Fluctuation bounds in the exponential bricklayers process

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Abstract

This paper is the continuation of our earlier paper [5], where we proved $t^{1/3}$-order of current fluctuations across the characteristics in a class of one dimensional interacting systems with one conserved quantity. We also claimed two models with concave hydrodynamic flux which satisfied the assumptions which made our proof work. In the present note we show that the *totally asymmetric exponential bricklayers process* also satisfies these assumptions. Hence this is the first example with *convex* hydrodynamics of a model with $t^{1/3}$-order current fluctuations across the characteristics. As such, it further supports the idea of universality regarding this scaling.

Keywords: Interacting particle systems, universal fluctuation bounds, $t^{1/3}$-scaling, second class particle, convexity, bricklayers process

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1 Introduction

It is conjectured that particle current through the characteristics of one dimensional stochastic interacting systems with one conserved quantity and concave or convex hydrodynamics show $t^{1/3}$-order fluctuations and Tracy-Widom type limit distributions in this order. Our earlier paper [5] provides a robust argument that proves this order of the fluctuations. We refer to that paper for the general framework and other results of the field. Very briefly, [5] works if one proves the following properties of a model (see the exact formulation therein):

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1. a strict domination of a second class particle of a denser system on one of a sparser system,
2. a non-strict, but tight, domination of a second class particle of a system on second class particles that are defined between the system in question and another system with a different density,
3. strictly concave or convex, in the second derivative sense, hydrodynamic flux function of the hyperbolic conservation law obtained by the Eulerian limiting procedure,
4. a tail bound of a second class particle in a(n essentially) stationary process.

Properties 1 and 2 form what we call the *microscopic concavity or convexity property*. Arguments in [5] are worked out for the concave setting, but everything works word-for-word in the convex case.

Two examples are claimed in [5]: the asymmetric simple exclusion process and a totally asymmetric zero range process with jump rates that increase with exponentially decaying slope. In this note we prove the above properties and hence the $t^{1/3}$ scaling for yet another system, the totally asymmetric exponential bricklayers process (TAEBLP). This model was introduced in [1], and its normal fluctuations off-characteristics were demonstrated in [2] (in case of general convex jump rates, not only exponential).

General convex increasing rates of a totally asymmetric bricklayers process allow couplings that prove properties 1 and 3 above. The exponential jump rates have a strong enough convexity property that will allow us to show property 2. To this order we repeat an argument somewhat similar to the one applied to the concave zero range process in [5].

Finally, property 4 is highly nontrivial when the jump rates have unbounded increments. We use a coupling based on property 1 and a recent result [4] that asserts that a second class particle of the exponential bricklayers process performs a simple (drifted) random walk under appropriate shock initial conditions. It is remarkable that exponential jump rates were also of fundamental importance in [4].

The case of exponential jump rates was constructed in [6]. The results of the note [7] are used by [5]. Those require strong construction results which are not provided by [6] and therefore, to our knowledge, are not available. To close that gap, we reproduce the results of [7] here for the TAEBLP.

The organization of this paper is the following: we repeat the introduction of the model, the fluctuation results and conclusions of [5], and the definition of the microscopic convexity property in Section 2. We construct the four process coupling and prove the microscopic convexity property in Section 3. Finally we show how to use [4] to prove property 4 in Section 4. The result of the note [7] is reproduced in the Appendix.

2 The model, properties and results

2.1 The model

The model we discuss is the totally asymmetric exponential bricklayers process (TAEBLP) introduced in [1], and also treated in [3] and [4]. The model is a member of the class in [5], here is a brief definition. The process describes the growth of a surface which we imagine as the top of a wall formed by columns of bricks over the interval $(i,\, i+1)$ for each pair of neighboring sites i and $i+1$ of $\mathbb{Z}$. The height h_i of this column is integer-valued. The components of a configuration $\underline{\omega} \in \Omega$ are the negative discrete gradients of the heights: $\omega_i = h_{i-1} - h_i \in \mathbb{Z}$. The configuration space is therefore

$$\Omega := \{\underline{\omega} = (\omega_i)_{i\in\mathbb{Z}} \ : \ \omega_i \in \mathbb{Z}\} = \mathbb{Z}^{\mathbb{Z}}.$$

At times it will be convenient to have notation for the increment configuration $\underline{\delta}_i \in \Omega$ with exactly one nonzero entry equal to 1:

$$(\underline{\delta}_i)_j = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j. \end{cases} \tag{2.1}$$

Bricklayers processes are characterized by a function $f : \mathbb{Z} \to \mathbb{R}^+$. We only consider the totally asymmetric version here, in which only deposition of bricks in the following way are allowed:

$$\left.\begin{aligned} (\omega_i,\, \omega_{i+1}) &\longrightarrow (\omega_i - 1,\, \omega_{i+1} + 1) \\ h_i &\longrightarrow h_i + 1 \end{aligned}\right\} \text{with rate } f(\omega_i) + f(-\omega_{i+1}). \tag{2.2}$$

Conditionally on the present state, these moves happen independently at all sites i. We can summarize this information in the formal infinitesimal generator L of the process $\underline{\omega}(\cdot)$:

$$(L\varphi)(\underline{\omega}) = \sum_{i\in\mathbb{Z}} [f(\omega_i) + f(-\omega_{i+1})] \cdot [\varphi(\dots,\, \omega_i - 1,\, \omega_{i+1} + 1,\, \dots) - \varphi(\underline{\omega})]\,. \tag{2.3}$$

L acts on bounded cylinder functions $\varphi : \Omega \to \mathbb{R}$ (this means that φ depends only on finitely many ω_i-values). The additive form of the rates gives rise to the bricklayers representation: at each site i stands a bricklayer who places a brick on the column on his left with rate $f(-\omega_i)$ and on the one on his right with rate $f(\omega_i)$.

Thus we have a Markov process $\{\underline{\omega}(t) : t \in \mathbb{R}^+\}$ of an evolving increment configuration and a Markov process $\{\underline{h}(t) : t \in \mathbb{R}^+\}$ of an evolving height configuration. The initial increments $\underline{\omega}(0)$ specify the initial height $\underline{h}(0)$ up to a vertical translation. We shall always normalize the height process so that $h_0(0) = 0$.

Attractivity of the process is essential for this paper. This is achieved by assuming that f is nondecreasing.

Finally, stationary translation-invariant product distributions for $\underline{\omega}(\cdot)$ are ensured by $f(z) \cdot f(1-z) = 1$ for each $z \in \mathbb{Z}$.

The totally asymmetric *exponential* bricklayers process (TAEBLP) is obtained by taking

$$f(z) = \mathrm{e}^{\beta(z-1/2)}. \tag{2.4}$$

The construction of the bricklayers process with any nondecreasing f that is bounded by an exponential function is given in [6] on a set of tempered configurations $\widetilde{\Omega}$. As certain desired semigroup properties are not fully proved, we avoid technical difficulties in the proofs of [7] by reproducing its results for the TAEBLP in the Appendix.

2.2 The basic coupling

We use a particularly simple form of the basic coupling which is made possible by the bricklayer representation: it is enough to define the structure of moves as described in [5] for a given side (left or right) of an individual bricklayer. Here is how to do it for a given bricklayer at site i. Given the present configurations $\underline{\omega}^1, \underline{\omega}^2, \ldots, \underline{\omega}^n \in \widetilde{\Omega}$, let $m \mapsto \ell(m)$ be a permutation that orders the ω_i values:

$$\omega_i^{\ell(m)} \le \omega_i^{\ell(m+1)}, \qquad 1 \le m < n.$$

For simplicity, set

$$p(m) := f(\omega_i^{\ell(m)}) \qquad \text{and} \qquad q(m) := f(-\omega_i^{\ell(m)}),$$

and the dummy variables $p(0) = q(n+1) = 0$. Recall that the function f is nondecreasing. Now the rule is that independently for each $m = 1, \ldots, n$, at rate $p(m) - p(m-1)$, precisely bricklayers of $\underline{\omega}^{\ell(m)}, \underline{\omega}^{\ell(m+1)}, \ldots, \underline{\omega}^{\ell(n)}$ place a brick on their right, and bricklayers of $\underline{\omega}^{\ell(1)}, \underline{\omega}^{\ell(2)}, \ldots, \underline{\omega}^{\ell(m-1)}$ do not. Independently, at rate $q(m) - q(m+1)$, precisely bricklayers of $\underline{\omega}^{\ell(1)}, \underline{\omega}^{\ell(2)}, \ldots, \underline{\omega}^{\ell(m)}$ place a brick on their left, and bricklayers of $\underline{\omega}^{\ell(m+1)}, \underline{\omega}^{\ell(m+2)}, \ldots, \underline{\omega}^{\ell(n)}$ do not. Given the configurations $\underline{\omega}^1, \underline{\omega}^2, \ldots, \underline{\omega}^n \in \widetilde{\Omega}$, bricklayers at different sites perform the above steps independently.

The combined effect of these joint rates creates the correct marginal rates, that is, the bricklayer of $\underline{\omega}^{\ell(m)}$ executes the move (2.2) with rate $p(m) = f(\omega_i^{\ell(m)})$, and the same move on column h_{i-1} with rate $q(m) = f(-\omega_i^{\ell(m)})$.

Notice also that, due to monotonicity of f, a jump of $\underline{\omega}^a$ without $\underline{\omega}^b$ on the column $[i,\, i{+}1]$ by the bricklayers at site i can only occur if $f(\omega_i^b) < f(\omega_i^a)$ which implies $\omega_i^a > \omega_i^b$. Also, a jump of $\underline{\omega}^a$ without $\underline{\omega}^b$ on the column $[i-1,\, i]$ by the bricklayers at site i can only occur if $f(-\omega_i^b) < f(-\omega_i^a)$ which implies $\omega_i^a < \omega_i^b$. The result of any of these steps then cannot increase the number of discrepancies between the two processes, hence the name *attractivity* for monotonicity of f. In particular, a sitewise ordering $\omega_i^a \le \omega_i^b \quad \forall i \in \mathbb{Z}$ is preserved by the basic coupling.

The differences between two processes are called *second class particles*. Their number is nonincreasing. In particular, if, for processes $\underline{\omega}^a$ and $\underline{\omega}^b$ we have $\omega_i^a \geq \omega_i^b$ for each $i \in \mathbb{Z}$, then the second class particles between them are conserved. A special case that is of key importance to us is the situation where only one second class particle is present between two processes.

2.3 Hydrodynamics and some exact identities

From now on, we restrict our attention to the TAEBLP. As described in [5], the process has product translation-invariant stationary distribution with marginals μ^θ that turn out to be of discrete Gaussian type, see [1] for the explicit formula. $\mathbf{P}^\theta$, $\mathbf{E}^\theta$, $\mathbf{Var}^\theta$, $\mathbf{Cov}^\theta$ will refer to laws of a process evolving in this stationary distribution. The density $\varrho(\theta) := \mathbf{E}^\theta(\omega) \in \mathbb{R}$ is a strictly increasing function of the parameter $\theta \in \mathbb{R}$, and can take on any real value by the Appendix of [5]. μ^ϱ, $\mathbf{P}^\varrho$, $\mathbf{E}^\varrho$, $\mathbf{Var}^\varrho$, $\mathbf{Cov}^\varrho$ will refer to laws of a density ϱ stationary process.

The hydrodynamic flux is

$$\mathcal{H}(\varrho) = \mathbf{E}^\varrho[f(\omega) + f(-\omega)] = \mathrm{e}^{\theta(\varrho)} + \mathrm{e}^{-\theta(\varrho)}.$$

As f (2.4) is convex and nonlinear, the Appendix of [5] applies and yields a convex hydrodynamic flux with

$$\mathcal{H}''(\varrho) > 0. \tag{2.5}$$

As in [5], we introduce

$$\widehat{\mu}^\varrho(y) := \frac{1}{\mathbf{Var}^\varrho(\omega_0)} \sum_{z=y+1}^{\infty} (z-\varrho)\mu^\varrho(z), \qquad y \in \mathbb{Z}. \tag{2.6}$$

The Appendix of [5] applies to show that both μ^ϱ and $\widehat{\mu}^\varrho$ are stochastically monotone in ϱ. Denote by $\mathbf{E}$ the expectation w.r.t. the evolution of a pair $(\underline{\omega}^-(\cdot), \underline{\omega}(\cdot))$ started with initial data (recall (2.1))

$$\underline{\omega}^-(0) = \underline{\omega}(0) - \underline{\delta}_0 \sim \Big(\bigotimes_{i\neq 0} \mu^\varrho\Big) \otimes \widehat{\mu}^\varrho, \tag{2.7}$$

and evolving under the basic coupling. This pair will always have a single second class particle whose position is denoted by $Q(t)$. In other words, $\underline{\omega}^-(t) = \underline{\omega}(t) - \underline{\delta}_{Q(t)}$. We reprove Corollaries 2.4 and 2.5 of [7] in the Appendix that state that for any $i \in \mathbb{Z}$ and $t \geq 0$,

$$\mathbf{Var}^\varrho(h_i(t)) = \mathbf{Var}^\varrho(\omega) \cdot \mathbf{E}|Q(t) - i| \tag{2.8}$$

and

$$\mathbf{E}(Q(t)) = V^\varrho \cdot t, \tag{2.9}$$

where $V^\varrho = \mathcal{H}'(\varrho)$ is the characteristic speed. Note in particular that in (2.8) the variances are taken in a stationary process, while the expectation of $Q(t)$ is taken in the coupling with initial distribution (2.7).

2.4 Results

We repeat the results of [5], valid now for the TAEBLP.

Theorem 2.1. *Fix any density $\varrho \in \mathbb{R}$, let the TAEBLP processes $(\underline{\omega}^-(t), \underline{\omega}(t))$ evolve in basic coupling with initial distribution (2.7) and let $Q(t)$ be the position of the second class particle between $\underline{\omega}^-(t)$ and $\underline{\omega}(t)$. Then there is a constant $C_1 = C_1(\varrho) \in (0, \infty)$ such that for all $1 \le m < 3$,*

$$\frac{1}{C_1} < \liminf_{t\to\infty} \frac{\mathbf{E}|Q(t) - V^\varrho t|^m}{t^{2m/3}} \le \limsup_{t\to\infty} \frac{\mathbf{E}|Q(t) - V^\varrho t|^m}{t^{2m/3}} < \frac{C_1}{3-m}. \tag{2.10}$$

Superdiffusivity of the second class particle is best seen with the choice $m = 2$: the variance of its position is of order $t^{4/3}$. Next some corollaries. Notation $\lfloor X \rfloor$ stands for the lower integer part of X.

