ON THE L-SERIES OF F. PELLARIN

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ABSTRACT. The calculation, by L. Euler, of the values at positive even integers of the Riemann zeta function, in terms of powers of π and rational numbers, was a watershed event in the history of number theory and classical analysis. Since then many important analogs involving L-values and periods have been obtained. In analysis in finite characteristic, a version of Euler's result was given by L. Carlitz [Ca2] in the 1930's which involved the period of a rank 1 Drinfeld module (the Carlitz module) in place of π. In a very original work [Pe2], F. Pellarin has quite recently established a "deformation" of Carlitz's result involving certain L-series and the deformation of the Carlitz period given in [AT1]. Pellarin works only with the values of this L-series at positive integral points. We show here how the techniques of [Go1] also allow these new L-series to be analytically continued – with associated trivial zeroes – and interpolated at finite primes.

1. INTRODUCTION

In 1734, after a number of attempts, Euler succeeded in giving a closed form formula for $\zeta(2)$ where $\zeta(s) = \sum_{n=0}^{\infty} n^{-s}$ is the Riemann zeta function. Indeed, at that time Euler obtained the famous formula $\zeta(2) = \pi^2/6$ (as well as $\zeta(4) = \pi^4/90$), see e.g., [Ay1]. This marks the beginning of the remarkably profound association between periods and L-series of classical motives; a subject being vigorously investigated to this day.

Let p be a prime and let $q = p^{m_0}$ for a fixed positive integer m_0. Set $A := \mathbb{F}_q[\theta]$ where θ is an indeterminate. Approximately 200 years after Euler, in the mid-1930's, L. Carlitz [Ca2] used characteristic p analysis to establish an analog for A of Euler's result. More precisely, let $k := \mathbb{F}_q(\theta)$ and $K := \mathbb{F}_q((1/\theta))$ (which is the completion of k at the infinite place). Let j be a positive integer and put

$$\zeta_A(j) := \sum_{f(\theta) \text{ monic}} f^{-j} \, ; \tag{1}$$

this series is readily seen to converge to nonzero element of K. Carlitz then established the existence of a nonzero constant $\tilde{\pi}$ so that if $(q-1) \mid j$, then $\zeta(j)/\tilde{\pi}^j \in k$. In the 1970's, this result was independently rediscovered by the present author in the context of Eisenstein series.

The constant $\tilde{\pi}$, which is uniquely defined up to an element of $\mathbb{F}_q^*$, is the period of the *Carlitz module* C [Ca1]; i.e., $\tilde{\pi}$ is generator of the rank 1 A-module of periods associated to the exponential function $e_C(z)$ of C. (As such, $\tilde{\pi}$ is actually analogous to $2\pi i$ in classical theory.)

Carlitz also gave a beautiful product formula for his period which was remarkably simplified in [AT1] (see also [ABP]) in the following fashion. Let θ_1 be a fixed choice of $(q-1)$-st

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root of $-\theta$. Then one has

$$\tilde{\pi} = \theta_1 \theta \prod_{i=1}^{\infty} \left(1 - \theta^{1-q^i}\right)^{-1}. \tag{2}$$

In [AT1] (and [ABP]), this formula is also "deformed" in the following fundamental fashion. Let t be another indeterminate and set

$$\Omega(t) := \theta_1^{-q} \prod_{i=1}^{\infty} \left(1 - t/\theta^{q^i}\right). \tag{3}$$

It follows immediately that the product for $\Omega(t)$ converges for all $t \in K$, and one has

$$\Omega(\theta)\tilde{\pi} = -1. \tag{4}$$

The function $\Omega(t)$ is essential for the study of tensor powers of the Carlitz module and special Γ-values. As such, it is also natural to search for a connection between it and characteristic p L-series. This has been very recently accomplished in seminal work by Federico Pellarin. Indeed, in [Pe2] (which depends of [Pe1]), Pellarin obtains the following elegant generalization of the formulas of Euler and Carlitz. Let $t \in K$ and let χ_t be the quasi-character of A given simply by

$$\chi_t(f) := f(t),$$

for $f(\theta) \in A$. Let j be a positive integer and set

$$L(\chi_t, j) := \sum_{f(\theta) \text{ monic}} \chi_t(f)/f^j. \tag{5}$$

It is easy to see that this converges for t sufficiently small.

