The Size of the Largest Part of Random Weighted Partitions of Large Integers

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Abstract

We consider partitions of the positive integer n whose parts satisfy the following condition: for a given sequence of non-negative numbers $\{b_k\}_{k\geq 1}$, a part of size k appears in exactly b_k possible types. Assuming that a weighted partition is selected uniformly at random from the set of all such partitions, we study the asymptotic behavior of the largest part X_n. Let $D(s)=\sum_{k=1}^{\infty} b_k k^{-s}, s=\sigma+iy$, be the Dirichlet generating series of the weights b_k. Under certain fairly general assumptions Meinardus (1954) has obtained the asymptotic of the total number of such partitions as $n\to\infty$. Using Meinardus scheme of conditions, we prove that X_n appropriately normalized, converges weakly to a random variable having Gumbel's distribution (i.e. its distribution function equals $e^{-e^{-t}}, -\infty < t < \infty$). This limit theorem extends some known results on particular types of partitions and on the Bose-Einstein model of ideal gas.

Mathematics Subject classifications: 05A17, 60C05, 60F05

1 Introduction and Statement of the Result

A weighted partition of the positive integer n is a multiset of size n whose decomposition into a union of disjoint components (parts) satisfies the following condition: for a given sequence of non-negative numbers $\{b_k\}_{k\geq 1}$, a part of size k appears in exactly one of b_k possible types. For more details on properties of multisets, we refer the reader e.g. to [3]. Weighted partitions are also associated with the generalized Bose-Einstein model of ideal gas, where $n(=E)$ is interpreted as the total energy of the system of particles. The weights $b_k, k\geq 1$, are viewed as counts of the distinct positions of the particles in the state space, where a particle in a given position has (rescaled) energy k (for more details on the relationship between combinatorial partitions and various models of ideal gas, see [19]). From combinatorial point of view, it is fairly natural to assume that $b_k, k \geq 1$, are integers (see e.g. the "money changing problem" discussed in detail in [22; Sect. 3.15]). On the other hand, it turns out that this requirement is not necessary for the analytical approach used in this paper. That is why, we assume that $b_k, k \geq 1$, are real non-negative numbers.

For a given sequence $b = \{b_k, k \geq 1\}$, let $\mathcal{P}_b(n)$ be the set of all weighted partitions of the positive integer n and let $p_b(n) =\mid \mathcal{P}_b(n) \mid$ be its cardinality. It is known that the generating function $f_b(x)$ of the numbers $p_b(n)$ is of Euler's type, namely,

$$f_b(x) = 1 + \sum_{n=1}^{\infty} p_b(n) x^n = \prod_{k=1}^{\infty} (1 - x^k)^{-b_k}, \quad \mid x \mid < 1 \tag{1.1}$$

(see [22; Sect. 3.14]). We introduce the uniform probability measure $P = P_{n,b}$ on the set of weighted partitions of n assuming that the probability $1/p_b(n)$ is assigned to each n-partition with weight sequence b. In this paper we focus on the size of the largest part X_n of a random weighted partition of n. With respect to the probability measure P, X_n becomes a random variable, defined on the set $\mathcal{P}_b(n)$. It is also well-known that

$$f_{m,b}(x) = 1 + \sum_{n=1}^{\infty} p_b(n) P(X_n \leq m) x^n = \prod_{k=1}^{m} (1 - x^k)^{-b_k}, \quad m \geq 1 \tag{1.2}$$

(see [22; Sect. 3.15]).

The asymptotic behavior of the combinatorial numbers $p_b(n)$ (the Taylor coefficients in (1.1)) will play important role in our further analysis. A fairly general scheme of assumptions on the parametric sequence b was proposed by Meinardus [11] (see also [2; Chap. 6]), who found an asymptotic expansion for the numbers $p_b(n)$ as $n \to \infty$. His approach is based on considering two generating series:

$$D(s) = \sum_{k=1}^{\infty} b_k k^{-s}, \quad s = \sigma + iy, \tag{1.3}$$

and

$$G(z) = \sum_{k=1}^{\infty} b_k z^k, \quad \mid z \mid \leq 1. \tag{1.4}$$

Below we give Meinardus' scheme of conditions. Throughout the paper by $\Re(z)$ and $\Im(z)$ we denote the real and imaginary part of the complex number z, respectively.

(M_1) The Dirichlet series (1.3) converges in the half-plane $\sigma > \rho > 0$ and there is a constant $C_0 \in (0, 1)$, such that the function $D(s)$ has an analytic continuation to the half-plane $\{s : \sigma \geq -C_0\}$ on which it is analytic except for the simple pole at $s = \rho$ with residue $A > 0$.

(M_2) There exists a constant $C_1 > 0$ such that

$$D(s) = O(\mid y \mid^{C_1}), \quad \mid y \mid \to \infty,$$

uniformly for $\sigma \geq -C_0$.