Corollary 2.2 (Current variance)**.** *There is a constant $C_1 = C_1(\varrho) > 0$, such that*

$$\frac{1}{C_1} < \liminf_{t\to\infty} \frac{\mathbf{Var}^\varrho(h_{\lfloor V^\varrho t\rfloor}(t))}{t^{2/3}} \le \limsup_{t\to\infty} \frac{\mathbf{Var}^\varrho(h_{\lfloor V^\varrho t\rfloor}(t))}{t^{2/3}} < C_1.$$

Corollary 2.3 (Weak Law of Large Numbers for the second class particle)**.** *In a density-ϱ stationary process,*

$$\frac{Q(t)}{t} \xrightarrow{d} V^\varrho. \tag{2.11}$$

Corollary 2.4 (Dependence of current on the initial configuration)**.** *For any $V \in \mathbb{R}$ and $\alpha > 1/3$ the following limit holds in the L^2 sense for a density-ϱ stationary process:*

$$\lim_{t\to\infty} \frac{h_{\lfloor Vt\rfloor}(t) - h_{\lfloor Vt\rfloor - \lfloor V^\varrho t\rfloor}(0) - t(\mathcal{H}(\varrho) - \varrho\mathcal{H}'(\varrho))}{t^\alpha} = 0. \tag{2.12}$$

Recall that

$$h_{\lfloor Vt\rfloor - \lfloor V^\varrho t\rfloor}(0) = \begin{cases} \sum_{i=\lfloor Vt\rfloor - \lfloor V^\varrho t\rfloor + 1}^{0} \omega_i(0), & \text{if } V < V^\varrho, \\ 0, & \text{if } V = V^\varrho, \\ -\sum_{i=1}^{\lfloor Vt\rfloor - \lfloor V^\varrho t\rfloor} \omega_i(0), & \text{if } V > V^\varrho \end{cases} \tag{2.13}$$

only depends on a finite segment of the initial configuration. Limit (2.12) shows that on the diffusive time scale $t^{1/2}$ only fluctuations from the initial distribution are visible: these fluctuations are translated rigidly at the characteristic speed V^ϱ.

Corollary 2.5 (Central Limit Theorem for the current)**.** *For any $V \in \mathbb{R}$ in a density-ϱ stationary process*

$$\lim_{t\to\infty} \frac{\mathbf{Var}^\varrho(h_{\lfloor Vt\rfloor}(t))}{t} = \mathbf{Var}^\varrho(\omega) \cdot |V^\varrho - V| =: D, \tag{2.14}$$

and the Central Limit Theorem also holds: the centered and normalized height $\widetilde{h}_{\lfloor Vt\rfloor}(t)/\sqrt{t \cdot D}$ converges in distribution to a standard normal.

For convex rate zero range and bricklayers processes Corollaries 2.3 and 2.5 were proved by M. Balázs [2].

The way we obtain the Theorem and the corollaries is simply proving that the assumptions formulated in [5] hold. For the sake of completeness, we repeat these, translated to our convex case.

2.5 Microscopic convexity

We start with the definition of *microscopic convexity.* This is just translated from the microscopic concavity property of [5], where more detailed explanations and comments can be found. Convexity (2.5) implies that the characteristic speed $V^\varrho = \mathcal{H}'(\varrho)$ is a nondecreasing function of the density ϱ:

$$\lambda < \varrho \implies V^\lambda \le V^\varrho. \tag{2.15}$$

The microscopic counterpart of a characteristic is the motion of a second class particle. Our key assumption that we term microscopic convexity is that the ordering (2.15) can also be realized at the particle level as an ordering between two second class particles introduced into two processes at densities λ and ϱ.

Let $\lambda < \varrho$ be two densities. Define $\widehat{\mu}^\varrho + 1$ as the measure that gives weight $\widehat{\mu}^\varrho(z-1)$ to an integer $z \in \mathbb{Z}$ (recall (2.6)). By the stochastic domination $\widehat{\mu}^\lambda \le \widehat{\mu}^\varrho$, we can let $\widehat{\mu}^{\lambda,\varrho}$ be a coupling measure with marginals $\widehat{\mu}^\lambda$ and $\widehat{\mu}^\varrho + 1$ and with the property

$$\widehat{\mu}^{\lambda,\varrho}\{(y,\, z) \,:\, y < z\} = 1. \tag{2.16}$$

Let also $\mu^{\lambda,\varrho}$ be a coupling measure of site-marginals μ^λ and μ^ϱ of the invariant distributions, with

$$\mu^{\lambda,\varrho}\{(y,\, z) \,:\, y \le z\} = 1. \tag{2.17}$$

Note the distinction that under $\widehat{\mu}^{\lambda,\varrho}$ the second coordinate is strictly above the first.

To have notation for inhomogeneous product measures on $\mathbb{Z}^\mathbb{Z}$, let $\underline{\lambda} = (\lambda_i)_{i\in\mathbb{Z}}$ and $\underline{\varrho} = (\varrho_i)_{i\in\mathbb{Z}}$ denote sequences of density values, with λ_i and ϱ_i assigned to site i. The product distribution with marginals $\widehat{\mu}^{\lambda_0,\varrho_0}$ at the origin and $\mu^{\lambda_i,\varrho_i}$ at other sites is denoted by

$$\underline{\widehat{\mu}}^{\underline{\lambda},\underline{\varrho}} := \Big(\bigotimes_{i\ne 0} \mu^{\lambda_i,\varrho_i}\Big) \otimes \widehat{\mu}^{\lambda_0,\varrho_0}. \tag{2.18}$$

Measure $\underline{\widehat{\mu}}^{\lambda,\varrho}$ gives probability one to the event

$$\{(\underline{\eta}(0),\, \underline{\omega}(0)) \,:\, \eta_0(0) < \omega_0(0), \text{ and } \eta_i(0) \le \omega_i(0) \text{ for } 0 \ne i \in \mathbb{Z}\}. \tag{2.19}$$

The initial configuration $(\underline{\eta}(0), \underline{\omega}(0))$ will always be assumed a member of this set, and the pair process $(\underline{\eta}(t), \underline{\omega}(t))$ evolves in basic coupling. Notice that $\underline{\widehat{\mu}}^{\lambda,\varrho}$ is in general *not* stationary for this joint evolution.

The discrepancies between these two processes are called the $\omega - \eta$ *(second class) particles*. The number of such particles at site i at time t is $\omega_i(t) - \eta_i(t)$. In the basic coupling the $\omega - \eta$ particles are conserved, in the sense that none are created or annihilated. We label the $\omega - \eta$ particles with integers, and let $X_m(t)$ denote the position of particle m at time t. The initial labeling is chosen to satisfy

$$\cdots \le X_{-1}(0) \le X_0(0) = 0 < X_1(0) \le \cdots . \tag{2.20}$$

We can specify that $X_0(0) = 0$ because under $\underline{\widehat{\mu}}^{\lambda,\varrho}$ there is an $\omega - \eta$ particle at site 0 with probability 1. During the evolution we keep the positions $X_i(t)$ of the $\omega - \eta$ particles ordered. To achieve this we stipulate that

(2.21)
whenever an $\omega - \eta$ particle jumps from a site,
if the jump is to the right the highest label moves,
and if the jump is to the left the lowest label moves.

Here is the precise form of microscopic convexity for this paper. The assumption states that a certain joint construction of processes (that is, a coupling) can be performed for a range of densities in a neighborhood of a fixed density ϱ. Recall (2.1) for the definition of the configuration $\underline{\delta}$.

Assumption 2.6. *Given a density $\varrho \in \mathbb{R}$, there exists $\gamma_0 > 0$ such that the following holds. For any $\underline{\lambda}$ and $\underline{\varrho}$ such that $\varrho - \gamma_0 \le \lambda_i \le \varrho_i \le \varrho + \gamma_0$ for all $i \in \mathbb{Z}$, a joint process $(\underline{\eta}(t), \underline{\omega}(t), y(t), z(t))_{t \ge 0}$ can be constructed with the following properties.*

- *Initially $(\underline{\eta}(0), \underline{\omega}(0))$ is $\underline{\widehat{\mu}}^{\lambda,\varrho}$-distributed and the joint process $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$ evolves in basic coupling.*
- *Processes $y(\cdot)$ and $z(\cdot)$ are integer-valued. Initially $y(0) = z(0) = 0$. With probability one*

$$y(t) \ge z(t) \text{ for all } t \ge 0. \tag{2.22}$$

- *Define the processes*

$$\underline{\omega}^-(t) := \underline{\omega}(t) - \underline{\delta}_{X_{y(t)}(t)} \quad \text{and} \quad \underline{\eta}^+(t) := \underline{\eta}(t) + \underline{\delta}_{X_{z(t)}(t)}. \tag{2.23}$$

Then both pairs $(\underline{\eta}, \underline{\eta}^+)$ and $(\underline{\omega}^-, \underline{\omega})$ evolve marginally in basic coupling.

- *For each $\gamma \in (0, \gamma_0)$ and large enough $t \geq 0$ there exists a probability distribution $\nu^{\varrho,\gamma}(t)$ on $\mathbb{Z}^+$ satisfying the tail bound*

$$\nu^{\alpha,\gamma}(t)\{y : y \geq y_0\} \leq Ct^{\kappa-1}\gamma^{2\kappa-3}y_0^{-\kappa} \tag{2.24}$$

for some fixed constants $3/2 \leq \kappa < 3$ and $C < \infty$, and such that if $\varrho - \gamma \leq \lambda_i \leq \varrho_i \leq \varrho + \gamma$ for all $i \in \mathbb{Z}$, then we have the stochastic bounds

$$y(t) \overset{d}{\geq} -\nu^{\varrho,\gamma}(t) \quad \textit{and} \quad z(t) \overset{d}{\leq} \nu^{\varrho,\gamma}(t). \tag{2.25}$$

Let us clarify some of the details in this assumption.

Equation (2.23) says that $Q^\eta(t) := X_{z(t)}(t)$ is the single second class particle between $\underline{\eta}$ and $\underline{\eta}^+$, while $Q(t) := X_{y(t)}(t)$ is the one between $\underline{\omega}^-$ and $\underline{\omega}$. The first three bullets say that it is possible to construct jointly four processes $(\underline{\eta}, \underline{\eta}^+, \underline{\omega}^-, \underline{\omega})$ with the specified initial conditions and so that each pair $(\underline{\eta}, \underline{\omega})$, $(\underline{\eta}, \underline{\eta}^+)$ and $(\underline{\omega}^-, \underline{\omega})$ has the desired marginal distribution, and most importantly so that

$$Q^\eta(t) = X_{z(t)}(t) \leq X_{y(t)}(t) = Q(t). \tag{2.26}$$

This is a consequence of (2.22) because the $\omega - \eta$ particles $X_i(t)$ stay ordered.

The tail bound (2.24) is formulated in this somewhat complicated fashion because this appears to be the weakest form our present proof allows. For the TAEBLP $\nu^{\varrho,\gamma}(t)$ will actually be a fixed geometric distribution.

The assumptions made imply $\underline{\eta}(t) \leq \underline{\omega}(t)$ a.s., and by (2.23)

$$\underline{\eta}(t) \leq \underline{\eta}^+(t) \leq \underline{\omega}(t) \quad \text{and} \quad \underline{\eta}(t) \leq \underline{\omega}^-(t) \leq \underline{\omega}(t) \quad \text{a.s.}$$

In our actual construction for the TAEBLP it turns out that while the triples $(\underline{\eta}, \underline{\eta}^+, \underline{\omega})$ and $(\underline{\eta}, \underline{\omega}^-, \underline{\omega})$ evolve also in basic coupling, the full joint evolution $(\underline{\eta}, \underline{\eta}^+, \underline{\omega}^-, \underline{\omega})$ does not.

As already explained, the microscopic convexity idea is contained in inequality (2.22). There is also a sense in which the tail bounds (2.25) relate to convexity of the flux. Consider the situation $\lambda_i \equiv \lambda < \varrho \equiv \varrho_i$. We would expect the $\omega - \eta$ particle $X_0(\cdot)$ to have average and long-term velocity

$$R(\lambda, \varrho) = \frac{\mathcal{H}(\varrho) - \mathcal{H}(\lambda)}{\varrho - \lambda},$$

the Rankine-Hugoniot or shock speed. By convexity $\mathcal{H}'(\varrho) = V^\varrho \geq R(\lambda, \varrho) \geq V^\lambda = \mathcal{H}'(\lambda)$. A strict microscopic counterpart would be $y(t) \geq 0 \geq z(t)$. But this condition is overly restrictive. The distributional bounds (2.25) are the natural relaxations we use.

Section 3 contains the proof of Assumption 2.6 for the TAEBLP. The proof of (2.25) makes use of the particular exponential form (2.4) of the rates. Unfortunately, we do not have an argument for more general convex rates at the moment.

There is one more assumption in [5] needed to state the main result. Constants $C_{\cdot}$, $\alpha_{\cdot}$ will not depend on time, but might depend on the density parameter ϱ, and their values can change from line to line.

Assumption 2.7. *Let* $(\underline{\omega}^-, \underline{\omega})$ *be a pair of processes in basic coupling, started from distribution* (2.7)*, with second class particle* $Q(t)$*. Then there exist constants* $0 < \alpha_0$, $C < \infty$ *such that*

$$\mathbf{P}\{|Q(t)| > K\} \leq C \cdot \frac{t^2}{K^3} \tag{2.27}$$

whenever $K > \alpha_0 t$ *and* t *is large enough.*

Such an assumption is natural and easy to prove if the jump rates have bounded increments. Since f (2.4) does not, this statement for the TAEBLP is nontrivial. We prove it in Section 4 for the TAEBLP.

3 Proof of microscopic convexity

In this section we verify that Assumption 2.6 can be satisfied. The task is to construct the processes $y(t)$ and $z(t)$ with the requisite properties. First let the processes $(\underline{\eta}(\cdot), \underline{\omega}(\cdot))$ evolve in the basic coupling so that $\eta_i(t) \leq \omega_i(t)$ for all $i \in \mathbb{Z}$ and $t \geq 0$. We consider as a background process this pair with the labeled and ordered $\omega - \eta$ second class particles $\cdots \leq X_{-2}(t) \leq X_{-1}(t) \leq X_0(t) \leq X_1(t) \leq X_2(t) \leq \cdots$.

