}^*_{\alpha_0}$ set WF_α. Let M be the transitive collapse of N and consider forcing over M with $\mathcal{Q}(\alpha+1)$. Since we are assuming MA(ctbl), for any $p \in \mathcal{Q}(\alpha+1)$ there is a G $\mathcal{Q}(\alpha+1)$-generic over the ground model M with $p \in G$. So take such a G with $H_G(\langle\rangle) = \alpha$. Then T_G is a well-founded tree of rank α and so $T_G \in WF_\alpha$. By absoluteness

$$M[G] \models T_G \in B$$

and so there must be a $p \in G$ such that

$$p \Vdash T \in B.$$

But consider $q = p(\alpha)$. Note that $h_q(\langle\rangle) = \infty$. Consequently, for any G' which is $\mathcal{Q}(\alpha+1)$-generic over M with $q \in G'$, the tree $T_{G'}$ is not even well-founded and hence

$$M[G'] \models T_{G'} \notin B.$$

But this means that

$$q \Vdash T \notin B$$

which contradicts Lemma 11.

QED

Next we prove an upper bound on the ω_1-Borel hierarchy in the Cohen real model. Our argument uses some ideas employed by Carlson [5].

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[9] The exact Borel class is computed in Stern [19] and Miller [12].

[10] For example $\kappa = \beth_\omega^+$.

Lemma 12 *In the Cohen real model for any ω_1-Borel set B there exists ω_1 ordinary Borel sets, $(B_\beta : \beta < \omega_1)$, such that B is their limit:*

$$B = \bigcup_{\alpha<\omega_1} \bigcap_{\beta>\alpha} B_\beta = \bigcap_{\alpha<\omega_1} \bigcup_{\beta>\alpha} B_\beta$$

Proof
Let B be coded by a well-founded tree $T \subseteq \omega_1^{<\omega_1}$ with basic clopen sets $(s_\sigma \in 2^{<\omega} : \sigma \in T^*)$ where T^* are the terminal nodes (or leaf nodes) of the tree T. Then $T, (s_\sigma : \sigma \in T^*)$ codes B as follows. Define

$$B(\sigma) = [s_\sigma] = \{x \in 2^\omega \ : \ s_\sigma \subseteq x\}$$

for $\sigma \in T^*$. Then for nonterminal nodes of T define

$$B(\sigma) = \bigcap \{\sim B(\sigma\hat{\ }\langle\alpha\rangle) : \alpha < \omega_1 \text{ and } \sigma\hat{\ }\langle\alpha\rangle \in T\}.$$

Finally, put $B = B(\langle\rangle)$.

Fix such a T for B and for any $\alpha < \omega_1$ define B_α inductively just as above but for the countable tree $T \cap \alpha^{<\omega}$.

We will show that for some closed unbounded set $C \subseteq \omega_1$ that B is the ω_1-limit of $(B_\beta : \beta \in C)$.

By the Cohen real model we mean an model obtained by forcing with $\mathrm{Fn}(\omega_2, 2)$, the finite partial maps from ω_2 into 2, over a model of ZFC+GCH. By standard arguments using the countable chain condition and product Lemma, we may without loss of generality assume that our code for B, $T, (s_\sigma : \sigma \in T^*)$, is in the ground model M a model of ZFC+GCH. For any $x \in M[G] \cap 2^\omega$ (where G is $\mathrm{Fn}(\omega_2, 2)$-generic over M there is an $H \in M[G]$ which is $\mathrm{Fn}(\omega, 2)$-generic over M and $x \in M[H]$.

Since the ground model M satisfies CH, there is a set of canonical names, CN, for elements of 2^ω in the extension $M[H]$ has size ω_1.

Working in the ground model M construct an continuous chain $(N_\alpha : \alpha < \omega_1)$ of countable elementary submodels of H_{ω_2}, with the code for B, $T, (s_\sigma : \sigma \in T^*)$, in N_0, $N_\alpha \preceq N_\beta$ and $N_\alpha \in N_\beta$ for $\alpha < \beta < \omega_1$. Note that it is automatically the case that every canonical name is in some N_α.

Now take for our club C the set

$$C = \{\omega_1 \cap N_\alpha \ : \ \alpha < \omega_1\}.$$

Suppose that $x = \tau^H$ where $\tau \in N_\alpha$ and H is $\mathrm{Fn}(\omega, 2)$-generic over M. Let M_α be the transitive collapse of N_α. By standard arguments H is $\mathrm{Fn}(\omega, 2)$-generic over M_α. Note that ordinal $\delta = N_\cap \omega_1$ is the ω_1 of M_α i.e.,

$$M_\alpha \models \delta = \omega_1.$$

Let $p \in \mathrm{Fn}(\omega, 2)$ be such that either

$$M_\alpha \models p \Vdash \tau \in B$$

or

$$M_\alpha \models p \Vdash \tau \in \sim B.$$

Assume the former. Note that $B^{M_\alpha[H]} = B_\delta \cap M[H]$. And since it is forced it must be that $x = \tau^H \in B_\delta$.

For every $\beta > \alpha$ the model N_β elementary superstructure of N_α and hence that

$$M_\beta \models p \Vdash \tau \in B$$

and for the same reason $x \in B_{\delta'}$ where δ' is the ω_1 of M_β.
QED

Remark 13 *Lemma 12 easily generalizes to the ω_1-Borel hierarchy giving that every ω_2-Borel set is the ω_2 limit of ω_1-Borel sets, and since each of them is at level $\omega_1 + 1$, we get an upperbound of $\omega_1 + 2$ for the length of the ω_2-Borel hierarchy.*

Remark 14 *Lemma 12 is also true in the random real model.*

Remark 15 *In Steprans [20] the hierarchy on the ω_1-Borel sets is defined by letting the bottom level, $\Pi_0^{\aleph_1} = \Sigma_0^{\aleph_1}$, be the family of all ordinary Borel sets. Lemma 12 shows that every ω_1-Borel set in the Cohen real model is $\Pi_2^{\aleph_1}$ and hence $\Sigma_2^{\aleph_1}$. It is easy to see that in this model there are $\Sigma_1^{\aleph_1}$ sets which are not $\Pi_1^{\aleph_1}$, for example, any nonmeager subset of 2^ω of size ω_1.*

Remark 16 *In Miller [11] Theorem 34 and 54, it is shown consistent for any countable ordinal $\alpha_0 \geq 2$ to have separable metric space X such that every subset of X is Borel and the Borel hierarchy on X has length exactly α_0. It is easy to show that if the set $X \subseteq 2^\omega$ has cardinality at least ω_2 that for each $\beta < \alpha_0$ the generic $\mathbf{\Pi}_\beta^0$ sets produced are not $\mathbf{\Sigma}_\beta^*$ relative to X.*

Hence these spaces have order α_0 in the relativized ω_1-Borel hierarchy. If we replace the use of almost disjoint forcing in Steprans model [20] Definition 2, by $\Pi^0_{\alpha_0}$-forcing from Miller [11] p. 236, then we get a model of ZFC in which every subset of 2^ω is ω_1-Borel and the ω_1-Borel hierarchy has length at least α_0 but no more than $\alpha_0 + 1$. Similarly if we change the Steprans model by using Π^0_α-forcing in the α model, then in the resulting model every subset of 2^ω is ω_1-Borel and the ω_1-Borel hierarchy has length at least ω_1 but no more than $\omega_1 + 1$.

Question 17 *Is possible to have a model of ZFC in which the ω_1-Borel hierarchy has length α where $\omega_1 + 2 \leq \alpha < \omega_2$?*

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