(M_3) There are constants $\epsilon > 0$ and $C_2 (= C_2(\epsilon) > 0)$, such that the function $g(\tau) = G(e^{-\tau}), \tau = \alpha + 2\pi i u$, u real and $\alpha > 0$ (see (1.4)) satisfies

$$\Re(g(\tau)) - g(\alpha) \leq -C_2 \alpha^{-\epsilon}, \quad | \arg(\tau) | > \pi/4, \quad 0 \neq | u | \leq 1/2,$$

for enough small values of α.

The first assumption (M1) specifies the domain, say $\mathcal{H}$, in which $D(s)$ has an analytic continuation. The second is related to the asymptotic behavior of $D(s)$, whenever $|\Im(s)| \to \infty$. Functions, which are bounded by $O(|\Im(s)|^c), 0 < c < \infty$, in certain domain, as $|\Im(s)| \to \infty$, are called functions of finite order. It is known that the sum of the Dirichlet series in (1.3) satisfies the finite order property in its half-plane of convergence $\sigma > \rho$ (see e.g. [18; Sect. 9.4]). Meinardus second condition requires that the same holds for the analytic continuation of $D(s)$ in the domain $\mathcal{H}$. Finally, Meinardus third condition implies a bound on $\Re(G(e^{-\tau}))$ (see (1.4)) for certain specific complex values of τ. In some cases its verification is technically complicated. Granovsky et al. [9] showed that it can be reformulated as follows:

(M_3')

$$\sum_{k=1}^{\infty} b_k e^{-k\alpha} \sin^2(\pi k u) \geq C_2 \alpha^{-\epsilon}, \quad 0 < \frac{\alpha}{2\pi} < | u | \leq 1/2,$$

for small enough α and some constants $C_2, \epsilon > 0$ $(C_2 = C_2(\epsilon))$.

Moreover, they proved that this inequality holds for any sequence $b_k, k \geq 1$, satisfying the inequality $b_k \geq C k^{\nu-1}, k \geq k_0$, for some $k_0 \geq 1$ and $C, \nu > 0$. We notice that if

$$b_k = C k^{\nu-1}, \quad k \geq 1, \tag{1.5}$$

then $D(s) = C\zeta(s - \nu + 1)$, where ζ denotes the Riemann zeta function. Therefore, $D(s)$ has a single pole at $s = \nu$ with residue $C > 0$ and a meromorphic analytic continuation to the whole complex plane [21; Sect. 13.13]. These facts show that Meinardus conditions $(M_1) - (M_3)$ are satisfied by the weights (1.5) with $\rho = \nu$ and $A = C$.

Throughout the paper we assume that conditions $(M_1) - (M_3))$ are satisfied. Our aim is to determine asymptotically, as $n \to \infty$, the distribution of the maximal part size X_n. Recalling (1.2), we also point out that our results may be interpreted in terms of the asymptotic of the combinatorial counts of partitions whose part sizes are $\leq m$, where the range of values of m is specified by the weak convergence of the random variable X_n to a non-degenerate random variable. In the brief review given below we summarize some known results on the limiting behavior of the random variable X_n.

Consider first the classical case of linear integer partitions, where the weights satisfy $b_k = 1, k \geq 1$. This kind of partitions were broadly studied by many authors in many respects. Their graphical representations by Ferrers diagrams show that their total number of parts and their maximal part size X_n are identically distributed for all n (see [2; Sect.1.3]). Erdös and Lehner [6] were apparently the first who applied a probabilistic approach to the study of integer partitions. As a matter of fact, they found an appropriate normalization for X_n in this case and showed that $\pi X_n/(6n)^{1/2} - \log((6n)^{1/2}/\pi)$ converges weakly, as $n \to \infty$, to a random variable having the extreme value (Gumbel's) distribution. The local version of their theorem was derived later by Auluck et al. [4]. Fristedt [8] studied linear integer partitions using a transfer method to functionals of independent and geometrically distributed random variables. Among other results, he obtained the limiting distribution of the kth largest part size whenever k is fixed. Finally, we notice that among weighted integer partitions only the linear ones possess the property that number of parts and maximum part size are identically distributed. The limiting distribution of the number of parts in the general case of random weighted partitions under Meinardus scheme of conditions is studied in [13]. It turns out that the limiting distribution laws depend on particular ranges in which the parameter ρ varies (see condition (M_1)).

Another important particular case of weighted partitions arises whenever $b_k = k, k \geq 1$. It turns out that in this case the generating function $f_b(x)$ (see (1.1)) enumerates the plane partitions. A plane partition of $n \geq 1$ is a matrix of non-negative integers arranged in non-increasing order from left to right and from top to bottom, so that their double sum of its elements equals n. Together with the largest part size X_n, consider also the counts of the non-zero rows and columns of the matrix of a plane partition. It turns out that these three quantities measure the sizes of the corresponding solid diagram of a plane partition in the $3D$ space. (The solid diagram is a heap of n unit cubes placed in the first octant of a coordinate system in a $3D$ space whose columns composed by stacked cubes have non-increasing heights along the x- and y-axis; the height of this heap along the z-axis is just X_n, the largest part size.) Similarly to Ferrers diagrams for linear integer partitions, the three sizes of this heap appear to be identically distributed for all $n \geq 1$ (for more details, see [17; p. 371]). Their joint limiting distribution was found in [15]. The marginal limiting distributions (including the limiting distribution of X_n) were obtained in [12]. For more details on various properties of plane partitions and their applications to combinatorics and analysis of algorithms, we refer the reader to [2; Chap. 11], [14; Chap. 11] and [17; Chap. 7].