At each time $t \geq 0$ this background induces a partition $\{\mathcal{M}_i(t)\}$ of the label space $\mathbb{Z}$ into intervals indexed by sites $i \in \mathbb{Z}$, with partition intervals given by

$$\mathcal{M}_i(t) := \{m : X_m(t) = i\}.$$

(For simplicity we assumed infinitely many second class particles in both directions, but no problem arises in case we only have finitely many of them.) $\mathcal{M}_i(t)$ contains the labels of the second class particles that reside at site i at time t, and can be empty. The labels of the second class particles that are at the same site as the one labeled m form the set $\mathcal{M}_{X_m(t)}(t) =: \{a^m(t), a^m(t)+1, \ldots, b^m(t)\}$. The processes $a^m(t)$ and $b^m(t)$ are always well-defined and satisfy $a^m(t) \leq m \leq b^m(t)$. Notice that

$$|\mathcal{M}_{X_m(t)}(t)| = b^m(t) - a^m(t) + 1 = \omega_{X_m(t)}(t) - \eta_{X_m(t)}(t). \tag{3.1}$$

Let us clarify these notions by discussing the ways in which $a^m(t)$ and $b^m(t)$ can change.

- A second class particle jumps from site $X_m(t-)-1$ to site $X_m(t-)$. Then this one necessarily has label $a^m(t-)-1$, and it becomes the lowest labeled one at site $X_m(t-) = X_m(t)$ after the jump. Hence $a^m(t) = a^m(t-) - 1$.
- A second class particle jumps from site $X_m(t-)+1$ to site $X_m(t-)$. Then this one necessarily has label $b^m(t-)+1$, and it becomes the highest labeled one at site $X_m(t-) = X_m(t)$ after the jump. Hence $b^m(t) = b^m(t-) + 1$.
- A second class particle, different from X_m, jumps from site $X_m(t-)$ to site $X_m(t-)+1$. Then this one is necessarily labeled $b^m(t-)$, and it leaves the site $X_m(t-)$, hence $b^m(t) = b^m(t-) - 1$.

- A second class particle, different from X_m, jumps from site $X_m(t-)$ to site $X_m(t-)-1$. Then this one is necessarily labeled $a^m(t-)$, and it leaves the site $X_m(t-)$, hence $a^m(t) = a^m(t-)+1$.
- The second class particle X_m is the highest labeled on its site, that is, $m = b^m(t-)$, and it jumps to site $X_m(t-)+1$. Then this particle becomes the lowest labeled in the set $\mathcal{M}_{X_m(t-)+1} = \mathcal{M}_{X_m(t)}$, hence $a^m(t) = m$. In this case $b^m(t)$ can be computed from (3.1), the number of second class particles at the site of X_m after the jump.
- The second class particle X_m is the lowest labeled on its site, that is, $m = a^m(t-)$, and it jumps to site $X_m(t-)-1$. Then this particle becomes the highest labeled in the set $\mathcal{M}_{X_m(t-)-1} = \mathcal{M}_{X_m(t)}$, hence $b^m(t) = m$. In this case $a^m(t)$ can be computed from (3.1), the number of second class particles at the site of X_m after the jump.

We fix initially $y(0) = z(0) = 0$. The evolution of (y, z) is superimposed on the background evolution $(\underline{\eta}, \underline{\omega}, \{X_m\})$ following the general rule below: Immediately after every move of the background process that involves the site where y resides before this move, y picks a new value from the labels on the site where it resides after the move. Thus y itself jumps only within partition intervals $\mathcal{M}_i$. But y joins a new partition interval whenever it is the highest X-label on its site and its "carrier" particle X_y is forced to move to the next site on the right, or it is the lowest X-label on its site and its "carrier" particle X_y is forced to move to the next site on the left.

These are the situations when $y(t-) = b^{y(t-)}(t-)$ and at time t an $\omega-\eta$ move from this site to the right happens, or $y(t-) = a^{y(t-)}(t-)$ and at time t an $\omega-\eta$ move from this site to the left happens. (Recall that the choice of X-particle to move is determined by rule (2.21).) All this works for z in exactly the same way.

Next we specify the probabilities that y and z use to refresh their values. Recall (2.4). To simplify notation, we abbreviate, given integers $\eta < \omega$,

(3.2)
$$p(\eta, \omega) = \frac{f(\omega) - f(\omega-1)}{f(\omega) - f(\eta)} = \frac{f(-\eta) - f(-\eta-1)}{f(-\eta) - f(-\omega)} = \frac{\mathrm{e}^{\beta(\omega-\eta)} - \mathrm{e}^{\beta(\omega-\eta-1)}}{\mathrm{e}^{\beta(\omega-\eta)} - 1}$$

and

(3.3)
$$q(\eta, \omega) = \frac{f(-\omega+1) - f(-\omega)}{f(-\eta) - f(-\omega)} = \frac{f(\eta+1) - f(\eta)}{f(\omega) - f(\eta)} = \frac{\mathrm{e}^{\beta} - 1}{\mathrm{e}^{\beta(\omega-\eta)} - 1}.$$

Notice that both $p(\eta, \omega)$ and $q(\eta, \omega)$ only depend on $\omega - \eta$. Therefore, with a little abuse of notation, we write $p(\omega-\eta) := p(\eta, \omega)$, $q(\omega-\eta) := q(\eta, \omega)$. Then

$$p(1) = q(1) = 1, \qquad p(d) \geq q(d), \qquad p(d) + q(d) \leq 1 \qquad \text{for } 2 \leq d \in \mathbb{Z}.$$

When y and z reside at separate sites, they refresh independently. When they are together in the same partition interval, they use the joint distribution in the third bullet below.

- Whenever any change occurs in either $\underline{\omega}$ or $\underline{\eta}$ at site $X_{y(t-)}(t-)$ and, as a result of the jump, $a^{y(t-)}(t) \neq a^{z(t-)}(t)$, that is, $y(t-)$ and $z(t-)$ belong to different parts *after* the jump, we abbreviate

$$p = p(\eta_{X_{y(t-)}(t)}(t),\, \omega_{X_{y(t-)}(t)}(t)) \qquad q = q(\eta_{X_{y(t-)}(t)}(t),\, \omega_{X_{y(t-)}(t)}(t))$$

  of (3.2) and (3.3) in the formulas below. These depend on the values of the respective processes at the site where the label y can be found right after the jump. In this case, independently of everything else,

$$y(t) := \begin{cases} a^{y(t-)}(t), & \text{with prob. } q, \\ b^{y(t-)}(t) - 1, & \text{with prob. } 1 - p - q, \\ b^{y(t-)}(t), & \text{with prob. } p, \end{cases} \tag{3.4}$$

  except for $y(t) := a^{y(t-)}(t) = b^{y(t-)}(t)$ when the difference $\omega_{X_{y(t-)}(t)}(t) - \eta_{X_{y(t-)}(t)}(t)$ is 1. Notice that the second line has probability zero when this difference is 2.

- Whenever any change occurs in either $\underline{\omega}$ or $\underline{\eta}$ at site $X_{z(t-)}(t-)$ and, as a result of the jump, $a^{y(t-)}(t) \neq a^{z(t-)}(t)$, that is, $y(t-)$ and $z(t-)$ belong to different parts *after* the jump, we abbreviate

$$p = p(\eta_{X_{z(t-)}(t)}(t),\, \omega_{X_{z(t-)}(t)}(t)) \qquad q = q(\eta_{X_{z(t-)}(t)}(t),\, \omega_{X_{z(t-)}(t)}(t))$$

  of (3.2) and (3.3) in the formulas below. These depend on the values of the respective processes at the site where the label z can be found right after the jump. In this case, independently of everything else,

$$z(t) := \begin{cases} a^{z(t-)}(t), & \text{with prob. } p, \\ a^{z(t-)}(t) + 1, & \text{with prob. } 1 - p - q, \\ b^{z(t-)}(t), & \text{with prob. } q, \end{cases} \tag{3.5}$$

  except for $z(t) := a^{z(t-)}(t) = b^{z(t-)}(t)$ when the difference $\omega_{X_{z(t-)}(t)}(t) - \eta_{X_{z(t-)}(t)}(t)$ is 1. Notice that the second line has probability zero when this difference is 2.

- Whenever any change occurs in either $\underline{\omega}$ or $\underline{\eta}$ at sites $X_{y(t-)}(t-)$ or $X_{z(t-)}(t-)$ and, as a result of the jump, $a^{y(t-)}(t) = a^{z(t-)}(t)$, that is, $y(t-)$ and $z(t-)$ belong to the same part *after* the jump, that is, $X_{y(t-)}(t) = X_{z(t-)}(t)$ then we have

$$\omega_{X_{y(t-)}(t)}(t) = \omega_{X_{z(t-)}(t)}(t) \qquad \text{and} \qquad \eta_{X_{y(t-)}(t)}(t) = \eta_{X_{z(t-)}(t)}(t),$$

and we abbreviate

$$p = p(\eta_{X_{y(t-)}(t)}(t),\, \omega_{X_{y(t-)}(t)}(t)) \qquad q = q(\eta_{X_{y(t-)}(t)}(t),\, \omega_{X_{y(t-)}(t)}(t))$$

of (3.2) and (3.3) in the formulas below. These depend on the values of the respective processes at the site where both the labels y and z can be found right after the jump. In this case, independently of everything else,

$$(3.6)\qquad \begin{pmatrix} y(t) \\ z(t) \end{pmatrix} := \begin{cases} \begin{pmatrix} a^{y(t-)}(t) \\ a^{y(t-)}(t) \end{pmatrix}, \text{ with prob. } q, \\ \begin{pmatrix} b^{y(t-)}(t)-1 \\ a^{y(t-)}(t) \end{pmatrix}, \text{ with prob. } (p-q)\wedge(1-p-q), \\ \begin{pmatrix} b^{y(t-)}(t) \\ a^{y(t-)}(t) \end{pmatrix}, \text{ with prob. } [2p-1]^+, \\ \begin{pmatrix} b^{y(t-)}(t)-1 \\ a^{y(t-)}(t)+1 \end{pmatrix}, \text{ with prob. } [1-2p]^+, \\ \begin{pmatrix} b^{y(t-)}(t) \\ a^{y(t-)}(t)+1 \end{pmatrix}, \text{ with prob. } (p-q)\wedge(1-p-q), \\ \begin{pmatrix} b^{y(t-)}(t) \\ b^{y(t-)}(t) \end{pmatrix}, \text{ with prob. } q, \end{cases}$$

except for $y(t) = z(t) := a^{y(t-)}(t) = b^{y(t-)}(t)$ when the difference $\omega_{X_{y(t-)}(t)}(t) - \eta_{X_{y(t-)}(t)}(t)$ is 1. Notice that the second, the fourth and the fifth lines have probability zero when this difference is 2.

The above moves for y and z always occur within labels at a given site. This determines whether the particle $Q(t) := X_{y(t)}(t)$ or $Q^\eta(t) := X_{z(t)}(t)$ is the one to jump if the next move out of the site is an $\omega - \eta$ move.

We prove that the above construction has the properties required in Assumption 2.6. First note that the refreshing rule (3.6) marginally gives the same moves and probabilities as (3.4) or (3.5) for $y(\cdot)$ or $z(\cdot)$, respectively.

Lemma 3.1. *The pair* $(\underline{\omega}^-, \underline{\omega}) := (\underline{\omega} - \underline{\delta}_{X_y}, \underline{\omega})$ *obeys basic coupling, as does the pair* $(\underline{\eta}, \underline{\eta}^+) := (\underline{\eta}, \underline{\eta} + \underline{\delta}_{X_z})$.

Proof. We write the proof for $(\underline{\omega}^-, \underline{\omega})$. We need to show that, given the configuration $(\underline{\eta}, \underline{\omega}, \{X_m\}, y)$, the jump rates of $(\underline{\omega}^-, \underline{\omega})$ are the ones prescribed in basic coupling (Section 2.2) and by (2.2). As mentioned in Section 2.2, the effect of bricklayers determine the evolution of processes. Notice first that an $\omega - \eta$ particle can only jump away from a site i if a bricklayer of ω or η moves. As the moves (3.4) or (3.6) by themselves never result in a change of $X_{y(\cdot)}(\cdot)$, any move of Q from a site i is a result of a bricklayer's move at site i. Therefore, we see that moves initiated by bricklayers of $\underline{\omega}$ at sites $i \neq Q$ happen as well to $\underline{\omega}^-$, as required by the basic coupling. The only point to consider is moves by the bricklayers at site $i = Q$. We start with them putting a brick on their right.

Since the last time any change occurred at site i, y chose values according to (3.4) or (3.6). Notice that (3.4) and (3.6) give the same marginal probabilities for this choice. Hence

(3.7)
$$y \text{ took a value less than } b^y \text{ with probability} \quad 1-p = \frac{f(\omega_i - 1) - f(\eta_i)}{f(\omega_i) - f(\eta_i)}$$

and

$$y \text{ took on value } b^y \text{ with probability} \quad p = \frac{f(\omega_i) - f(\omega_i - 1)}{f(\omega_i) - f(\eta_i)}, \tag{3.8}$$

as given in (3.4). Notice that (3.7) happens with probability zero if $\omega_i = \eta_i + 1$. According to the basic coupling of $\underline{\eta}$ and $\underline{\omega}$, the following right moves of bricklayers at i can occur:

- With rate $f(\omega_i) - f(\eta_i)$, $\underline{\omega}$ jumps without $\underline{\eta}$. The highest labeled second class particle, X_{b^y} jumps from site i to site $i+1$.
  - With probability (3.8) $X_y = Q$ jumps with X_{b^y}. In this case
  $$\omega_i^-(t-) = \omega_i(t-) - 1 = \omega_i(t) = \omega_i^-(t)$$
  since the difference Q disappears from site i. Also,
  $$\omega_{i+1}^-(t-) = \omega_{i+1}(t-) = \omega_{i+1}(t) - 1 = \omega_{i+1}^-(t),$$
  since the difference Q appears at site $i+1$. So in this case $\underline{\omega}$ undergoes a jump but $\underline{\omega}^-$ does not, and the rate is
  $$[f(\omega_i) - f(\eta_i)] \cdot \frac{f(\omega_i) - f(\omega_i - 1)}{f(\omega_i) - f(\eta_i)} = f(\omega_i) - f(\omega_i^-).$$
  - With probability (3.7) $X_y = Q$ does not jump with X_{b^y}, since it has label less than b^y (this probability is zero if $\omega_i = \eta_i + 1$). In this case $\underline{\omega}^-$ and $\underline{\omega}$ perform the same jump and it occurs with rate
  $$[f(\omega_i) - f(\eta_i)] \cdot \frac{f(\omega_i - 1) - f(\eta_i)}{f(\omega_i) - f(\eta_i)} = f(\omega_i^-) - f(\eta_i).$$
- With rate $f(\eta_i)$, both bricklayers of $\underline{\eta}$ and $\underline{\omega}$ at site i move. No change occurs in the $\omega - \eta$ particles, hence no change occurs in Q. This implies that the process $\underline{\omega}^-$ jumps as well.