Let j now be congruent to 1 modulo $q - 1$. Then, also using Eisenstein series (or, rather, vectorial Eisenstein series), Pellarin shows the existence of a rational function $b_j(\theta, t) \in \mathbb{F}_q(\theta, t)$ such that

$$L(\chi_t, j) = b_j(\theta, t)\tilde{\pi}^j \Omega(t). \tag{6}$$

(N.B.: The normalized function $\tilde{\pi}\Omega(t)$ is independent of the choice of θ_1.) As $\tilde{\pi}\Omega(\theta) = -1$, Equation 6 is clearly a deformation of the result of Carlitz.

One can compute that $b_1(\theta, t) \equiv -1$. As such,

$$\lim_{t \to \theta} L(\chi_t, 1) = 1 \tag{7}$$

in agreement with previous results on zeta values.

In his paper, Pellarin only considers positive values of j. Clearly, however, one would like these "Pellarin L-series" to have the good analytic properties possessed by the L-series of Drinfeld modules. It is our purpose here to establish these properties. Indeed, we show that the needed results follow from the elementary estimates of Lemma 8.8.1 in [Go1].

The L-series considered here are but the first in a long line of functions one now wants to understand. In fact, the quasi-character χ_t may be used to deform any L-series of any Drinfeld module, A-module, etc., (see e.g., [Bo1], [Bo2], [Go2]) and all such functions must be studied.

Moreover, $A = \mathbb{F}_q[\theta]$ is but the simplest base ring in the general theory of Drinfeld modules. One also expects similar results for general A where there are obviously some subtleties involved. Still any such formula will involve the periods of the Hayes modules [Ha1]; these objects are normalized rank 1 Drinfeld A-modules which were constructed by David Hayes

as the correct generalizations of the Carlitz module. Indeed for $A = \mathbb{F}_q[\theta]$, the Hayes module is the Carlitz module. Therefore, it is only fitting that Hayes' ideas appear in the present context as his work has been absolutely essential in moving the arithmetic theory from the simplest case of $\mathbb{F}_q[\theta]$ to Drinfeld's arbitrary base rings A. In fact, without his construction in the rank 1 case almost no arithmetic in general is even possible. As such, it is my pleasure to dedicate this paper to David's memory and mathematical legacy.

I also express my gratitude to Federico Pellarin for his remarks on a previous version of this work.

2. Definition of the Pellarin L-series

As before, let $A = \mathbb{F}_q[\theta]$, $q = p^{m_0}$; we put $k := \mathbb{F}_q(\theta)$, $K := k_\infty = \mathbb{F}_q((1/\theta))$. Let A_+ be the set of monic elements of A. For d a nonnegative integer, we let $A_+(d)$ be the monics of degree d in A and $A(d)$ the vector space of all polynomials of degree $\leq d$. As usual, K comes equipped with the absolute value $|?|_\infty$ such that $|\theta|_\infty = q$. We let $\overline{K}$ be a fixed algebraic closure of K equipped with the canonical extension of $|?|_\infty$ and we let $\mathbb{C}_\infty$ be its completion. Let $v_\infty(?)$ be the associated canonical additive valuation on $\mathbb{C}_\infty$ extending the valuation v_∞ on K with $v_\infty(1/\theta) = 1$.

Let π be a fixed uniformizer in K of the form $\pi = 1/\theta + \{\text{higher terms}\}$. (In fact, as will be seen in Remark 1 just below, the reader may take $\pi = 1/\theta$ without any loss of generality.) Let $a \in A_+$ and put

$$\langle a \rangle_\pi := \pi^{\deg a} a \,. \tag{8}$$

Clearly, the map $a \mapsto \langle a \rangle_\pi$ is a homomorphism from the multiplicative monoid A_+ into the group of 1-units of K. The Binomial Theorem therefore allows one to raise $\langle a \rangle_\pi$ to any power $y \in \mathbb{Z}_p$.