Our study is also closely related to some recent results on the maximal particle energy in the Bose-Einstein model of ideal gas. The general setting and the probabilistic frame of problems from statistical mechanics and their relationship with enumerative combinatorics was given by Vershik [19]. In the context of the infinite product formula (1.1), he studied the Bose-Einstein model by a family of probability measures $\mu_v, v \in (0,1)$, defined on the set of all b-weighted partitions $\mathcal{P}_b = \cup_{n \geq 0} \mathcal{P}_b(n)$. So, for a partition $\lambda = (\lambda_1, ..., \lambda_l) \in \mathcal{P}, \lambda_1 \geq ... \geq \lambda_l > 0$, we let $r_k(\lambda) = \{j : \lambda_j = k\}$ to denote the number of parts of λ that are equal to $k \geq 1$. Then μ_v is defined by

$$\mu_v(\{\lambda \in \mathcal{P} : r_k(\lambda) = j\}) = \binom{b_k + j - 1}{j} v^{kj}(1 - v^k)^{b_k}, \quad 0 < v < 1.$$

The key feature in the study of this kind of measures is the fact that a kind of a conditional probability measure on $\mathcal{P}_b(n)$ turns out to be independent of

v for all n and coincides with the uniform probability measure $P = P_{n,b}$ (for more details, see [19]). In [20] Vershik and Yakubovich studied the limiting distribution of the maximal particle energy $X(\lambda)$, or, which is the same, the largest part size $X(\lambda), \lambda \in \mathcal{P}$, with respect to the measure μ_v, as $v \to 1^-$. In particular, under the assumption that the weight sequence b satisfies (1.5), they proved that

$$\lim_{v\to 1^-} \mu_v(\{\lambda \in \mathcal{P} : (1-v)X(\lambda) - \nu \mid \log(1-v) \mid$$
$$-(\nu - 1)\log \mid \log(1-v) \mid - (\nu-1)\log\nu - \log C \le t\})$$
$$= e^{-e^{-t}}, \quad -\infty < t < \infty. \tag{1.6}$$

As it was mentioned before, the weight sequence (1.5) satisfies Meinardus conditions $(M_1) - (M_3)$. Vershik and Yakubovich [20] studied also a more realistic model of quantum ideal gas in a d-dimensional space, for which the weights satisfy $\sum_{j=1}^{k} b_j = c_d k^{d/2} + O(k^{\kappa_d})$, as $k \to \infty$ (c_d and $\kappa_d < d/2$ are computable constants).

The main result of this paper is obtained in terms of the uniform probability measure $P = P_{n,b}$ on the set $\mathcal{P}_b(n)$. Before stating it, for the sake of brevity, we introduce the following notation:

$$a_n(\rho, A) = \left(\frac{A\Gamma(\rho+1)\zeta(\rho+1)}{n}\right)^{\frac{1}{\rho+1}}, \quad n \ge 1, \tag{1.7}$$

where the constants ρ and A are defined by condition $(M1)$.

Theorem 1 *If the weight sequence b, satisfies conditions $(M1) - (M3)$, then, for all real t, the limiting distribution of the largest part size X_n is given by*

$$\lim_{n\to\infty} P(a_n X_n + \rho \log a_n - (\rho-1)\log \mid \log a_n \mid - (\rho-1)\log\rho - \log A \le t)$$
$$= e^{-e^{-t}}. \tag{1.8}$$

Remark. One can easily compare (1.8) with (1.6) setting in the latter one $v = 1 - a_n, \nu = \rho$ and $C = A$ and observe the coinciding normalizations. We also notice that the limiting results for linear and plane partitions (see [6,12]) follow from (1.8) with $A = 1$ and $\rho = 1$ and 2, respectively.

The method of our proof combines Hayman's theorem for estimating coefficients of admissible power series [10] (see also [7; Sect. VIII.5], a generalization of Perron's formula, expressing partial sums of a Dirichlet series by a complex integral of the inverse Mellin transform applied to the Dirichlet series itself (see Thm 3.1 from the Supplement of [16]) and some Mellin transform computations.

We organize our paper as follows. Section 2 includes some auxiliary facts that we need further. Some proofs are omitted since they are given in [9,11]. In Section 3 we present the proof of Theorem 1. The Appendix contains some technical details related to the application of the generalized Perron's formula [16].

2 Preliminary Results

We start with a lemma establishing an asymptotic estimate for infinite product representation (1.1) of the generating function $f_b(x)$. It has been proved by Meinardus [11] (see also [2; Lemma 6.1]).