Summarizing we see that the rate for the bricklayers of $(\underline{\omega}^-, \underline{\omega})$ at site i to lay brick on their rights together is $f(\omega_i^-)$, and the rate for the one of $\underline{\omega}$ to move without $\underline{\omega}^-$ is $f(\omega_i) - f(\omega_i^-)$. This is exactly what basic coupling requires.

Consider now bricklayers at site $i = Q$ putting a brick on their left. Since the last time any change occurred at site i, y chose values according to (3.4) or (3.6). Hence

$$(3.9) \qquad y \text{ took on value } a^y \text{ with probability} \quad q = \frac{f(-\omega_i + 1) - f(-\omega_i)}{f(-\eta_i) - f(-\omega_i)}$$

and

$$(3.10) \qquad y \text{ took a value higher than } a^y \text{ with probability} \quad 1 - q = \frac{f(-\eta_i) - f(-\omega_i + 1)}{f(-\eta_i) - f(-\omega_i)},$$

as given in (3.4). Notice that (3.10) happens with probability zero if $\omega_i = \eta_i + 1$. According to the basic coupling of $\underline{\eta}$ and $\underline{\omega}$, the following left moves of bricklayers at i can occur:

- With rate $f(-\eta_i) - f(-\omega_i)$, $\underline{\eta}$ jumps without $\underline{\omega}$. The lowest labeled second class particle, X_{a^y} jumps from site i to site $i - 1$.
  - With probability (3.9) $X_y = Q$ jumps with X_{a^y}. In this case
    $$\omega_i^-(t-) = \omega_i(t-) - 1 = \omega_i(t) - 1 = \omega_i^-(t) - 1$$
    since the difference Q disappears from site i. Also,
    $$\omega_{i-1}^-(t-) = \omega_{i-1}(t-) = \omega_{i-1}(t) = \omega_{i-1}^-(t) + 1,$$
    since the difference Q appears at site $i + 1$. So in this case $\underline{\omega}^-$ undergoes a jump but $\underline{\omega}$ does not, and the rate is
    $$[f(-\eta_i) - f(-\omega_i)] \cdot \frac{f(-\omega_i + 1) - f(-\omega_i)}{f(-\eta_i) - f(-\omega_i)} = f(-\omega_i^-) - f(-\omega_i).$$
  - With probability (3.10) $X_y = Q$ does not jump with X_{a^y}, since it has label more than a^y (this probability is zero if $\omega_i = \eta_i + 1$). In this case none of $\underline{\omega}^-$ or $\underline{\omega}$ move; this occurs with rate
    $$[f(-\eta_i) - f(-\omega_i)] \cdot \frac{f(-\eta_i) - f(-\omega_i + 1)}{f(-\eta_i) - f(\omega_i)} = f(-\eta_i) - f(-\omega_i^-).$$
- With rate $f(-\omega_i)$, both bricklayers of $\underline{\eta}$ and $\underline{\omega}$ at site i move. No change occurs in the $\omega - \eta$ particles, hence no change occurs in Q. This implies that the process $\underline{\omega}^-$ jumps as well.

Summarizing we see that the rate for the bricklayers of $(\underline{\omega}^-, \underline{\omega})$ at site i to lay brick on their rights together is $f(-\omega_i)$, and the rate for the one of $\underline{\omega}^-$ to move without $\underline{\omega}$ is $f(-\omega_i^-) - f(-\omega_i)$. This is exactly what basic coupling requires.

A very similar argument can be repeated for $(\underline{\eta}, \underline{\eta}^+)$. □

Lemma 3.2. *Inequality* (2.22) $y \geq z$ *holds in the above construction.*

Proof. Since no jump of y or z moves one of them into a new partition interval, the only situation that can jeopardize (2.22) is the simultaneous refreshing of y and z in a common partition interval. But this case is governed by step (3.6) which by definition ensures that $y \geq z$. (When $b^{y(t-)}(t) = a^{y(t-)}(t) + 1$, we have, by (3.1), $\omega_{X_{y(t-)}(t)}(t) - \eta_{X_{y(t-)}(t)}(t) = 2$, and hence p of (3.6) is more than $1/2$. Therefore the probability of the step in line 4 of (3.6) is zero.) $\square$

Define the geometric distribution

$$\nu(m) := \begin{cases} \mathrm{e}^{-\beta m}(1 - \mathrm{e}^{-\beta}), & m \geq 0 \\ 0, & m < 0. \end{cases} \tag{3.11}$$

Lemma 3.3. *Conditioned on the process* $(\underline{\eta}, \underline{\omega})$, *the bounds* $y(t) \overset{d}{\geq} -\nu$ *and* $z(t) \overset{d}{\leq} \nu$ *hold for all* $t \geq 0$.

To avoid unnecessary complications with negative values, we show the proof for $z(t)$. Notice that both the statement and the behavior of $y(t)$ is reflected compared to $z(t)$, hence the proof is the same for the two processes. The argument consists of three steps.

Lemma 3.4. *Let* Z *be a random variable with distribution* ν, *and fix integers* $a \leq b$ *and* $\eta < \omega$ *so that* $\omega - \eta = b - a + 1$. *Apply the following operation to* Z*:*

(i) *if* $a \leq Z \leq b$, *apply the probabilities from* (3.5) *with parameters* a, b, η, ω *to pick a new value for* Z*;*

(ii) *if* $Z < a$ *or* $Z > b$ *then do not change* Z.

Then the resulting distribution ν^* *is stochastically dominated by* ν.

Proof. There is nothing to prove when $b = a$, hence we assume $b > a$ or, equivalently, $\omega - \eta = b - a + 1 \geq 2$. It is also clear that $\nu^*(m) = \nu(m)$ for $m < a$ or $m > b$. We need to prove, in view of the distribution functions,

$$\sum_{\ell=a}^{m} \nu^*(\ell) \geq \sum_{\ell=a}^{m} \nu(\ell)$$

for all $a \leq m \leq b$. Notice that ν^* gives zero weight on values $a + 1 < m < b$ (if any), and also that the display becomes an equality if $m = b$. Therefore, it is enough to prove the inequality for $m = a$:

$$\nu^*(a) \geq \nu(a), \tag{3.12}$$

and $m = b - 1$:

$$\sum_{\ell=a}^{b-1} \nu^*(\ell) \geq \sum_{\ell=a}^{b-1} \nu(\ell) \qquad \text{that is,} \qquad \nu^*(b) \leq \nu(b). \tag{3.13}$$

Notice that (3.12) is trivially true for $a < 0$. For $a \geq 0$ we start with rewriting the left hand-side of (3.12) with the use of (3.5), (3.2), and the abbreviation $d = \omega - \eta = b - a + 1$:

$$\begin{aligned}
\nu^*(a) &= p(d) \cdot \sum_{\ell=a}^{b} \nu(\ell) \\
&= \frac{\mathrm{e}^{\beta d} - \mathrm{e}^{\beta(d-1)}}{\mathrm{e}^{\beta d} - 1} \cdot (\mathrm{e}^{-\beta a} - \mathrm{e}^{-\beta(b+1)}) \\
&= \mathrm{e}^{-\beta a} \cdot \frac{\mathrm{e}^{\beta d} - \mathrm{e}^{\beta(d-1)}}{\mathrm{e}^{\beta d} - 1} \cdot (1 - \mathrm{e}^{-\beta d}) = \nu(a).
\end{aligned}$$

As for (3.13), both sides become zero if $b < 0$. For $b \geq 0$ we have

$$\begin{aligned}
\nu^*(b) &= q(d) \cdot \sum_{\ell=a}^{b} \nu(\ell) \\
&\leq \frac{\mathrm{e}^{\beta} - 1}{\mathrm{e}^{\beta d} - 1} \cdot (\mathrm{e}^{-\beta a} - \mathrm{e}^{-\beta(b+1)}) \\
&= \mathrm{e}^{-\beta b} \cdot \frac{\mathrm{e}^{\beta} - 1}{\mathrm{e}^{\beta d} - 1} \cdot (\mathrm{e}^{\beta(d-1)} - \mathrm{e}^{-\beta}) = \nu(b).
\end{aligned}$$

□

Lemma 3.5. *The dynamics defined by* (3.5) *is attractive.*

Proof. Following the same realizations of (3.5), we see that two copies of $z(\cdot)$ under a common environment can be coupled so that whenever they get to the same part $\mathcal{M}_i$, they move together from that moment. □

Proof of Lemma 3.3. Initially $z(0) = 0$ by definition, which is clearly a distribution dominated by ν of (3.11). Now we argue recursively: by time t the distribution of $z(t)$ was a.s. only influenced by finitely many jumps of the environment, which resulted in distributions ν_1, then ν_2, then ν_3, etc. Suppose $\nu_k \overset{\mathrm{d}}{\leq} \nu$, and let ν^* be the distribution that would result from ν by the $k+1^{\mathrm{st}}$ jump. Then $\nu_{k+1} \overset{\mathrm{d}}{\leq} \nu^*$ by $\nu_k \overset{\mathrm{d}}{\leq} \nu$ and Lemma 3.5, while $\nu^* \overset{\mathrm{d}}{\leq} \nu$ by Lemma 3.4. □

4 A tail bound for the second class particle

In this section we prove that Assumption 2.7 holds for the TAEBLP model. The difficulty comes from the fact that jump rates of the second class particle, being the increments of the growth rates (2.4), are unbounded. First recall the coupling measure $\mu^{\lambda,\varrho}$ of (2.17) and notice that it gives weight one on pairs of the form $(y,\, y)$ if $\lambda = \varrho$. Define also $\mu^{\mathrm{shock}\,\varrho}$ by

$$\mu^{\mathrm{shock}\,\varrho}(y,\, z) = \begin{cases} \mu^{\varrho}(y), & \text{if } z = y + 1, \\ 0, & \text{otherwise.} \end{cases}$$

With these marginals we define the shock product distribution

$$\underline{\mu}^{\text{shock}\,\varrho} := \bigotimes_{i<0} \mu^{\varrho+1,\varrho+1} \cdot \bigotimes_{i=0} \mu^{\text{shock}\,\varrho} \cdot \bigotimes_{i>0} \mu^{\varrho,\varrho}, \tag{4.1}$$

a measure on a pair of coupled processes with a single second class particle at the origin.

Remark 4.1. *The first marginal of $\underline{\mu}^{shock\,\varrho}$ is the product distribution*

$$\bigotimes_{i<0} \mu^{\varrho+1} \cdot \bigotimes_{i\geq 0} \mu^{\varrho},$$

while the second marginal is

$$\bigotimes_{i\leq 0} \mu^{\varrho+1} \cdot \bigotimes_{i<0} \mu^{\varrho}. \tag{4.2}$$

Proof. The first part of the statement and the second part, apart from $i = 0$, follow from the definitions. The nontrivial part is

$$\mu^{\varrho+1}(z) = \mu^{\varrho}(z-1), \qquad z \in \mathbb{Z},$$

valid for the second marginal at $i = 0$. This is specific to the exponential rates (2.4), and to prove it we write, with $\theta = \theta(\varrho)$,

$$\mu^{\varrho}(z-1) = \frac{f(z)}{\mathrm{e}^{\theta}} \cdot \frac{\mathrm{e}^{\theta z}}{f(z)!} \cdot \frac{1}{Z(\theta)} = \frac{\mathrm{e}^{(\theta+\beta)z}}{f(z)!} \cdot \frac{1}{\mathrm{e}^{\theta+\beta/2} Z(\theta)}.$$

Summing this up for all $z \in \mathbb{Z}$ gives one on the left hand-side, hence leads to

$$Z(\theta+\beta) = \sum_{z=-\infty}^{\infty} \frac{\mathrm{e}^{(\theta+\beta)z}}{f(z)!} = \mathrm{e}^{\theta+\beta/2} Z(\theta),$$

which also implies

$$\varrho(\theta+\beta) = \frac{\mathrm{d}}{\mathrm{d}\theta} \log(Z(\theta+\beta)) = \varrho(\theta) + 1.$$

We conclude that

$$\mu^{\varrho}(z-1) = \frac{\mathrm{e}^{(\theta+\beta)z}}{f(z)!} \cdot \frac{1}{Z(\theta+\beta)} = \mu^{\varrho(\theta+\beta)}(z) = \mu^{\varrho+1}(z),$$

which finishes the proof of the remark. □

The translation of $\underline{\mu}^{\text{shock}\,\varrho}$ is denoted by

$$\tau_k \underline{\mu}^{\text{shock}\,\varrho} := \bigotimes_{i<k} \mu^{\varrho+1,\varrho+1} \cdot \bigotimes_{i=k} \mu^{\text{shock}\,\varrho} \cdot \bigotimes_{i>k} \mu^{\varrho,\varrho}.$$

The main tool we use is Theorem 1 from [4], which we reformulate here. $\mu S(t)$ will just denote the time evolution of a measure μ under the process dynamics:

Theorem 4.2. *In the sense of bounded test functions on $\Omega \times \Omega$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tau_k \underline{\mu}^{shock\,\varrho})S(t) = \big(\mathrm{e}^{\theta(\varrho+1)} - \mathrm{e}^{\theta(\varrho)}\big) \cdot (\tau_{k+1}\underline{\mu}^{shock\,\varrho} - \underline{\mu}^{shock\,\varrho}) + \big(\mathrm{e}^{-\theta(\varrho)} - \mathrm{e}^{-\theta(\varrho+1)}\big) \cdot (\tau_{k-1}\underline{\mu}^{shock\,\varrho} - \underline{\mu}^{shock\,\varrho}). \tag{4.3}$$

The first interesting consequence of this theorem is that the measure $\underline{\mu}^{\mathrm{shock}\,\varrho}$ on a coupled pair evolves into a linear combination of its shifted versions. Second, notice that (4.3) is the Kolmogorov equation for an asymmetric simple random walk. Indeed, this theorem implies the following

Corollary 4.3. *Let the pair $(\underline{\xi}^-(0), \underline{\xi}(0))$ have initial distribution $\underline{\mu}^{shock\,\varrho}$ defined by* (4.1)*. Then its later distribution evolves into a linear combination of translated versions of $\underline{\mu}^{shock\,\varrho}$: at time t the pair $(\underline{\xi}^-(t), \underline{\xi}(0))$ has distribution*

$$\underline{\mu}^{shock\,\varrho}S(t) = \sum_{k=-\infty}^{\infty} P_k(t) \cdot \tau_k \underline{\mu}^{shock\,\varrho},$$

where $P_k(t)$ is the transition probability at time t from the origin to k of a continuous time asymmetric simple random walk with jump rates

$$\mathrm{e}^{\theta(\varrho+1)} - \mathrm{e}^{\theta(\varrho)} \text{ to the right and } \mathrm{e}^{-\theta(\varrho)} - \mathrm{e}^{-\theta(\varrho+1)} \text{ to the left.}$$

In particular, $Q^\xi(\cdot)$, started from an environment $\underline{\mu}^{shock\,\varrho}$, is a continuous time asymmetric simple random walk with these rates.

