A NEW FORMULA FOR THE NATURAL LOGARITHM OF A NATURAL NUMBER

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Abstract. For every natural number T, we write $\operatorname{Ln} T$ as a series, generalizing the known series for $\operatorname{Ln} 2$.

1. Introduction

The Euler-Mascheroni constant γ, [1], is given by the limit

$$\gamma = \lim_{n\to\infty} A_n, \tag{1}$$

where for every $n \geq 1$, $A_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \operatorname{Ln} n$. An elementary way to show the convergence of $\{A_n\}_{n=1}^{\infty}$ is to consider the series $\sum_{n=0}^{\infty}(A_{n+1}-A_n)$. (Here $A_0 := 0$.) Indeed, by Lagrange's Mean Value Theorem, there exists for every $n \geq 1$ a number θ_n, $0 < \theta_n < 1$ such that

$$A_{n+1} - A_n = \frac{1}{n+1} - \operatorname{Ln}(n+1) + \operatorname{Ln} n = \frac{1}{n+1} - \frac{1}{n+\theta_n} = \frac{\theta_n - 1}{(n+1)(n+\theta_n)},$$

and thus $0 > A_{n+1} - A_n > \frac{-1}{n(n+1)}$ and the series converges to some limit γ.

2. The new formula

Let $T \geq 2$ be an integer. We have

$$A_{nT} = \sum_{k=0}^{n-1}\sum_{j=1}^{T}\frac{1}{kT+j} - \operatorname{Ln}(nT) \underset{n\to\infty}{\to} \gamma. \tag{2}$$

By subtracting (1) from (2) and using $\operatorname{Ln}(nT) = \operatorname{Ln} n + \operatorname{Ln} T$, we get

$$\sum_{k=0}^{n-1}\left(\sum_{j=1}^{T}\frac{1}{kT+j} - \frac{1}{k+1}\right) \underset{n\to\infty}{\to} \operatorname{Ln} T,$$

that is,

$$\operatorname{Ln} T = \sum_{k=0}^{\infty}\left(\frac{1}{kT+1} + \frac{1}{kT+2} + \cdots + \frac{1}{kT+(T-1)} - \frac{(T-1)}{kT+T}\right). \tag{3}$$

We observe that (3) generalizes the formula $\operatorname{Ln} 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$.

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We can write (3) also as

$$\text{Ln}\, T = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{T} - 1\right) + \left(\frac{1}{T+1} + \frac{1}{T+2} + \cdots + \frac{1}{2T} - \frac{1}{2}\right) + \ldots \tag{4}$$

and this gives $\text{Ln}\, T$ as a rearrangement of the conditionally convergent series $1-1+\frac{1}{2}-\frac{1}{2}+\frac{1}{3}-\frac{1}{3}+\ldots$. The formula (4) holds also for $T=1$. Formulas (3) and (4) can be applied also to introduce $\text{Ln}\, Q$ as a series for any positive rational $Q = \frac{M}{L}$ since $\text{Ln}\, \frac{M}{L} = \text{Ln}\, M - \text{Ln}\, L$.

Now, for any $k \geq 0$, the nominators of the k-th element in (3) are the same and their sum is 0. This fact is not random. For every constant $a_1, a_2, \ldots a_T$, the sum

$$S_T(a_1, \ldots, a_T) := \sum_{k=0}^{\infty} \left(\frac{a_1}{kT+1} + \frac{a_2}{kT+2} + \cdots + \frac{a_T}{(k+1)T}\right) \tag{5}$$

converges if and only $a_1 + a_2 + \cdots + a_T = 0$. This follows by comparison to the series $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$. By (3) and the notation (5), $\text{Ln}\, T = S_T(1, 1, \ldots, 1, T-1)$.

For $T \geq 2$, let us denote by $\Sigma(T)$ the collection of all sums of rational series of type (5), i.e.,

$$\Sigma(T) = \big\{S_T(a_1, \ldots, a_T) : a_i \in Q,\, 1 \leq i \leq T, a_1 + \cdots + a_T = 0\big\}.$$

The collection $\Sigma(T)$ is a linear space of real numbers over $\mathbb{Q}$ (or over the field of algebraic numbers if we would define $\Sigma(T)$ to be with algebraic coefficients instead of rational coefficients), and $\dim \Sigma(T) \leq T-1$. A spanning set of $T-1$ elements of $\Sigma(T)$ is

$$\big\{S_T(1, -1, 0, 0, \ldots, 0), S_T(0, 1, -1, 0, 0, \ldots, 0), \ldots, S_T(0, \ldots, 0, 1, -1)\big\}.$$

Also, if T is not a prime number, then $\dim \Sigma(T) < T-1$. If $Q = \frac{M}{L}$ is a positive rational number and $P_1, P_2, \ldots, P_k$ are all the prime factors of M and L together, then $\text{Ln}\, Q \in \Sigma(P_1 P_2 \ldots P_k)$.