Lemma 1 *Suppose that sequence b is such that the associated Dirichlet series (1.3) satisfies conditions (M_1) and (M_2). If $\tau = \alpha + i\theta$, then*

$$f_b(e^{-\tau}) = \exp\left(A\Gamma(\rho)\zeta(\rho+1)\tau^{-\rho} - D(0)\log\tau + O(\alpha^{C_0})\right)$$

as $\alpha \to 0^+$ uniformly for $|\theta| \le \pi$ and $|\arg\tau| \le \pi/4$. ■

Our first goal is to show that Meinardus conditions $(M_1) - (M_3)$ imply that the generating function $f_b(x)$ possesses Hayman's admissibility properties [10] (see also [7; Sect. VIII.5]) in the unit disc. Hence, for $0 < r < 1$, we introduce the functions:

$$F_b(r) = \log f_b(r) = -\sum_{k=1}^{\infty} b_k \log(1 - r^k), \tag{2.1}$$

$$\mathcal{A}_b(r) = rF_b'(r) = r\frac{f_b'(r)}{f_b(r)}, \tag{2.2}$$

$$\mathcal{B}_b(r) = r^2F_b''(r) + rF_b'(r) = r\frac{f_b'(r)}{f_b(r)} + r^2\frac{f_b''(r)}{f_b(r)} - r^2\left(\frac{f_b'(r)}{f_b(r)}\right)^2. \tag{2.3}$$

Furthermore, setting in (2.1)-(2.3) $r = e^{-\alpha}$, we shall obtain their asymptotic expansions as $\alpha \to 0^+$. For the sake of convenience, we also set

$$h = h(\rho, A) = A\Gamma(\rho+1)\zeta(\rho+1). \tag{2.4}$$

The proof of the next lemma is contained in [9; Lemma 2].

Lemma 2 *Meinardus conditions (M_1) and (M_2) imply the following asymptotic expansions:*

$$\mathcal{A}_b(e^{-\alpha}) = h\alpha^{-\rho-1} + D(0)\alpha^{-1} + O(\alpha^{C_0-1}), \tag{2.5}$$

$$\mathcal{B}_b(e^{-\alpha}) = \frac{d}{d\alpha}(-\mathcal{A}_b(e^{-\alpha})) = h(\rho+1)\alpha^{-\rho-2} - D(0)\alpha^{-2} + O(\alpha^{C_0-2}), \tag{2.6}$$

$$F_b''' = O(\alpha^{-\rho-3}), \tag{2.7}$$

as $\alpha \to 0^+$, where $F_b, \mathcal{A}_b$ and $\mathcal{B}_b$ are defined by (2.1)-(2.3), respectively. Moreover, from (2.5) it follows that the equation

$$\mathcal{A}_b(e^{-\alpha}) = n, \quad n \ge 1, \tag{2.8}$$

has a unique solution $\alpha = \alpha_n$, such that $\alpha_n \to 0$ as $n \to \infty$. An asymptotic expansion of this solution, as $n \to \infty$, is given by

$$\alpha_n = a_n + \frac{D(0)}{(\rho+1)n} + O(n^{-1-\beta}), \tag{2.9}$$

where $\beta = \min\left(\frac{C_0}{\rho+1}, \frac{\rho}{\rho+1}\right)$ and a_n are the normalizing constants given by (1.7). ■

We notice that (2.5), (2.6) and (2.9) imply that

$$\mathcal{A}_b(e^{-\alpha_n}) \to \infty, \quad \mathcal{B}_b(e^{-\alpha_n} \to \infty, \quad n \to \infty, \tag{2.10}$$

that is, Hayman's "capture" condition [7; p. 565] is satisfied with $r = r_n = e^{-\alpha_n}$. Our next step is to establish Hayman's "locality" condition, which implies the asymptotic behavior of $f_b(x)$ in a suitable neighborhood of $x = 1$.

Lemma 3 *Suppose that the weight sequence b satisfies Meinardus conditions (M_1) and (M_2) and α_n is the solution of (2.8) given by (2.9). Let*

$$\delta_n = \alpha_n^{1+\rho/3}/\omega(n), \quad n \geq 1, \tag{2.11}$$

where $\omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. Then

$$e^{-i\theta n}\frac{f_b(e^{-\alpha_n+i\theta})}{f_b(e^{-\alpha_n})} = e^{-\theta^2\mathcal{B}_b(e^{-\alpha_n})/2}(1 + O(1/\omega^3(n)) \tag{2.12}$$

uniformly for $\mid \theta \mid \leq \delta_n$.