Although the corollary is quite natural, let us give a formal proof here. First some notation. $(\underline{\xi}^-(\cdot), \underline{\xi}(\cdot))$ will denote a pair of processes evolving under the basic coupling, g will be a bounded function on the path space of such a pair, and for shortness we introduce Θ_t for the whole random path, shifted to time t: $\Theta_t = (\underline{\xi}^-(t+\cdot), \underline{\xi}(t+\cdot))$. Expectation of the process, started from $\tau_k\underline{\mu}^{\mathrm{shock}\,\varrho}$, will be denoted by $\mathbf{E}^{(k)}$. Notice that under $\mathbf{E}^{(k)}$ we a.s. have the position $Q^\xi(t)$ of the single conserved second class particle. With some abuse of notation we also use $\mathbf{E}^{(\underline{\xi}^-, \underline{\xi})}$ for the evolution of the pair $(\underline{\xi}^-(\cdot), \underline{\xi}(\cdot))$, started from the specific initial state $(\underline{\xi}^-, \underline{\xi})$.

We aim for proving the semigroup property of $S(\cdot)$. The first step is

Lemma 4.4. *Given times $0 < s < t$ and $k \in \mathbb{Z}$,*

$$\mathbf{E}^{(0)}\big[g(\Theta_t) \,|\, Q^\xi(s) = k\big] = \mathbf{E}^{(k)}[g(\Theta_{t-s})].$$

Proof. The left hand-side is

$$
\begin{aligned}
&\frac{\mathbf{E}^{(0)}\big[g(\Theta_t)\,;\,Q^\xi(s)=k\big]}{\mathbf{P}^{(0)}\{Q^\xi(s)=k\}}\\
&\qquad\qquad = \frac{\mathbf{E}^{(0)}\big[\mathbf{E}^{(\underline{\xi}^-(s),\,\underline{\xi}(s))}g(\Theta_{t-s})\,;\,Q^\xi(s)=k\big]}{\mathbf{P}^{(0)}\{Q^\xi(s)=k\}}\\
&\qquad\qquad = \frac{\sum\limits_{j\in\mathbb{Z}}\mathbf{P}^{(0)}\{Q^\xi(s)=j\}\mathbf{E}^{(j)}\big[\mathbf{E}^{(\underline{\xi}^-(0),\,\underline{\xi}(0))}g(\Theta_{t-s})\,;\,Q^\xi(0)=k\big]}{\mathbf{P}^{(0)}\{Q^\xi(s)=k\}}\\
&\qquad\qquad = \frac{\mathbf{P}^{(0)}\{Q^\xi(s)=k\}\mathbf{E}^{(k)}\big[\mathbf{E}^{(\underline{\xi}^-(0),\,\underline{\xi}(0))}g(\Theta_{t-s})\,;\,Q^\xi(0)=k\big]}{\mathbf{P}^{(0)}\{Q^\xi(s)=k\}}\\
&\qquad\qquad = \mathbf{E}^{(k)}[g(\Theta_{t-s})],
\end{aligned}
$$

where in the second equality we used that the distribution at time s is a linear combination of shifted versions of $\underline{\mu}^{\mathrm{shock}\,\varrho}$. □

Next we prove the Markov property for $Q^\xi(\cdot)$.

Lemma 4.5. *Let $n>0$ be an integer, φ_i, $i=0,\ldots,n$ bounded functions on $\mathbb{Z}$, and $0=t_0<t_1<\cdots<t_n$. Then*

$$
\mathbf{E}^{(0)}\prod_{i=1}^{n}\varphi_i\big(Q^\xi(t_i)-Q^\xi(t_{i-1})\big)=\prod_{i=1}^{n}\mathbf{E}^{(0)}\varphi_i\big(Q^\xi(t_i-t_{i-1})\big).
$$

Proof. The statement is trivially true for $n=1$. We proceed by induction, and assume the statement is true for $n-1$. Then

$$
\begin{aligned}
&\mathbf{E}^{(0)}\prod_{i=1}^{n}\varphi_i\big(Q^\xi(t_i)-Q^\xi(t_{i-1})\big)\\
&=\sum_{j\in\mathbb{Z}}\mathbf{P}^{(0)}\{Q^\xi(t_1)=j\}\varphi_1(j)\cdot\mathbf{E}^{(0)}\Big[\prod_{i=2}^{n}\varphi_i\big(Q^\xi(t_i)-Q^\xi(t_{i-1})\big)\,|\,Q^\xi(t_1)=j\Big]\\
&=\sum_{j\in\mathbb{Z}}\mathbf{P}^{(0)}\{Q^\xi(t_1)=j\}\varphi_1(j)\cdot\mathbf{E}^{(j)}\prod_{i=2}^{n}\varphi_i\big(Q^\xi(t_i-t_1)-Q^\xi(t_{i-1}-t_1)\big)\\
&=\sum_{j\in\mathbb{Z}}\mathbf{P}^{(0)}\{Q^\xi(t_1)=j\}\varphi_1(j)\cdot\mathbf{E}^{(0)}\prod_{i=2}^{n}\varphi_i\big(Q^\xi(t_i-t_1)-Q^\xi(t_{i-1}-t_1)\big)\\
&=\sum_{j\in\mathbb{Z}}\mathbf{P}^{(0)}\{Q^\xi(t_1)=j\}\varphi_1(j)\cdot\prod_{i=2}^{n}\mathbf{E}^{(0)}\varphi_i\big(Q^\xi(t_i-t_{i-1})\big)\\
&=\prod_{i=1}^{n}\mathbf{E}^{(0)}\varphi_i\big(Q^\xi(t_i-t_{i-1})\big).
\end{aligned}
$$

The second equality uses Lemma 4.4, the third one uses the fact that ϕ's only depend on Q^ξ-differences, and the fourth one follows from the induction hypothesis. $\square$

Proof of Corollary 4.3. We know that at any fixed time $t > 0$ the distribution of $(\underline{\xi}^-(t),\, \underline{\xi}(t))$ is a linear combination of shifted versions of $\underline{\mu}^{\mathrm{shock}\,\varrho}$. The shift is traced by the second class particle $Q^\xi(t)$, therefore the differential equation

$$
\begin{aligned}
(4.4)\quad & \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{P}^{(0)}\{Q^\xi(t) = k\} \\
&= \big(\mathrm{e}^{\theta(\varrho+1)} - \mathrm{e}^{\theta(\varrho)}\big) \cdot \big(\mathbf{P}^{(0)}\{Q^\xi(t) = k+1\} - \mathbf{P}^{(0)}\{Q^\xi(t) = k\}\big) \\
&\quad + \big(\mathrm{e}^{-\theta(\varrho)} - \mathrm{e}^{-\theta(\varrho+1)}\big) \cdot \Big(\mathbf{P}^{(0)}\{Q^\xi(t) = k-1\} - \mathbf{P}^{(0)}\{Q^\xi(t) = k\}\Big)
\end{aligned}
$$

follows from (4.3). In the above lemmas, we also proved that $Q^\xi(t)$ is Markovian (annealed w.r.t. the initial distribution of $(\underline{\xi}^-,\, \underline{\xi})$). As there exists only one Markovian process with Kolmogorov equation (4.4) of the simple asymmetric random walk, we conclude that the process $Q^\xi(\cdot)$ with initial environment $\underline{\mu}^{\mathrm{shock}\,\varrho}$ is an asymmetric simple random walk with rates as stated in the Corollary. $\square$

Lemma 4.6. *Let $(\underline{\omega}^-,\, \underline{\omega})$ be a pair of processes in basic coupling, started from distribution* (2.7)*, with second class particle $Q(t)$. Then there exist constants $0 < \alpha_0$, $C < \infty$ such that*

$$\mathbf{P}\{|Q(t)| > K\} \le \mathrm{e}^{-CK}$$

whenever $K > \alpha_0 t$ and t is large enough.

Notice that this implies that Assumption 2.7 holds for the TAEBLP.

Proof. The proof uses auxiliary processes to connect the above arguments to the setting of Assumption 2.7. Define the pair

$$
(\lambda_i,\, \varrho_i) := \begin{cases} (\varrho,\, \varrho+1), & \text{for } i \le 0, \\ (\varrho,\, \varrho), & \text{for } i > 0. \end{cases}
$$

Draw the pair $(\underline{\zeta}(0),\, \underline{\xi}(0))$ from the product distribution of coupling measures (2.17)

$$\bigotimes_{i\in\mathbb{Z}} \mu^{\lambda_i,\varrho_i}.$$

Then $\underline{\xi}(0)$ has distribution

$$\bigotimes_{i\le 0} \mu^{\varrho+1} \cdot \bigotimes_{i<0} \mu^{\varrho},$$

in agreement with (4.2).

Let now the pair $(\underline{\zeta}(\cdot),\, \underline{\xi}(\cdot))$ evolve in the basic coupling, and let them play the role of $(\underline{\eta}(\cdot),\, \underline{\omega}(\cdot))$ of Section 3. This results in the pair $(\underline{\zeta}(\cdot),\, \underline{\zeta}^{+}(\cdot))$ with a second class particle $Q^{\zeta}(\cdot)$ and the pair $(\underline{\xi}^{-}(\cdot),\, \underline{\xi}(\cdot))$ with a second class particle $Q^{\xi}(\cdot)$ such that $Q^{\zeta}(t) \le Q^{\xi}(t)$, see Lemma 3.2. Therefore the random walk result in Corollary 4.3 on $Q^{\xi}(\cdot)$ yields the desired estimate for $Q^{\zeta}(t)$. Finally, notice that the distribution of $\underline{\omega}^{-}(0)$ in Assumption 2.7 and of $\underline{\xi}(0)$ $\underline{\zeta}(0)$ above only differ by $\omega_0^-(0) \sim \widehat{\mu}^{\varrho}$, while $\zeta_0(0) \sim \mu^{\varrho}$. Therefore

$$
\begin{aligned}
\mathbf{P}\{Q(t) > K\} &= \sum_{z=-\infty}^{\infty} \mathbf{P}\{Q(t) > K \,|\, \omega_0^-(0) = z\} \cdot \mu^{\varrho}(z)^{\frac{1}{2}} \Big(\frac{\widehat{\mu}^{\varrho}(z)^2}{\mu^{\varrho}(z)}\Big)^{\frac{1}{2}} \\
&= \sum_{z=-\infty}^{\infty} \mathbf{P}\{Q^{\zeta}(t) > K \,|\, \zeta_0(0) = z\} \cdot \mu^{\varrho}(z)^{\frac{1}{2}} \Big(\frac{\widehat{\mu}^{\varrho}(z)^2}{\mu^{\varrho}(z)}\Big)^{\frac{1}{2}} \\
&\le \Big[\sum_{z=-\infty}^{\infty} \mathbf{P}\{Q^{\zeta}(t) > K \,|\, \zeta_0(0) = z\} \cdot \mu^{\varrho}(z)\Big]^{\frac{1}{2}} \cdot \Big[\sum_{y=-\infty}^{\infty} \frac{\widehat{\mu}^{\varrho}(y)^2}{\mu^{\varrho}(y)}\Big]^{\frac{1}{2}} \\
&= \mathbf{P}\{Q^{\zeta}(t) > K\}^{\frac{1}{2}} \cdot \Big[\sum_{y=-\infty}^{\infty} \frac{\widehat{\mu}^{\varrho}(y)^2}{\mu^{\varrho}(y)}\Big]^{\frac{1}{2}}.
\end{aligned}
$$

We are done as soon as we show that $\widehat{\mu}^{\varrho}(y)/\mu^{\varrho}(y)$ is uniformly bounded in y. With the exponential rates (2.4) one obtains from (2.6)

$$
\frac{\widehat{\mu}^{\varrho}(y)}{\mu^{\varrho}(y)} = C \sum_{z=y+1}^{\infty} (z-\varrho)\mathrm{e}^{-\frac{\beta}{2}(z-\frac{\theta}{\beta})^2 + \frac{\beta}{2}(y-\frac{\theta}{\beta})^2} = C \sum_{k=1}^{\infty} (k+y-\varrho)\mathrm{e}^{-\frac{\beta}{2}k^2 - \beta k y + \theta k}.
$$

This is uniformly bounded for large y's since then $y\mathrm{e}^{-\beta y} < 1$. For large negative y's one uses the equivalent form

$$
\widehat{\mu}^{\varrho}(y) := \frac{1}{\mathbf{Var}^{\varrho}(\omega_0)} \sum_{z=-\infty}^{y} (\varrho - z)\mu^{\varrho}(z)
$$

of (2.6) and writes

$$
\frac{\widehat{\mu}^{\varrho}(y)}{\mu^{\varrho}(y)} = C \sum_{z=-\infty}^{y} (\varrho - z)\mathrm{e}^{-\frac{\beta}{2}(z-\frac{\theta}{\beta})^2 + \frac{\beta}{2}(y-\frac{\theta}{\beta})^2} = C \sum_{k=1}^{\infty} (k-y+\varrho)\mathrm{e}^{-\frac{\beta}{2}k^2 + \beta k y - \theta k}
$$

which is again uniformly bounded for large negative y values. $\square$

Appendices

A Covariance identities for bricklayer process with exponential rates

The purpose of this appendix is to prove the variance formula for stationary BLP under the following exponential bound assumption on rates: for some $C, \beta < \infty$,

$$f(\omega_0) \le Ce^{\beta|\omega_0|}.$$

Assume the height process is normalized initially by $h_0(0) = 0$.