Definition 1. We set

$$\mathbb{S}_\infty := \mathbb{C}_\infty^* \times \mathbb{Z}_p \,. \tag{9}$$

The space $\mathbb{S}_\infty$ is a commutative group whose operation will be written additively. Let $a \in A_+$ and $s = (x, y) \in \mathbb{S}_\infty$. Then we define

$$a^s := x^{\deg a} \langle a \rangle_\pi^y \,. \tag{10}$$

Let j be an integer and set $s_j := (\pi^{-j}, j) \in \mathbb{S}_\infty$. One then readily checks that $a^{s_j} = a^j$ where "a^j" has the usual meaning. We will freely refer to s_j as "j".

The space $\mathbb{S}_\infty$ is the domain of characteristic p L-series at the place ∞ of k.

Remark 1. Let π_1 and π_2 be as above. Note that for $a \in A_+$ of degree d, and $y \in \mathbb{Z}_p$, we have $\langle a \rangle_{\pi_1}^y = (\pi_1/\pi_2)^{dy} \langle a \rangle_{\pi_2}^y$. Clearly $|(\pi_1/\pi_2)^{dy}|_\infty = 1$. Therefore, as we are interested here in establishing certain estimates, this factor is harmless. *As such, we now set $\pi := 1/\theta$ for the rest of this paper and drop the reference to π.*

Definition 2. Let $t \in \mathbb{C}_\infty$ and $f \in A$. Then we define $\chi_t(f) := f(t)$.

Obviously, χ_t is just the evaluation map of $f(\theta)$ at $t \in \mathbb{C}_\infty$, $\theta \mapsto t$, and thus is clearly an $\mathbb{F}_q$-algebra morphism from A to $\mathbb{C}_\infty$. We view it as a quasi-character of A.

Definition 3. For β a nonnegative integer and $s \in \mathbb{S}_\infty$, we formally define

$$L(\chi_t^\beta, s) := \sum_{a \in A_+} \chi_t(a)^\beta a^{-s} = \prod_{f \text{ monic prime}} (1 - \chi_t^\beta(f) f^{-s})^{-1} \,. \tag{11}$$

We call $L(\chi_t^\beta, s)$ the *Pellarin L-series associated to* χ_t^β.

Lemma 1. *Let* $t \in \mathbb{C}_\infty$ *and set* $\lambda_t := \max\{1, |t|_\infty\}$. *Let* $f(\theta) \in A$ *have degree* d. *Then* $|f(t)|_\infty \leq \lambda_t^d$.

Proof. Obvious.   □

Proposition 1. *Let* $t \in \mathbb{C}_\infty$ *and* λ_t *be as Lemma 1. Then the Euler-product for the L-series* $L(\chi_t^\beta, s)$ *converges on the "half-plane" of* $\mathbb{S}_\infty$ *given by* $\{s = (x, y) \mid |x|_\infty > \lambda_t^\beta\}$.

Proof. This follows immediately from the definitions and Lemma 1.   □

In Theorem 1 we establish that $L(\chi_t^\beta, s)$ can be analytically continued to an entire function on $\mathbb{S}_\infty$ in the sense of Section 8.5 of [Go1].

3. Review of estimates

We recall here the basic estimates from Section 8.8 of [Go1] necessary for us and refer the reader there for their derivations. Let J_0 and J_1 be two fields over $\mathbb{F}_q$. Let $W \subseteq J_0$ be a finite dimensional $\mathbb{F}_q$-vector space of dimension d. Further let $\{\mathcal{L}_1, \ldots, \mathcal{L}_t\}$ be $\mathbb{F}_q$-linear maps of J_0 into J_1. Let $x \in J_0$ and let $\{i_1, \ldots, i_t\}$ be nonnegative integers so that

$$\sum_{h=1}^t i_h < (q-1)d. \tag{12}$$

Lemma 2. *Under the above assumptions we have*

$$\sum_{w \in W} \left(\prod_{h=1}^t \mathcal{L}_h(x+w)^{i_h} \right) = 0. \tag{13}$$

Assume now that J_1 has an additive discrete valuation v with $v(\mathcal{L}_h(w)) > 0$ for all h and w. Let $\{i_h\}$ now be an arbitrary collection of non-negative integers, and for $j > 0$ put

$$W_j := \{w \in W \mid v(\mathcal{L}_h(w)) \geq j \text{ for all } h\}.$$

Finally, set $Q := \sum_j \dim_{\mathbb{F}_q} W_j$.