Proof. Applying Lemma 1, we observe that

$$\begin{aligned} & e^{-i\theta n}\frac{f_b(e^{-\alpha_n+i\theta})}{f_b(e^{-\alpha_n})} \\ & = \exp\left(\frac{h}{\rho}((\alpha_n - i\theta)^{-\rho} - \alpha_n^{-\rho}) - D(0)\log\left(1 - \frac{i\theta}{\alpha_n}\right) - i\theta n + O(\alpha_n^{C_0})\right) \end{aligned} \tag{2.13}$$

where h is given by (2.4). Expanding $(\alpha_n - i\theta)^{-\rho}$ and $\log(1 - i\theta/\alpha_n)$ by Taylor's formula and using (2.5), (2.6) and (2.8), we obtain

$$\begin{aligned} & \frac{h}{\rho}((\alpha_n - i\theta)^{-\rho} - \alpha_n^{-\rho}) - D(0)\log\left(1 - \frac{i\theta}{\alpha_n}\right) - i\theta n \\ & = i\theta(h\alpha_n^{-\rho-1} + D(0)\alpha_n^{-1} - n) - \frac{\theta^2}{2}h(\rho+1)\alpha_n^{-\rho-2} - \frac{D(0)\theta^2}{2\alpha_n^2} + O(\mid \theta \mid^3 \alpha_n^{-3-\rho}) \\ & = \imath\theta(\mathcal{A}_b(e^{-\alpha_n}) - n + O(\alpha_n^{C_0-1})) - \frac{\theta^2}{2}(\mathcal{B}_b(e^{-\alpha_n}) + O(\alpha_n^{C_0-2})) + O(\mid \theta \mid^3 \alpha_n^{-3-\rho}) \\ & = -\frac{\theta^2}{2}\mathcal{B}_b(e^{-\alpha_n}) + O(\delta_n\alpha_n^{C_0-1}) + O(\delta_n^2\alpha_n^{C_0-2}) + O(\delta_n^3\alpha_n^{-3-\rho}). \end{aligned}$$

Substituting this into (2.13) and taking into account (2.11), we obtain (2.12). ■

To complete our analysis, we also need to study the behavior of $f_b(e^{-\alpha_n+i\theta})$ outside the range $-\delta_n < \theta < \delta_n$. The next lemma shows that Hayman's last ("decay") condition [7; p. 565] is also valid.

Lemma 4 *Suppose that $f_b(x)$ satisfies Meinardus conditions $(M_1)-(M_3)$. Then, for sufficiently large n,*

$$\mid f_b(e^{-\alpha_n+i\theta}) \mid \leq f_b(e^{-\alpha_n})e^{-C_2\alpha_n^{-\epsilon}}$$

uniformly for $\delta_n \leq \theta < \pi$, where $C_2, \epsilon > 0$ are the constants defined in condition (M_3).

Proof. First, we notice that

$$\frac{|\ f_b(e^{-\alpha_n+i\theta})\ |}{f_b(e^{-\alpha_n})} = \exp\left(\Re(\log f_b(e^{-\alpha_n+i\theta})) - \log f_b(e^{-\alpha_n})\right). \tag{2.14}$$

Then, setting $\theta = 2\pi u$, we almost repeat the argument from [9;p. 324]:

$$\begin{aligned}
&\Re(\log f_b(e^{-\alpha_n+i\theta})) - \log f_b(e^{-\alpha_n})\\
&\Re\left(\sum_{k=1}^{\infty} b_k \log\left(\frac{1-e^{-\alpha_n+2\pi iu}}{1-e^{-\alpha_n}}\right)\right)\\
&= -\frac{1}{2}\sum_{k=1}^{\infty} b_k \log\left(\frac{1-2e^{-k\alpha_n}\cos(2\pi uk)+e^{-2\alpha_n k}}{(1-e^{-\alpha_n k})^2}\right)\\
&= -\frac{1}{2}\sum_{k=1}^{\infty} b_k \log\left(1+\frac{4e^{-\alpha_n k}\sin^2(\pi uk)}{(1-e^{-\alpha_n k})^2}\right)\\
&\le -\frac{1}{2}\sum_{k=1}^{\infty} b_k \log\left(1+4e^{-\alpha_n k}\sin^2(\pi uk)\right)\\
&\le -\frac{\log 5}{2}\sum_{k=1}^{\infty} b_k e^{-\alpha_n k}\sin^2(\pi uk),
\end{aligned} \tag{2.15}$$

where the last inequality follows from the fact that $\log(1+v) \ge \left(\frac{\log 5}{4}\right) v, 0 \le v \le 4$. Substituting this estimate into (2.14) and applying condition (M_3'), we obtain the required inequality. ■

We now recall (2.6) from Lemma 2. It implies that

$$\mathcal{B}_b^{1/2}(e^{-\alpha_n}) \sim (h(\rho+1))^{1/2}\alpha_n^{-1-\rho/2}, \quad n \to \infty.$$

Combining this asymptotic equivalence with the result of Lemma 4, we obtain Hayman's "decay" condition [7; p. 565], namely,

$$|\ f_b(e^{-\alpha_n+i\theta})\ | = o(f_b(e^{\alpha_n})/\mathcal{B}_b^{1/2}(e^{-\alpha_n})), \quad n \to \infty, \tag{2.16}$$

uniformly for $\delta_n \le |\ \theta\ | < \pi$.

Eqs. (2.10), (2.12) and (2.16) show that the function $f_b(x)$ is admissible in the sense of Hayman. Therefore, we can apply Thm. VIII.4 of [7] for its coefficients. We state this result in the next lemma.