Theorem A.1. *Fix $z \in \mathbb{Z}$. In the stationary infinite volume process with time-marginal distribution $\underline{\omega}(t) \sim \underline{\mu}^\theta$,*

$$\mathbf{Var}[h_z(t)] = \sum_{n\in\mathbb{Z}} |n - z| \mathbf{Cov}[\omega_n(t), \omega_0(0)]. \tag{A.1}$$

Formula (A.1) was already proved in [7] for a general class of processes that includes ZRP and BLP. However, the proofs in [7] were carried out under the assumption that certain semigroup calculations work. This presents no problem when the single-site state space is compact (such as exclusion processes) for then one has a strongly continuous semigroup on the Banach space of continuous functions on the state space of the process. For BLP with superlinear rates, only some rudimentary features of the usual semigroup picture have been established in [6]. Hence the need for additional justification. We use the finite-volume auxiliary processes as introduced in [6].

To prove Theorem A.1 we show that the infinite-volume stationary process is a limit of finite-volume $(\ell, \mathfrak{r}, \theta)$ processes as $-\ell, \mathfrak{r} \to \infty$. A preliminary form of (A.1) is true for an $(\ell, \mathfrak{r}, \theta)$ process by simple counting. (See (A.13) below and its expanded form on lines (A.15)–(A.18).) The technical work goes into establishing moment bounds that are uniform over $\ell < 0 < \mathfrak{r}$. These in turn permit us to take the $-\ell, \mathfrak{r} \to \infty$ limit in the proto-formula (A.15)–(A.18).

This appendix is based on the construction of the infinite-volume BLP $\underline{h}(t)$ given in [6]. Article [6] utilized two types of finite-volume processes: the $[\ell, \mathfrak{r}]$ *processes* whose height variables were denoted by $\underline{h}^{[\ell, \mathfrak{r}]}(t)$, and the $(\ell, \mathfrak{r}, \theta)$ *processes* with height variables denoted by $\underline{g}^{(\ell, \mathfrak{r}, \theta)}(t)$. The $[\ell, \mathfrak{r}]$ evolution is a straightforward restriction of the full system into the finite interval $[\ell, \mathfrak{r}]$. Its virtue is a monotone dependence on the interval $[\ell, \mathfrak{r}]$. The $(\ell, \mathfrak{r}, \theta)$ process has also the right boundary currents that make the i.i.d. product measures $\underline{\mu}^\theta$ invariant for the finite collection of increment variables $\{\omega_i^{(\ell, \mathfrak{r}, \theta)} : \ell \le i \le \mathfrak{r}\}$.

In [6] the infinite-volume process $\underline{h}(\cdot)$ was defined as the a.s. increasing limit of the processes $\underline{h}^{[\ell, \mathfrak{r}]}(t)$ as $-\ell, \mathfrak{r} \to \infty$.

For a concrete construction of the processes, we imagine that bricklayer at site i has two unit rate Poisson processes $N_i^{(L)}$ and $N_i^{(R)}$ on the first quadrant

$\mathbb{R}_+^2$ of the plane. These govern his brick-laying action to the left (L) and right (R). A Poisson point (t, y) in $N_i^{(L)}$ such that $y \le f(-\omega_i(t-))$ signals a brick to be laid on $[i-1, i]$ at time t, while a Poisson point (t, y) in $N_i^{(R)}$ such that $y \le f(\omega_i(t-))$ signals a brick to be laid on $[i, i+1]$ at time t. Shift of these planar Poisson processes by time t will be denoted by $S_t N$.

The construction of the finite systems in terms of these Poisson processes provides the usual jump chain-holding time construction of a continuous time Markov chain on a countable state space. After each jump, the holding time and the next state are read from the Poisson processes. By the strong Markov property of the Poisson processes this is equivalent to looking at freshly sampled exponential variables with appropriate rates. In both $[\ell, \mathfrak{r}]$ processes and $(\ell, \mathfrak{r}, \theta)$ processes the increment $H(t) - H(0)$ of the maximal height

$$H(t) = \max_{\ell-1 \le j \le \mathfrak{r}} h_j(t) \tag{A.2}$$

is bounded by a Poisson process. Hence explosions do not happen [6, Sect. 3.1-3.2].

In the next section we show that the $(\ell, \mathfrak{r}, \theta)$ processes converge to the infinite volume stationary process. Then we develop moment bounds through martingales, uniformly in $\ell < 0 < \mathfrak{r}$. After Section A.1 we drop the superscripts $[\ell, \mathfrak{r}]$ and $(\ell, \mathfrak{r}, \theta)$ to ease notation.

A.1 Convergence of $(\ell, \mathfrak{r}, \theta)$ processes

The infinite-volume process $\underline{h}(\cdot)$ is defined as the a.s. increasing limit of the processes $\underline{h}^{[\ell, \mathfrak{r}]}(\cdot)$ [6, Theorem 2.2]. Lemma 7.1 in [6] shows that if the initial state $\underline{\omega}(0)$ is $\underline{\mu}^\theta$-distributed then the increment process $\underline{\omega}(\cdot)$ is stationary.

Calculations in this appendix are done in a stationary finite-volume $(\ell, \mathfrak{r}, \theta)$ process. Hence we need to show that this process also converges as $-\ell, \mathfrak{r} \to \infty$ to the stationary infinite-volume process.

It will be convenient to represent the processes by measurable mappings of the initial configuration $\underline{h}$ and the Poisson clocks $\underline{N}$ on $(0, t]$:

$$\underline{g}^{(\ell, \mathfrak{r}, \theta)}(t) = \Psi_t^{(\ell, \mathfrak{r}, \theta)}(\underline{h}, \underline{N}), \ \underline{h}^{[\ell, \mathfrak{r}]}(t) = \Phi_t^{[\ell, \mathfrak{r}]}(\underline{h}, \underline{N}) \quad \text{and} \quad \underline{h}(t) = \Phi_t(\underline{h}, \underline{N}).$$

Then the construction of the process $\underline{h}(t)$ in [6] can be expressed as follows: for any initial $\underline{h} \in \widetilde{\Omega}$, any $m, T < \infty$,

$$\Phi_{t,i}(\underline{h}, \underline{N}) = \Phi_{t,i}^{[\ell, \mathfrak{r}]}(\underline{h}, \underline{N}) \quad \text{for } -m \le i \le m \text{ and } 0 \le t \le T \tag{A.3}$$

for all large enough $-\ell, \mathfrak{r}$, almost surely.

Lemma A.2. *Let $\underline{h}(t)$ be the infinite-volume process whose increment process $\underline{\omega}(t)$ is stationary with marginal $\underline{\omega}(t) \sim \underline{\mu}^\theta$. As $-\ell, \mathfrak{r} \to \infty$, $\underline{g}^{(\ell, \mathfrak{r}, \theta)}(\cdot) \to \underline{h}(\cdot)$ in the following sense: given any $m, T < \infty$,*

$$g_i^{(\ell, \mathfrak{r}, \theta)}(t) = h_i(t) \text{ for } -m \le i \le m \text{ and } 0 \le t \le T$$

for all large enough $-\ell, \mathfrak{r}$.

Proof. The proof of Lemma 7.1 on p. 1243 of [6] shows that there exists a (nonrandom) time $t_0 = t_0(\theta) > 0$ such that, for any m, $g_i^{(\ell,\,\mathfrak{r},\,\theta)}(t) = h_i^{[\ell,\,\mathfrak{r}]}(t)$ for $-m \le i \le m$ and $0 \le t \le t_0$ if $-\ell$ and $\mathfrak{r}$ are large enough. Combined with (A.3) we have the statement on the time interval $[0, t_0]$.

Assume the claim has been proved up to time kt_0. By this induction assumption and (A.3), given $\ell < 0 < \mathfrak{r}$, there exist finite (random) $a(\ell, \mathfrak{r})$ and $b(\ell, \mathfrak{r})$ such that

$$g_i^{(a,\,b,\,\theta)}(s) = h_i^{[a,\,b]}(s) = h_i(s) \quad \text{for } -\ell \le i \le \mathfrak{r} \text{ and } 0 \le s \le kt_0$$

for all $a \le a(\ell, \mathfrak{r})$ and $b \ge b(\ell, \mathfrak{r})$.

Let $\mathcal{B}$ be the event that in the process $\{\underline{g}^{(a,\,b,\,\theta)}(kt_0 + t) : 0 \le t \le t_0\}$ either all columns in the range $\{\lfloor 3\ell/4 \rfloor, \ldots, \lfloor \ell/2 \rfloor + 1\}$ grew, or all columns in the range $\{\lfloor \mathfrak{r}/2 \rfloor, \ldots, \lceil 3\mathfrak{r}/4 \rceil\}$ grew. Then by [6, Corollary 5.5],

$$\mathbf{P}(\mathcal{B}) \le C_1 e^{-C_2(|\ell| \wedge \mathfrak{r})}$$

and this bound is independent of a, b.

Fix a column i. Since $\underline{h}(kt_0) \in \widetilde{\Omega}$ a.s., by (A.3) we can take $-\ell, \mathfrak{r}$ large enough so that $\ell < i < \mathfrak{r}$ and for $t \in [0, t_0]$

$$\Phi_{t,i}^{[\ell,\,\mathfrak{r}]}(\underline{h}(kt_0), S_{kt_0}\underline{N}) = \Phi_{t,i}(\underline{h}(kt_0), S_{kt_0}\underline{N}) = h_i(kt_0 + t).$$

The second equality above is simply the Markov property of $\underline{h}(\cdot)$. Then let $a \le a(\ell, \mathfrak{r})$ and $b \ge b(\ell, \mathfrak{r})$.

On the event $\mathcal{B}^c$ for $t \in [0, t_0]$,

$$\begin{aligned} g_i^{(a,\,b,\,\theta)}(kt_0 + t) &= \Psi_{t,i}^{(a,\,b,\,\theta)}(\underline{g}^{(a,\,b,\,\theta)}(kt_0), S_{kt_0}\underline{N}) = \Phi_{t,i}^{[\ell,\,\mathfrak{r}]}(\underline{g}^{(a,\,b,\,\theta)}(kt_0), S_{kt_0}\underline{N}) \\ &= \Phi_{t,i}^{[\ell,\,\mathfrak{r}]}(\underline{h}(kt_0), S_{kt_0}\underline{N}) = h_i(kt_0 + t). \end{aligned}$$

The first equality above is the Markov property.

The second equality above is true on the event $\mathcal{B}^c$. The reason is that when some column(s) in both ranges $[3\ell/4, \ell/2]$ and $[\mathfrak{r}/2, 3\mathfrak{r}/4]$ failed to grow, the $(a,\, b,\, \theta)$ and $[\ell,\, \mathfrak{r}]$ processes started from a common initial configuration are indistinguishable inside $(\ell/2, \mathfrak{r}/2)$. In equivalent terminology, second class particles or antiparticles that originate at the edges ℓ and $\mathfrak{r}$ have not had a chance to penetrate into $(\ell/2, \mathfrak{r}/2)$. Note that by the monotonicity properties of these processes the columns of the $(\ell,\, \mathfrak{r},\, \theta)$ process never go below those of the $[\ell,\, \mathfrak{r}]$ process [6, Lemma 3.2], hence it is enough to use the event $\mathcal{B}^c$ to suppress growth in the $(\ell,\, \mathfrak{r},\, \theta)$ process.

The third equality is true by the choice of a, b since the $\Phi_t^{[\ell,\,\mathfrak{r}]}$ evolution reads its initial condition only in the range $[\ell,\, \mathfrak{r}]$. The last equality is by choice of $\ell, \mathfrak{r}$.

Thus $g_i^{(a,\,b,\,\theta)}(kt_0 + t) = h_i(kt_0 + t)$ fails for some $t \in (0, t_0]$ with probability that is exponentially small in $|\ell| \wedge \mathfrak{r}$. Now we can take $-\ell, \mathfrak{r} \to \infty$. This extends the statement to the time interval $[0, (k + 1)t_0]$. □

A.2 Martingales

First a general result about countable Markov chains. Let S be a countable state space, Q a generator matrix, $q_x = -q_{x,x} = \sum_{y:y\neq x} q_{x,y}$ the total rate to jump away from state $x \in S$. Let $P(t)$ be the semigroup associated to Q, in other words the minimal positive solution of the backward equation $P'(t) = QP$, $P(0) = I$ [8]. This differential equation is interpreted as a system of equations, one equation for each derivative $p'_{x,y}(t)$. It is also shown in [8] that $P(t)$ is the minimal positive solution of the forward equation $P'(t) = PQ$, $P(0) = I$, and the series

$$p'_{x,y}(t) = \sum_z p_{x,z}(t) q_{z,y}$$

converges uniformly over $t \in [0, T]$ [8, p. 101].

Lemma A.3. *Let ν be an initial distribution on S and $T < \infty$. Assume*

$$\int_0^T E^\nu[q_{X_s}]\,\mathrm{d}s = \int_0^T \sum_x \nu(x) \sum_y p_{x,y}(s) q_y \,\mathrm{d}s < \infty. \tag{A.4}$$

Let φ be a bounded function on S. Then under P^ν, for $t \in [0, T]$, the process

$$\sigma(t) = \int_0^t Q\varphi(X_s)\,\mathrm{d}s$$

is well-defined and in $L^1(P^\nu)$, and the process

$$M_t = \varphi(X_t) - \varphi(X_0) - \int_0^t Q\varphi(X_s)\,\mathrm{d}s \tag{A.5}$$

is a martingale.

Proof. Since φ is bounded,

$$\sum_y |q_{x,y}\varphi(y)| \le \|\varphi\|_\infty \sum_y |q_{x,y}| = 2\|\varphi\|_\infty q_x.$$

Hence for $t \in [0, T]$

$$|\sigma(t)| \le \int_0^T \sum_y |q_{X_s,y}\varphi(y)|\,\mathrm{d}s \le C \int_0^T q_{X_s}\,\mathrm{d}s$$

and so by assumption (A.4)

$$E^\nu|\sigma(t)| \le C \int_0^T E^\nu[q_{X_s}]\,\mathrm{d}s < \infty.$$

Thus $\{\sigma(t) : 0 \le t \le T\}$ is a well-defined integrable process.