Lemma 3. *Under the above assumptions, we have*

$$v \left(\sum_{w \in W} \prod_{h=1}^t \mathcal{L}_h(w)^{i_h} \right) \geq (q-1)Q. \tag{14}$$

4. The main theorem

We now establish the analytic continuation of $L(\chi_t^\beta, s)$ using Lemma 3. We note that it is not always possible to apply this result directly and so we proceed in a slightly indirect fashion.

The first step is to rewrite the Pellarin L-series in the usual fashion by summing according to degrees. That is, upon unraveling the definitions, we find in the half-plane of convergence that

$$L(\chi_t^\beta, s) = \sum_{j=0}^\infty x^{-j} \left(\sum_{a \in A_+(j)} \chi_t(a)^\beta \langle a \rangle^{-y} \right). \tag{15}$$

For each fixed $y \in \mathbb{Z}_p$, the goal is to establish that $L(\chi_t, x, y)$ is an entire power series in x^{-1} with the resulting function on $\mathbb{S}_\infty$ having good continuity properties.

Let $t \in \mathbb{C}_\infty$ and set $\delta_t := \max\{-v_\infty(t) + 1, 1\}$. Let $0 \neq \alpha \in \mathbb{C}_\infty$ be chosen so that $-v_\infty(\alpha) \geq \delta_t$.

Lemma 4. *Let $\{t, \alpha\}$ be as above. Let d be a nonnegative integer and $0 \leq i \leq d$. Then, under the above assumption on α, we have*

$$v_\infty(t^i/\alpha^d) \geq d - i. \tag{16}$$

Proof. Obviously one has

$$v_\infty(t^i/\alpha^d) = iv_\infty(t) - v_\infty(\alpha)d. \tag{17}$$

There are then two cases: $v_\infty(t) \geq 0$ and $v_\infty(t) < 0$. In the first case, by assumption, $-v_\infty(\alpha) \geq 1$ and the result follows directly. In the second case, we have $-v_\infty(\alpha) \geq -v_\infty(t) + 1$. Thus,

$$v_\infty(t^i/\alpha^d) = iv_\infty(t) - v_\infty(\alpha)d \geq (-v_\infty(t) + 1)d + iv_\infty(t) = d + v_\infty(t)(i - d) \geq d - i.$$

$\square$

With the above choices of $\{t, \alpha\}$ we now temporarily set

$$L_\alpha(\chi_t^\beta, s) := L(\chi_t^\beta, \alpha^\beta x, y) = \sum_{j=0}^\infty x^{-j} \left(\sum_{a \in A_+(j)} (\chi_t(a)/\alpha^j)^\beta \langle a \rangle^{-y} \right). \tag{18}$$

Let $\mathbb{F}_q[1/\theta](j)$ the $\mathbb{F}_q$-vector space of polynomials in $1/\theta$ of degree at most j. Note that if $a \in A_+(j)$ then $\langle a \rangle \in \mathbb{F}_q[1/\theta](j)$. We now define two $\mathbb{F}_q$-linear maps $\{\mathcal{L}_1, \mathcal{L}_2\}$ from $\mathbb{F}_q[1/\theta](j)$ to $\mathbb{C}_\infty$ as follows: $\mathcal{L}_1$ is simply the identity map, while

$$\mathcal{L}_2(\sum_{n=0}^j c_n \theta^{-n}) := \sum_{n=0}^j c_n t^{j-n}/\alpha^j. \tag{19}$$

We can now establish our main result.

Theorem 1. *The Pellarin L-series $L(\chi_t^\beta, s)$ analytically continues to an entire function on $\mathbb{S}_\infty$.*

Proof. It is clearly sufficient to establish that $L_\alpha(\chi_t^\beta, s)$ is entire.

Let $W(j) := \{f \in \mathbb{F}_q[1/\theta](j) \mid f(0) = 0\}$ so that $\dim_{\mathbb{F}_q} W(j) = j$. Let $w \in W(j)$ with $v_\infty(w) = i$. The auxiliary constant α has been chosen *precisely* to guarantee $\mathcal{L}_2(w) \geq i$ via Lemma 4.