We can get a non-trivial series for $x = 0$: $\text{Ln}\, 4 = 2\,\text{Ln}\, 2 = S_2(2, -2) = S_4(2, -2, 2, -2)$, and also $\text{Ln}(4) = S_4(1, 1, 1, -3)$. Hence

$$\begin{aligned} 0 &= S_4(2, -2, 2, -2) - S_4(1, 1, 1-3) = S_4(1, -3, 1, 1) \\ &= \left(\frac{1}{1} - \frac{3}{2} + \frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{3}{6} + \frac{1}{7} + \frac{1}{8}\right) + \ldots. \end{aligned}$$

3. The integral approach

The formula (3) can as well be deduced in the following way.

$$\begin{aligned}
\operatorname{Ln} T &= \lim_{x\to 1^-} \operatorname{Ln}(1+x+\dots+x^{T-1}) = \lim_{x\to 1^-} \operatorname{Ln}\left(\frac{1-x^T}{1-x}\right) \\
&= \lim_{x\to 1^-} (\operatorname{Ln}(1-x^T) - \operatorname{Ln}(1-x)) = \lim_{x\to 1^-} \int_0^x \left[\frac{Tu^{T-1}}{u^T-1} + \frac{1}{1-u}\right] du \\
&= \lim_{x\to 1^-} \int_0^x \frac{Tu^{T-1} - (1+u+\dots+u^{T-1})}{u^T-1} du \\
&= \lim_{x\to 1^-} \int_0^x \frac{-1-u-u^2-\dots-u^{T-2}+(T-1)u^{T-1}}{u^T-1} du \\
&= \lim_{x\to 1^-} \left[1\cdot \int_0^x \frac{u-1}{u^T-1} du + 2\cdot \int_0^x \frac{u^2-u}{u^T-1} du + 3\int_0^x \frac{u^3-u^2}{u^T-1} du + \dots \right. \\
&\quad \left. + (T-2)\int_0^x \frac{u^{T-2}-u^{T-3}}{u^T-1} du + (T-1)\int_0^x \frac{u^{T-1}-u^{T-2}}{u^T-1} du\right]. \qquad (6)
\end{aligned}$$

For every $1 \le j \le T-1$,

$$\begin{aligned}
\lim_{x\to 1^-} \int_0^x \frac{u^j - u^{j-1}}{u^T-1} du &= \lim_{x\to 1^-} \int_0^x \left[u^{j-1}\sum_{k=0}^\infty u^{kT} - u^j \sum_{k=0}^\infty u^{kT}\right] du \\
&= \lim_{x\to 1^-} \int_0^x \left(\sum_{k=0}^\infty u^{kT+j-1} - \sum_{k=0}^\infty u^{kT+j}\right) du = \lim_{x\to 1^-} \sum_{k=0}^\infty \left(\frac{x^{kT+j}}{kT+j} - \frac{x^{kT+j+1}}{kT+j+1}\right).
\end{aligned}$$

The series in the last expression converges at $x = 1$, and thus it defines a continuous function in $[0, 1]$ and so the limit is

$$\int_0^1 \frac{u^j - u^{j-1}}{u^T-1} du = \sum_{k=0}^\infty \left(\frac{1}{kT+j} - \frac{1}{kT+j+1}\right). \qquad (7)$$

By (6), we now get that

$$\begin{aligned}
\operatorname{Ln} T &= \sum_{k=0}^\infty \left(\frac{1}{kT+1} - \frac{1}{kT+2}\right) + 2\sum_{k=0}^\infty \left(\frac{1}{kT+2} - \frac{1}{kT+3}\right) + \dots \\
&\quad + (T-2)\sum_{k=0}^\infty \left(\frac{1}{kT+T-1} - \frac{1}{kT+T-1}\right) + (T-1)\sum_{k=0}^\infty \left(\frac{1}{kT+T-1} - \frac{1}{(k+1)T}\right) \\
&= \sum_{k=0}^\infty \left(\frac{1}{kT+1} + \frac{1}{kT+2} + \dots + \frac{1}{kT+T-1} - \frac{(T-1)}{(k+1)T}\right),
\end{aligned}$$

and this is formula (3).

If we put $T = 3$, $j = 1$ into (7), we get that

$$\int_0^1 \frac{u-1}{u^3-1} du = \sum_{k=0}^\infty \left(\frac{1}{3k+1} - \frac{1}{3k+2}\right) = S_3(1,-1,0). \qquad (8)$$

On the other hand,

$$\int \frac{u-1}{u^3-1}du = \frac{2}{\sqrt{3}}\arctan\left(\frac{2u+1}{\sqrt{3}}\right),$$

and together with (8), this gives

$$\frac{2}{\sqrt{3}}\left(\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right) = S_3(1,-1,0)$$

or

$$\pi = 3\sqrt{3}\cdot S_3(1,-1,0) = 3\sqrt{3}\left[\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\dots\right].$$

References

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