Write the forward equation in integrated form as

$$p_{x,y}(t) - \delta_{x,y} = \int_0^t \sum_z p_{x,z}(u) q_{z,y} \, \mathrm{d}u. \tag{A.6}$$

In order to justify applications of Fubini below, consider the assumption

$$x \text{ satisfies} \quad \int_0^t \sum_y p_{x,y}(u) q_y \, \mathrm{d}u < \infty. \tag{A.7}$$

on a state x. Under assumption (A.7) we can multiply equation (A.6) by $\varphi(y)$, sum over y, and use Fubini on the right-hand side. This gives the equality

$$\begin{aligned} \sum_y p_{x,y}(t)\varphi(y) - \varphi(x) &= \sum_y \int_0^t \sum_z p_{x,z}(u) q_{z,y} \varphi(y) \, \mathrm{d}u \\ &= \int_0^t \sum_{z,y} p_{x,z}(u) q_{z,y} \varphi(y) \, \mathrm{d}u \end{aligned}$$

or in operator form

$$G_t(x) \;\equiv\; P(t)\varphi(x) - \varphi(x) - \int_0^t P(u) Q\varphi(x) \, \mathrm{d}u \;=\; 0. \tag{A.8}$$

Another application of Fubini on the last integral above, also justified by assumption (A.7), gives the identity

$$G_t(x) = E^x\Big\{\varphi(X_t) - \varphi(X_0) - \int_0^t Q\varphi(X_u) \, \mathrm{d}u\Big\}. \tag{A.9}$$

Next we claim that for $s+t \le T$ we have $G_t(X_s) \in L^1(P^\nu)$. First an obvious bound from the definition (A.8) of G_t:

$$|G_t(X_s)| \le 2\|\varphi\|_\infty \Big(1 + \int_0^t \sum_z p_{X_s\,,\,z}(u) q_z \, \mathrm{d}u \Big)$$

and then utilize assumption (A.4) for the last term inside parentheses:

$$\begin{aligned} E^\nu \int_0^t \sum_z p_{X_s\,,\,z}(u) q_z \, \mathrm{d}u &= \sum_{x,y} \nu(x) p_{x,y}(s) \int_0^t \sum_z p_{y,z}(u) q_z \, \mathrm{d}u \\ &= \int_s^{s+t} \sum_{x,z} \nu(x) p_{x,z}(u) q_z \, \mathrm{d}u < \infty. \end{aligned}$$

This gives $E^\nu|G_t(X_s)| < \infty$.

Finally we need to observe that assumption (A.7) and thereby identities (A.8) and (A.9) are valid for $\nu P(s)$-a.e. x if $s + t \le T$. For this we check that

$$E^\nu \int_0^t \sum_y p_{X_s\,,\,y}(s) q_y \,\mathrm{d}s < \infty$$

which is exactly what we did in the previous display.

Now we combine the parts established. Let $A \in \mathcal{F}_s$ and $s < t \le T$. Since $G_{t-s}(X_s) \in L^1(P^\nu)$ and $G_{t-s}(X_s) = 0$ P^ν-a.s. we can start with

$$\begin{aligned} 0 &= E^\nu[\mathbf{1}_A \cdot G_{t-s}(X_s)] \\ &= E^\nu\big[\mathbf{1}_A \cdot E^{X(s)}\big\{\varphi(X_{t-s}) - \varphi(X_0) - \int_0^{t-s} Q\varphi(X_u)\,\mathrm{d}u\big\}\big] \\ &= E^\nu\big[\mathbf{1}_A \cdot \big(\varphi(X_t) - \varphi(X_s) - \int_s^t Q\varphi(X_u)\,\mathrm{d}u\big)\big]. \end{aligned}$$

The second equality utilizes (A.9) a.s. under $\nu P(s)$, and the last equality is the Markov property. The vanishing of the last line is the martingale property that we wished to prove. $\square$

Now apply this to the $[\ell,\,\mathfrak{r}]$ and $(\ell,\,\mathfrak{r},\,\theta)$ processes. To simplify notation, let

$$f_i(\underline{\omega}(s)) = f(\omega_i(s)) + f(-\omega_{i+1}(s))$$

be the rate of growth of the column height $h_i(s)$.

Lemma A.4. *Fix $\ell < 0 < \mathfrak{r}$. Consider either the $[\ell,\,\mathfrak{r}]$ process or the $(\ell,\,\mathfrak{r},\,\theta)$ process and in either case denote the height variables by $h_i(t)$. Let ν be an initial distribution such that for some $c > \beta$*

$$E^\nu\big(e^{c|h_0| + c\sum_{i=\ell}^{\mathfrak{r}}|\omega_i|}\big) < \infty. \tag{A.10}$$

Then for any $1 \le p < \infty$ and any index i this process is a martingale:

$$M_t = h_i^p(t) - h_i^p(0) - \int_0^t f_i(\underline{\omega}(s))\big((h_i(s)+1)^p - h_i^p(s)\big)\,\mathrm{d}s. \tag{A.11}$$

Proof. In the $[\ell,\,\mathfrak{r}]$ process the total jump rate out of state $\underline{h}$ is

$$q_{\underline{h}} = \sum_{i=\ell}^{\mathfrak{r}-1} f_i(\underline{\omega}) \le 2\sum_{i=\ell}^{\mathfrak{r}} e^{\beta|\omega_i|} \le 2(\mathfrak{r} - \ell + 1)e^{\beta\sum_{i=\ell}^{\mathfrak{r}}|\omega_i|}.$$

In the $(\ell, \mathfrak{r}, \theta)$ process the total jump rate out of state $\underline{g}$ is

$$q^{\theta}_{\underline{g}} = q_{\underline{g}} + f(-\omega_\ell) + f(\omega_{\mathfrak{r}}) + e^{\theta} + e^{-\theta} \le 2(\mathfrak{r} - \ell + 1)e^{\beta \sum_{i=\ell}^{\mathfrak{r}} |\omega_i|} + e^{\theta} + e^{-\theta}.$$

In the $[\ell, \mathfrak{r}]$ process under a fixed initial configuration $\underline{h}$,

$$\mathbf{E}^{\underline{h}}\Big[\sup_{s\in[0,T]} e^{\beta \sum_{i=\ell}^{\mathfrak{r}} |\omega_i(s)|} \Big] \le e^{\beta \sum_{i=\ell}^{\mathfrak{r}} |\omega_i|} \mathbf{E}[\, e^{\beta Y(T)} \,]$$

where $Y(\cdot)$ is a Poisson process of rate $\lambda = 2f(0)(\mathfrak{r} - \ell)$. This comes from the observation that the process

$$v(t) = \sum_{i=\ell}^{\mathfrak{r}} |\omega_i(t)| \; - \; \sum_{i=\ell}^{\mathfrak{r}} |\omega_i(0)|$$

increases only when a local maximum column grows, and this happens at rate at most $2f(0)(\mathfrak{r} - \ell)$. Then, under the initial distribution ν,

$$\begin{aligned} \mathbf{E}^{\nu}\Big[\sup_{s\in[0,T]} q_{\underline{h}(s)} \Big] &\le \mathbf{E}^{\nu}\Big[\sup_{s\in[0,T]} e^{\beta \sum_{i=\ell}^{\mathfrak{r}} |\omega_i(s)|} \Big] \\ &\le \mathbf{E}^{\nu}\big(e^{\beta \sum_{i=\ell}^{\mathfrak{r}} |\omega_i|} \big) \cdot \exp\{\lambda(e^{\beta T} - 1)\} < \infty. \end{aligned} \tag{A.12}$$

In the $(\ell, \mathfrak{r}, \theta)$ process the same idea works except the rate for the process that bounds $v(t)$ is $\lambda^{\theta} = 2f(0)(\mathfrak{r} - \ell + 1) + e^{\theta} + e^{-\theta}$.

Let $F(x) = (b \wedge x) \vee (-b)$ be a truncation function. Now that assumption (A.4) has been verified, for any integer $0 < b < \infty$ and $\ell \le i \le \mathfrak{r} - 1$ (A.5) gives the martingale

$$M_t^{(b)} = F(h_i(t))^p \; - \; F(h_i(0))^p \; - \; \int_0^t f_i(\underline{\omega}(s))\big(F(h_i(s)+1)^p - F(h_i(s))^p\big)\,\mathrm{d}s.$$

So for an event $A \in \mathcal{F}_s$ and $s < t$

$$\begin{aligned} &\mathbf{E}^{\nu}\big[F(h_i(t))^p \mathbf{1}_A\big] - \mathbf{E}^{\nu}\big[F(h_i(s))^p \mathbf{1}_A\big] \\ &\qquad = \int_s^t \mathbf{E}^{\nu}\big[f_i(\underline{\omega}(u))\big(F(h_i(u)+1)^p - F(h_i(u))^p\big)\mathbf{1}_A\big]\,\mathrm{d}u. \end{aligned}$$

As $b \nearrow \infty$ dominated convergence takes each term to the desired limit. This is justified by the following. Restrict to an interval $s, t \in [0, T]$. The rate in the last expectation is bounded as in

$$f_i(\underline{\omega}(u)) \le \sup_{s\in[0,T]} e^{\beta(|\omega_i(s)| + |\omega_{i+1}(s)|)}$$

and the random variable on the right has a finite L^p-norm for some $p > 1$ by a bound of the type in (A.12) and the assumption that $c > \beta$ in (A.10). For the height we have

$$|h_i(t)| = h_i^+(t) + h_i^-(t) \le H(t) + h_i^-(0) \le (H(t) - H(0)) + H(0) + |h_i(0)|$$

where $H(t)$ is the maximal height of (A.2). The increment $H(t) - H(0)$ is stochastically bounded by a Poisson process while $H(0)$ and $|h_i(0)|$ have all moments by assumption (A.10). We conclude that (A.11) is a martingale. $\square$

A.3 Bounds for the $(\ell, \mathfrak{r}, \theta)$ process

Henceforth consider stationary $(\ell, \mathfrak{r}, \theta)$ processes with μ^θ marginals and initial height normalized by $h_0(0) = 0$. The increment process is denoted by $\underline{\omega}(t)$ and the height process by $\underline{h}(t)$. In this process the column heights $h_i(t)$ for $i \le \ell - 2$ and $i \ge \mathfrak{r} + 1$ are frozen at their initial values. We start with moment bounds that hold uniformly in $\ell < 0 < \mathfrak{r}$.

Lemma A.5. *Fix θ. For $1 \le p < \infty$ there is a constant $C = C(p, \theta)$ such that in all stationary $(\ell, \mathfrak{r}, \theta)$ processes with marginal distribution μ^θ, for all $t \ge 0$ and $i \in \mathbb{Z}$,*

$$\mathbf{E}\big[(h_i(t) - h_i(0))^p\big] \le e^{Ct}.$$

The bound is valid also for the boundary columns $i = \ell - 1$ and $i = \mathfrak{r}$, and also for the infinite-volume stationary process with marginal distribution μ^θ.

Proof. Abbreviate $\bar{h}_i(t) = h_i(t) - h_i(0)$. It suffices to consider the $(\ell, \mathfrak{r}, \theta)$ processes because the bound extends to the $-\ell, \mathfrak{r} \to \infty$ limit by Fatou's lemma and Lemma A.2. The increments ω_i have all exponential moments under μ^θ so assumption (A.10) is satisfied. In particular, $\bar{h}_i(t)$ has all moments.

Fix $\ell < 0 < \mathfrak{r}$. Columns h_i for $i \notin [\ell - 1, \mathfrak{r}]$ are frozen at their initial values and need no argument. Consider the case $\ell \le i \le \mathfrak{r} - 1$. It suffices to consider a positive integer p. In the next calculation, use martingale (A.11), the fact that $\bar{h}_i(s)$ is a nonnegative integer to bound a lower power with a higher one, and apply Hölder's inequality:

$$\begin{aligned}
\mathbf{E}\bar{h}_i(t)^p &= \mathbf{E}\int_0^t f_i(\underline{\omega}(s)) \sum_{k=0}^{p-1} \binom{p}{k} \bar{h}_i^k(s)\,\mathrm{d}s \\
&\le C_p \int_0^t \mathbf{E}\big[f_i(\underline{\omega}(s))\big(1 \vee \bar{h}_i(s)\big)^{p-1}\big]\,\mathrm{d}s \\
&\le C_{p,\theta} \int_0^t \big(\mathbf{E}[f_i(\underline{\omega}(s))^p]\big)^{1/p} \big(\mathbf{E}[1 \vee \bar{h}_i(s))^p]\big)^{(p-1)/p}\,\mathrm{d}s \\
&\le C_{p,\theta} \int_0^t \big(\mathbf{E}\bar{h}_i(s)^p + 1\big)\,\mathrm{d}s.
\end{aligned}$$

Rewrite this as

$$\mathbf{E}\bar{h}_i(t)^p + 1 \le 1 + C_{p,\theta} \int_0^t \big(\mathbf{E}\bar{h}_i^p(s) + 1\big)\,\mathrm{d}s$$

and now Gronwall's inequality gives the conclusion.

The boundary columns $i = \ell - 1$ and $i = \mathfrak{r}$ are handled similarly, with the only difference that the rates include also the constant terms e^{θ} or $e^{-\theta}$. $\square$

Fix a path $z(t)$ in $\mathbb{Z}$ with $z(0) = 0$ and since t is fixed abbreviate $z = z(t)$. Define

$$I_+(t) = \sum_{n=z(t)+1}^{\mathfrak{r}+1} \omega_n(t) \quad \text{and} \quad I_-(t) = \sum_{n=\ell-1}^{z(t)} \omega_n(t).$$

Due to the frozen columns at the boundaries and the normalization $h_0(0) = 0$ we have the identity

$$h_z(t) = I_+(t) - I_+(0) = -I_-(t) + I_-(0)$$

from which

$$\mathbf{Var}[h_z(t)] = \mathbf{Cov}\big[-I_-(t) + I_-(0)\,,\, I_+(t) - I_+(0)\big]. \tag{A.13}$$

For each time t the increment variables $\{\omega_i(t) : \ell \le i \le \mathfrak{r}\}$ are i.i.d. μ^{θ}-distributed. The process is independent of the initial values $\omega_{\ell-1}(0)$ and $\omega_{\mathfrak{r}+1}(0)$ of the boundary increments. This is because the bricklayer at site $\ell - 1$ lays bricks to his right at rate e^{θ} regardless of the value of the increment at site $\ell - 1$, and similarly the left action of the bricklayer at site $\mathfrak{r} + 1$ has constant rate $e^{-\theta}$. But the time increment

$$\omega_{\ell-1}(t) - \omega_{\ell-1}(0) = -h_{\ell-1}(t) + h_{\ell-1}(0) \tag{A.14}$$

and the same for $\omega_{\mathfrak{r}+1}(t)$ are needed for $h_z(t)$. Expanding and removing vanishing covariances from (A.13) leaves

$$\mathbf{Var}[h_z(t)] = \sum_{\substack{\ell \le i \le 0 \\ z < j \le \mathfrak{r}}} \mathbf{Cov}(\omega_i(0), \omega_j(t)) \;+\; \sum_{\substack{\ell \le i \le z \\ 0 < j \le \mathfrak{r}}} \mathbf{Cov}(\omega_i(t), \omega_j(0)) \tag{A.15}$$

$$+ \sum_{\ell \le i \le 0} \mathbf{Cov}(\omega_i(0), \omega_{\mathfrak{r}+1}(t)) \;+\; \sum_{0 < j \le \mathfrak{r}} \mathbf{Cov}(\omega_{\ell-1}(t), \omega_j(0)) \tag{A.16}$$

$$- \sum_{\ell \le i \le z} \mathbf{Cov}(\omega_i(t), \omega_{\mathfrak{r}+1}(t)) \;-\; \sum_{z < j \le \mathfrak{r}} \mathbf{Cov}(\omega_{\ell-1}(t), \omega_j(t)) \tag{A.17}$$

$$- \;\mathbf{Cov}(\omega_{\ell-1}(t), \omega_{\mathfrak{r}+1}(t)). \tag{A.18}$$

We argue that the sums on lines (A.16)–(A.18) vanish as $-\ell, \mathfrak{r} \to \infty$ by showing that the covariances decay exponentially in the spatial distance.