Lemma 5 *Suppose that the weight sequence b satisfies Meinardus conditions $(M_1) - (M_3)$. Then, the asymptotic for the total number of weighted partitions is given by*

$$p_b(n) \sim \frac{e^{-n\alpha_n} f_b(e^{-\alpha_n})}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}} \tag{2.17}$$

as $n \to \infty$, where α_n is the unique solution of (2.8) whose asymptotic expansion is given by (2.9) and $\mathcal{B}_b(e^{-\alpha_n})$ is defined by (2.6). ■

Remark. The asymptotic equivalence (2.17) is in fact Meinardus asymptotic formula [11] for the number of weighted partitions of n. It was also established by Granovsky et al. [9] under a condition weaker than (M_3). Here we give the formula in a slightly different form, which is more convenient for our further asymptotic analysis. One can easily show the coincidence of (2.17) with the Meinardus original formula, applying the result of Lemma 1 to $f_b(e^{-\alpha_n})$ and replacing α_n and $\mathcal{B}_b(e^{-\alpha_n})$ by (2.9) and (2.6), respectively.

Further, we also need a bound on the rate of growth of the weights b_k, as $k \to \infty$. Granovsky et al. [9; p. 310] obtained it using Tauberian theorem technique.

Lemma 6 *If the sequence of weights b satisfies Meinardus conditions (M_1) and (M_2), then*

$$b_k = o(k^\rho)$$

as $k \to \infty$. ■

Now we recall formula (1.2) for the truncated products $f_{m,b}(x), m \geq 1$. Similarly to (2.1), we set

$$F_{m,b}(x) = \log f_{m,b}(x) = -\sum_{k=1}^{m} b_k \log(1 - x^k). \tag{2.18}$$

(Here we consider the main branch of the logarithmic function, assuming that $\log y < 0$ for $0 < y < 1$). Further on, when computing the derivatives of (2.1) and (2.18), we shall write

$$F_b^{(j)}(e^{-\alpha_n}) = F_b^{(j)}(x) \mid_{x=e^{-\alpha_n}}, \quad F_{m,b}^{(j)}(e^{-\alpha_n}) = F_{m,b}^{(j)}(x) \mid_{x=e^{-\alpha_n}}, j = 1, 2, 3.$$

Our next lemma establishes estimates on the tails $F_b^{(j)}(e^{-\alpha_n}) - F_{m,b}^{(j)}(e^{-\alpha_n})$ for some specific values of m.

Lemma 7 *Suppose that the weight sequence b satisfies Meinardus conditions (M_1) and (M_2) and $\alpha_n, n \geq 1$, is defined by eq. (2.9). Moreover, let*

$$m \sim \rho \alpha_n^{-1} \log \alpha_n^{-1}, \quad n \to \infty. \tag{2.19}$$

Then

$$F_b^{(j)}(e^{-\alpha_n}) - F_{m,b}^{(j)}(e^{-\alpha_n}) = O(\alpha_n^{-j} \log^{\rho+j} \alpha_n^{-1}), \quad j = 1, 2, 3.$$

Proof. We shall consider the case $j = 1$. The other two cases are studied in a similar way.

First, we choose an integer m_1 that satisfies the asymptotic equivalence

$$m_1 \sim (\rho + 1) \alpha_n^{-1} \log \alpha_n^{-1} \tag{2.20}$$

and decompose the difference of the first derivatives in the following way:

$$F'_b(e^{-\alpha_n}) - F'_{m,b}(e^{-\alpha_n}) = \sum_{k=m+1}^{\infty} \frac{kb_k e^{-k\alpha_n}}{1-e^{-k\alpha_n}} = S_1 + S_2, \tag{2.21}$$

where

$$S_1 = \sum_{k=m+1}^{m_1} \frac{kb_k e^{-k\alpha_n}}{1-e^{-k\alpha_n}}, \quad S_2 = \sum_{k=m_1+1}^{\infty} \frac{kb_k e^{-k\alpha_n}}{1-e^{-k\alpha_n}}.$$

From Lemma 6, (2.19) and (2.20) it follows that

$$\begin{aligned} S_1 &\le \frac{1}{1-\alpha_n^\rho} \sum_{k=m+1}^{m_1} kb_k e^{-k\alpha_n} = o(m_1^{\rho+1}) \frac{e^{-(m+1)\alpha_n}}{1-e^{-\alpha_n}} \\ &= o(m_1^{\rho+1} e^{-m\alpha_n}) = o(\alpha_n^{-(\rho+1)} (\log^{\rho+1} \alpha_n^{-1}) \alpha_n^\rho) \\ &= o(\alpha_n^{-1} \log^{\rho+1} \alpha_n^{-1}). \end{aligned} \tag{2.22}$$