Let $y \in \mathbb{Z}_p$ be fixed and let $c_j(y)$ be the coefficient of x^{-j} in the expansion of $L_\alpha(\chi_t^\beta, x, y)$. Lemma 3 now immediately implies that $v_\infty(c_j(y)) \geq (q-1)j(j+1)/2$. This quadratic growth is easily seen to be sufficient to establish the result. $\square$

Remark 2. One also readily deduces that $L(\chi_t, s)$ is continuous in t.

5. Special polynomials and trivial zeroes

In this section, we examine the behavior of $L(\chi_t^\beta, s)$ when $s = (x, y)$ and $y \in \mathbb{Z}_p$ is a nonpositive integer. So let $y = -j$, for $j \geq 0$.

Definition 4. We set

$$z(\chi_t^\beta, x, -j) := L(\chi_t^\beta, x\pi^j, -j) = \sum_{e=0}^\infty x^{-e} \left(\sum_{a \in A_+(e)} \chi_t^\beta(a) a^j \right). \qquad (20)$$

Our next result will show that for each such j, $z(\chi_t^\beta, x, -j)$ is a *polynomial* in x^{-1} called the *special polynomial* at $-j$.

Theorem 2. *Let j be as above. Then $z(\chi_t^\beta, x, -j) \in A[t][x^{-1}]$.*

Proof. As in the proof of Theorem 1, we need to define two $\mathbb{F}_q$-linear maps $\mathcal{L}_1$ and $\mathcal{L}_2$ on $A(e)$. Also as before, we set $\mathcal{L}_1$ to be the identity, and now we set $\mathcal{L}_2 := \chi_t$. Notice that $A_+(e) = \{\theta^e + h\}$ where h runs over $A(e-1)$ and $\dim_{\mathbb{F}_q} A(e-1) = e$. Thus, Lemma 2 immediately implies that the coefficient of x^{-e} in $z(\chi_t^\beta, x, -j)$ vanishes when $e > (\beta + j)/(q-1)$. □

5.1. **Trivial zeroes.** Let $t = \zeta \in \mathbb{C}_\infty$ be a root of unity. In this case, as was pointed out in [Pe1], the quasi-character χ_ζ is actually a character factoring through $A/(p(\theta))$ where $p(\theta)$ is the minimal polynomial associated to ζ. As such, the techniques of Section 8.13 of [Go1] are easily altered to equip $L(\chi_\zeta^\beta, s)$ with a trivial zero at $s = -\lambda$ where $\lambda > \beta$ is a positive integer such that $\lambda \equiv -\beta \pmod{q-1}$.

Proposition 2. *Let $\lambda > \beta$ be a positive integer congruent to $-\beta \pmod{q-1}$. Then $L(\chi_t^\beta, s) = 0$ for $s = -\lambda = (\pi^\lambda, -\lambda) \in \mathbb{S}_\infty$.*

Proof. Theorem 2 immediately implies that $z(\chi_t^\beta, 1, -\lambda)$ is an element of $A[t]$. On the other hand, this element vanishes for all $t = \zeta$, where ζ is a root of unity as above. Thus it is identically 0. □

Remark 3. A combinatorial proof of Proposition 2 has also been given by R. Perkins.

6. The $\mathfrak{v}$-adic theory

Let $\mathfrak{v}$ be a nontrivial prime of A of degree d and let $\mathbb{C}_\mathfrak{v}$ be the $\mathfrak{v}$-adic version of $\mathbb{C}_\infty$. We very briefly indicate here how the above techniques may be altered to give the $\mathfrak{v}$-adic interpolation of the above special polynomials to entire functions on the $\mathfrak{v}$-adic analog of $\mathbb{S}_\infty$.

Let $P(\theta)$ be the monic generator of $\mathfrak{v}$ and let $t \in \mathbb{C}_\mathfrak{v}$, so that we have the $\mathfrak{v}$-adic quasi-character χ_t of A. We now choose the auxiliary constant $\alpha \in \mathbb{C}_\mathfrak{v}$ small enough so that $v_\mathfrak{v}(\alpha^{-d}P(t)) > 1$. With this choice the results now follow directly (see also Section 8.9 of [Go1]).

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