We illustrate with a term $\mathbf{Cov}(\omega_{i_0}(t), \omega_{\mathfrak{r}+1}(t))$ for a fixed $i_0 \le z$ from the first sum on line (A.17). Consider $-\ell, \mathfrak{r}$ large so that $\ell < i_0 < z < \mathfrak{r}$. Let $w = \lfloor (i_0 + \mathfrak{r})/2 \rfloor$ be a lattice point at or next to the midpoint between i_0 and $\mathfrak{r}$. Let $\underline{\zeta}(t)$ and $\underline{\xi}(t)$ be two further processes whose initial configurations agree with those of $\underline{\omega}(t)$, both for the increments and for the heights:

$$\underline{\zeta}(0) = \underline{\xi}(0) = \underline{\omega}(0) \quad \text{and} \quad \underline{h}^{\zeta}(0) = \underline{h}^{\xi}(0) = \underline{h}(0). \tag{A.19}$$

Processes $\underline{\zeta}(t)$ and $\underline{\xi}(t)$ follow the $(\ell,\, \mathfrak{r},\, \theta)$ evolution, except that the columns $\{\underline{h}^{\zeta}_i(t) : i \geq w-1\}$ with bases in $[w-1, \infty)$ are not permitted to grow, and the columns $\{\underline{h}^{\xi}_i(t) : i \leq w\}$ based in $(-\infty, w]$ are similarly frozen. (Equivalently, replace the Poisson clocks of the corresponding bricklayers with empty point measures.) To determine their dynamics, in addition to disjoint collections of Poisson clocks, $\underline{\zeta}(t)$ requires initial increments $\{\omega_i(0) : i \leq w-1\}$ while $\underline{\xi}(t)$ requires initial increments $\{\omega_i(0) : i \geq w+1\}$. Thus $\{\zeta_i(t) : i \leq w-1\}$ and $\{\xi_i(t) : i \geq w+1\}$ are independent processes. The height processes $\{h^{\zeta}_i(t) : i \leq w-2\}$ and $\{h^{\xi}_i(t) : i \geq w+1\}$ are not independent. For example if $z > 0$ the initial heights $\{h^{\xi}_i(0) : i \geq w+1\}$ are very much influenced by the initial increments $\{\zeta_i(0) : 0 < i \leq z\}$.

On the sites where $\underline{\zeta}(t)$ and $\underline{\xi}(t)$ lay bricks they obey the same realizations of Poisson clocks as the $(\ell,\, \mathfrak{r},\, \theta)$ process $\underline{\omega}(t)$. This coupling preserves the inequalities

$$\underline{h}^{\zeta}(t) \leq \underline{h}(t) \text{ and } \underline{h}^{\xi}(t) \leq \underline{h}(t). \tag{A.20}$$

Let

$$A = \{\omega_i(t) = \zeta_i(t) \text{ for } i \leq i_0\} \quad \text{and} \quad B = \{\omega_{\mathfrak{r}+1}(t) = \xi_{\mathfrak{r}+1}(t)\}. \tag{A.21}$$

Let us say that the space-time window $[a, b] \times (s_0, s_1]$ *is a block* if $h_i(s_1) = h_i(s_0)$ for some $\lceil a \rceil \leq i \leq \lfloor b \rfloor - 1$, in other words, if some column failed to grow inside the space interval $[a, b]$ during time interval $(s_0, s_1]$. The space-time window is not a block if every column inside grew during $(s_0, s_1]$. The sense of the terminology is that a block does not permit a discrepancy to pass. For convenience we do not require a, b to be integers. We restate Corollary 5.5 from [6]. This lemma is valid in the stationary process because there is control over the spatial averages of the increments ω_i and thereby control over rates.

Lemma A.6. [6, Cor. 5.5] *There exist constants $k_0 < \infty$ and $t_0 > 0$ and a function $0 < G(s) < \infty$ for $s \in (0, t_0)$ such that $G(s) \nearrow \infty$ as $s \searrow 0$, and this bound holds: if $s_1 - s_0 \leq t_0$ and $b - a \geq k_0$ then*

$$\mathbf{P}\{[a, b] \times (s_0, s_1] \text{ is not a block}\} \leq e^{-(b-a)\cdot G(s_1 - s_0)}.$$

For a given θ this bound is valid for all $(\ell,\, \mathfrak{r},\, \theta)$ processes.

Recalling that t is fixed in the present discussion, fix a positive integer m and real $\tau > 0$ so that

$$t = m\tau \text{ and } \tau \in (0, t_0].$$

On the event A^c there must exist a sequence of times $0 < t_{w-2} < t_{w-3} < t_{w-4} < \cdots < t_{i_0} \leq t$ such that column h_i grew at time t_i, $i_0 \leq i \leq w-2$. This results from the observation that in the range $i \leq w-2$ the first discrepancy $h_i - h^{\zeta}_i > 0$ in column heights at i can appear only next to an existing discrepancy. Thus the leftmost discrepancy $Q(t) = \min\{i : h_i(t) - h^{\zeta}_i(t) > 0\}$ starts from the value $Q(0)=\infty$ due to (A.19), arrives at the boundary $w-1$ of the frozen region of $\underline{h}^\zeta$ at time t_{w-1} when h_{w-1} first grows, and then moves to the left one step at a time and always with a jump of the column $h_{Q(t)-1}(t)$ that is not matched by a jump in $h^\zeta_{Q(t)-1}(t)$.

Consequently event A^c implies that at least one of the windows

$$\Big[w-2-j\frac{w-2-i_0}{m}, w-2-(j-1)\frac{w-2-i_0}{m}\Big]\times[(j-1)\tau, j\tau], \quad 1\le j\le m,$$

is not a block. For if all these windows were blocks, the sequence of growths over intervals $[w-i, w-i+1]$ at times t_{w-i}, $2\le i\le w-i_0$, could not happen. This gives the bound

$$\mathbf{P}(A^c)\le me^{-C(w-2-i_0)/m}\le me^{-C(\mathfrak{r}-i_0)/m}$$

where for the second inequality we assumed that $\mathfrak{r}$ is large enough relative to z.

The same argument for the propagation of discrepancies can be repeated for event B in (A.21) to improve the bound to

$$\mathbf{P}(A^c\cup B^c)\le 2me^{-C(\mathfrak{r}-i_0)/m}\le C_1e^{-C_2(\mathfrak{r}-i_0)} \tag{A.22}$$

where we subsumed m in the constants.

Next we turn these bounds into bounds on covariances.

$$\begin{aligned}\mathbf{Cov}(\omega_{i_0}(t),\omega_{\mathfrak{r}+1}(t)) &= \mathbf{Cov}(\zeta_{i_0}(t),\xi_{\mathfrak{r}+1}(t))\\ &\quad+\mathbf{E}\big[(\omega_{i_0}(t)\omega_{\mathfrak{r}+1}(t)-\zeta_{i_0}(t)\xi_{\mathfrak{r}+1}(t))\mathbf{1}_{A^c\cup B^c}\big]\\ &\quad-\mathbf{E}\big[(\omega_{i_0}(t)-\zeta_{i_0}(t))\mathbf{1}_{A^c}\big]\cdot\mathbf{E}\omega_{\mathfrak{r}+1}(t)\\ &\quad-\ \mathbf{E}\big[(\omega_{\mathfrak{r}+1}(t)-\xi_{\mathfrak{r}+1}(t))\mathbf{1}_{B^c}\big]\cdot\mathbf{E}\zeta_{i_0}(t).\end{aligned} \tag{A.23}$$

By independence of $\zeta_{i_0}(t)$ and $\xi_{\mathfrak{r}+1}(t)$ the first covariance after the equality sign vanishes. We claim that for $\ell\le i\le\mathfrak{r}$,

$$\mathbf{E}|\omega_i(t)|^p\ +\ \mathbf{E}|\zeta_i(t)|^p\ +\ \mathbf{E}|\omega_{\mathfrak{r}+1}(t)|^p\ +\ \mathbf{E}|\xi_{\mathfrak{r}+1}(t)|^p\le e^{Ct}. \tag{A.24}$$

Granting this for the moment, we apply Hölder's inequality to the other expectations in (A.23), then (A.22) and the moment bounds (A.24) to arrive at

$$|\mathbf{Cov}(\omega_{i_0}(t),\omega_{\mathfrak{r}+1}(t))|\le C_1e^{-C_2(\mathfrak{r}-i_0)}. \tag{A.25}$$

Now to verify (A.24) term by term. For $\omega_i(t)$ this is simply a finite moment under the μ^θ distribution. The other increment variables we express so that Lemma A.5 applies to each case. For the boundary increment $\omega_{\mathfrak{r}+1}(t)$ write

$$\begin{aligned}\omega_{\mathfrak{r}+1}(t)=h_{\mathfrak{r}}(t)-h_{\mathfrak{r}+1}(t) &= h_{\mathfrak{r}}(t)-h_{\mathfrak{r}}(0)+h_{\mathfrak{r}}(0)-h_{\mathfrak{r}+1}(0)\\ &= h_{\mathfrak{r}}(t)-h_{\mathfrak{r}}(0)+\omega_{\mathfrak{r}}(0)\end{aligned} \tag{A.26}$$

utilizing the frozen column $h_{\mathfrak{r}+1}$ as in (A.14). Utilizing (A.19) and (A.20)

$$\zeta_i(t)=h^\zeta_{i-1}(t)-h^\zeta_i(t)\le h_{i-1}(t)-h_i(0)=h_{i-1}(t)-h_{i-1}(0)+\omega_i(0)$$

with a similar upper bound for $-\zeta_i(t)$, which together give

$$|\zeta_i(t)| \le \big(h_{i-1}(t) - h_{i-1}(0)\big) \vee \big(h_i(t) - h_i(0)\big) + |\omega_i(0)|.$$

For $\xi_{\mathfrak{r}+1}(t)$ use stochastic monotonicity of columns to begin with

$$\xi_{\mathfrak{r}+1}(t) = h^{\zeta}_{\mathfrak{r}}(t) - h^{\zeta}_{\mathfrak{r}+1}(t) \le h_{\mathfrak{r}}(t) - h_{\mathfrak{r}+1}(0)$$

and continue as in (A.26). The completes the verification of (A.24).

We have proved the exponential decay of the covariance in (A.25). The same arguments work for all the covariances on lines (A.15)–(A.18), and we state the lemma in this generality. The bounds extend also to the infinite-volume stationary process because Lemma (A.5) gives moment bounds that ensure uniform integrability.

Lemma A.7. *There exist constants $C_i = C_i(t, \theta) \in (0, \infty)$ such that, for all stationary (ℓ, $\mathfrak{r}$, θ) processes and all $i, j \in \mathbb{Z}$ and $s \in [0, t]$,*

$$\mathbf{Cov}(\omega_i(s), \omega_j(t)) \le C_1 e^{-C_2|i-j|}. \tag{A.27}$$

The bound is also valid for the infinite-volume stationary process.

Now we can complete the proof of the covariance formula.

Proof of Theorem A.1. Since the bound (A.27) hold uniformly as $-\ell, \mathfrak{r} \to \infty$, the sums on lines (A.16)–(A.18) vanish while the sums on line (A.15) converge to

$$\sum_{i\le 0\,,\,j>z} \mathbf{Cov}(\omega_i(0), \omega_j(t)) \;+\; \sum_{i\le z\,,\,j>0} \mathbf{Cov}(\omega_i(t), \omega_j(0)).$$

Translation invariance of the stationary infinite volume process turns the above sum into the right-hand side of (A.1). The left-hand side of (A.15) converges to the left-hand side of (A.1) as $-\ell, \mathfrak{r} \to \infty$ by Lemma A.2 and by the uniform integrability given by Lemma A.5. □

References

[1] M. Balázs. Microscopic shape of shocks in a domain growth model. *J. Stat. Phys.*, 105(3/4):511–524, 2001.

[2] M. Balázs. Growth fluctuations in a class of deposition models. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(4):639–685, 2003.

[3] M. Balázs. Multiple shocks in bricklayers' model. *J. Stat. Phys.*, 117:77–98, 2004.

[4] M. Balázs, Gy. Farkas, P. Kovács, and A. Rákos. Random walk of second class particles in product shock measures. *J. Stat. Phys.*, 139(2):252–279, 2010.

[5] M. Balázs, J. Komjáthy, and T. Seppäläinen. Microscopic concavity and fluctuation bounds in a class of deposition processes. http://arxiv.org/abs/0808.1177, *accepted for publication in the Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, 2011.

[6] M. Balázs, F. Rassoul-Agha, T. Seppäläinen, and S. Sethuraman. Existence of the zero range process and a deposition model with superlinear growth rates. *Ann. Probab.*, 35(4):1201–1249, 2007.

[7] M. Balázs and T. Seppäläinen. Exact connections between current fluctuations and the second class particle in a class of deposition models. *J. Stat. Phys.*, 127(2):431–455, 2007.

[8] J. R. Norris. *Markov Chains*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1997.