S_2 can be estimated in a similar way, using the asymptotic expansion of the incomplete gamma function $\Gamma(a,z)$ as $z \to \infty$ (see [1; Sect. 6.5]). We have

$$\begin{aligned} S_2 &= O\left(\sum_{k=m_1+1}^{\infty} \frac{k^{\rho+1} e^{-k\alpha_n}}{1-e^{-k\alpha_n}} \right) = O\left(\sum_{k=m_1+1}^{\infty} k^{\rho+1} e^{-k\alpha_n} \right) \\ &= O\left(\alpha_n^{-\rho-2} \int_{m_1\alpha_n}^{\infty} u^{\rho+1} e^{-u} du \right) = O(\alpha_n^{-\rho-2} (m_1\alpha_n)^{\rho+1} e^{-m_1\alpha_n}) \\ &= O(\alpha_n^{-1} m_1^{\rho+1} e^{-(\rho+1)\log \alpha_n^{-1}}) = O(\alpha_n^{-1} \log^{\rho+1} \alpha_n^{-1}). \end{aligned} \tag{2.23}$$

The required estimate now follows from (2.21)-(2.23).■

Our last forthcoming lemma supplies us with integral representations for $F(e^{-\alpha})$ and $F_m(e^{-\alpha}), \alpha > 0$, using Dirichlet series (1.3) and its partial sums

$$D_m(s) = \sum_{k=1}^{m} b_k k^{-s}, \quad s = \sigma + iy, \quad m \ge 1, \tag{2.24}$$

respectively. The proof is based on Mellin transforms and can be found in [11], [9; Lemma 2(ii)] and [2; Sect. 6.2].

Lemma 8 *For any $\alpha, \Delta > 0$, we have*

$$F_{m,b}(e^{-\alpha}) = \frac{1}{2\pi i} \int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha^{-s} \Gamma(s) \zeta(s+1) D_m(s) ds \tag{2.25}$$

and

$$F_b(e^{-\alpha}) = \frac{1}{2\pi i} \int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha^{-s} \Gamma(s) \zeta(s+1) D(s) ds, \tag{2.26}$$

where $D_m(s)$ and $D(s)$ are defined by (2.24) and (1.3), respectively.■

3 Proof of the Main Result

We apply first Cauchy coefficient formula to (1.2) using the circle $x = e^{-\alpha_n + i\theta}, \pi < \theta \le \pi$, as a contour of integration (α_n is determined by (2.9)). We obtain

$$p_b(n)P(X_n \le m) = \frac{e^{-n\alpha_n}}{2\pi} \int_{-\pi}^{\pi} f_{m,b}(e^{-\alpha_n + i\theta}) e^{-i\theta n} d\theta.$$

Then, we break up the range of integration as follows:

$$p_b(n)P(X_n \le m) = J_1(m,n) + J_2(m,n), \tag{3.1}$$

where

$$J_1(m,n) = \frac{e^{-n\alpha_n}}{2\pi} \int_{-\delta_n}^{\delta_n} f_{m,b}(e^{-\alpha_n + i\theta}) e^{-i\theta n} d\theta$$
$$= \frac{e^{-n\alpha_n + F_{m,b}(e^{-\alpha_n})}}{2\pi} \int_{-\delta_n}^{\delta_n} \frac{f_{m,b}(e^{-\alpha_n + i\theta})}{f_{m,b}(e^{-\alpha_n})} e^{-i\theta n} d\theta, \tag{3.2}$$

$$J_2(m,n) = \frac{e^{-n\alpha_n + F_{m,b}(e^{-\alpha_n})}}{2\pi} \int_{\delta_n < \theta \le \pi} \frac{f_{m,b}(e^{-\alpha_n + i\theta})}{f_{m,b}(e^{-\alpha_n})} e^{-i\theta n} d\theta, \tag{3.3}$$

where δ_n is the sequence defined by (2.11).

We start with an estimate for $J_1(m,n)$, expanding the integrand of (3.2) by Taylor's formula:

$$\frac{f_{m,b}(e^{-\alpha_n + i\theta})}{f_{m,b}(e^{-\alpha_n})} = \exp\{(e^{i\theta} - 1)e^{-\alpha_n} F'_{m,b}(e^{-\alpha_n})$$
$$+ \frac{1}{2}(e^{i\theta} - 1)^2 e^{-2\alpha_n} F''_{m,b}(e^{-\alpha_n}) + O(|\ \theta\ |^3\ F'''_{m,b}(e^{-\alpha_n}))\}.$$

Hence, we can rewrite (3.2) as follows:

$$J_1(m,n) = \frac{e^{-n\alpha_n}}{\sqrt{2\pi}} e^{F_{m,b}(e^{-\alpha_n})} I_n, \tag{3.4}$$

where

$$I_n = \frac{1}{\sqrt{2\pi}} \int_{-\delta_n}^{\delta_n} \exp\{(e^{i\theta} - 1)e^{-\alpha_n} F'_{m,b}(e^{-\alpha_n})$$
$$+ \frac{1}{2}(e^{i\theta} - 1)^2 e^{-2\alpha_n} F''_{m,b}(e^{-\alpha_n}) + O(|\ \theta\ |^3\ F'''_{m,b}(e^{-\alpha_n})) - i\theta n\} d\theta.$$

Lemma 7 shows that, for those m satisfying (2.19), we can replace the derivatives $F^{(j)}_{m,b}(e^{-\alpha_n})$ by $F^{(j)}_b(e^{-\alpha_n}), j = 1,2,3$, at the expense of a negligible error term. In fact, combining Lemma 7 with (2.11), we have

$$(e^{i\theta} - 1)^j F^{(j)}_{m,b}(e^{-\alpha_n}) = (e^{i\theta} - 1)^j F^{(j)}_b(e^{-\alpha_n}) + O(\delta_n^j \alpha_n^{-j} \log^{\rho+j} \alpha_n^{-1})$$
$$= (e^{i\theta} - 1)^j F^{(j)}_b(e^{-\alpha_n}) + O(\alpha_n^{\rho j/3} \log^{\rho+j} \alpha_n^{-1} / \omega^j(n)), \quad j = 1,2,$$

and

$$O(|\ \theta\ |^3\ F'''_{m,b}(e^{-\alpha_n})) = O(\alpha_n^{\rho}(\log^{\rho+3}\alpha_n^{-1})/\omega^3(n)),$$

for any function $\omega(n) \to \infty$ as $n \to \infty$ arbitrarily slowly. Since all error terms above tend to 0, we obtain

$$\begin{aligned}
I_n &= \frac{1+o(1)}{\sqrt{2\pi}} \int_{-\delta_n}^{\delta_n} \exp\{(e^{i\theta}-1)e^{-\alpha_n}F'_b(e^{-\alpha_n}) \\
&+\frac{1}{2}(e^{i\theta}-1)^2 e^{-2\alpha_n}F''_b(e^{-\alpha_n}) + O(|\ \theta\ |^3\ F'''_b(e^{-\alpha_n})) - i\theta n\}d\theta \\
&= \frac{1+o(1)}{\sqrt{2\pi}} \int_{-\delta_n}^{\delta_n} \exp\{F_b(e^{-\alpha_n+i\theta}) - F_b(e^{-\alpha_n}) - i\theta n\}d\theta \\
&= \frac{1+o(1)}{\sqrt{2\pi}} \int_{-\delta_n}^{\delta_n} \frac{f_b(e^{-\alpha_n+i\theta})}{f_b(e^{-\alpha_n})} e^{-i\theta n} d\theta,
\end{aligned}$$

where the last equality follows from a similar Taylor's expansion for $F_b(e^{-\alpha_n+i\theta})$. Now, from Lemma 3 it follows that

$$\begin{aligned}
I_n &\sim \int_{-\delta_n}^{\delta_n} e^{-\theta^2\mathcal{B}_b(e^{-\alpha_n})/2}d\theta \\
&= \frac{1}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}} \int_{-\delta_n\sqrt{\mathcal{B}_b(e^{-\alpha_n})}}^{\delta_n\sqrt{\mathcal{B}_b(e^{-\alpha_n})}} e^{-y^2/2}dy \sim \frac{1}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}} \int_{-\infty}^{\infty} e^{-y^2/2}dy \\
&= \frac{1}{\sqrt{\mathcal{B}_b(e^{-\alpha_n})}}, \quad n \to \infty.
\end{aligned}$$

The last asymptotic equivalence follows from (2.6) of Lemma 2, which implies that

$$\delta_n\sqrt{\mathcal{B}_b(e^{-\alpha_n})} \sim \frac{\alpha_n^{-\rho/6}}{\omega(n)}\sqrt{h(\rho+1)} \to \infty,$$

if $\omega(n) \to \infty$ slower than $\alpha_n^{-\rho/6}$.

Substituting the asymptotic equivalence for I_n into (3.4), we conclude that

$$J_1(m,n) \sim \frac{e^{-n\alpha_n}}{\sqrt{2\pi\mathcal{B}(e^{-\alpha_n})}} e^{F_{m,b}(e^{-\alpha_n})}, \tag{3.5}$$

whenever m satisfies (2.19) as $n \to \infty$.

For the estimate of $J_2(m,n)$, we recall (3.3) and the proof of Lemma 4. Using an argument, similar to that presented in (2.15), for any real u, we obtain

$$\begin{aligned}
&\Re(F_{m,b}(e^{-\alpha_n+2\pi iu})) - F_{m,b}(e^{-\alpha_n}) \le -\frac{\log 5}{2}\sum_{k=1}^{m} b_k e^{-\alpha_n k}\sin^2(\pi uk) \\
&= -\frac{\log 5}{2}\left(\sum_{k=1}^{\infty} b_k e^{-\alpha_n k}\sin^2(\pi uk) - \sum_{k=m+1}^{\infty} b_k e^{-\alpha_n k}\sin^2(\pi uk)\right). \quad (3.6)
\end{aligned}$$

Combining Lemma 6 with (2.19) and repeating the argument given in the proof of Lemma 7, we also get the following estimate:

$$\sum_{k=m+1}^{\infty} b_k e^{-\alpha_n k} \sin^2(\pi k u) \le \sum_{k=m+1}^{\infty} b_k e^{-\alpha_n k} = O(\log^{\rho} \alpha_n^{-1}).$$

Replacing the second term of the right-hand side of (3.6) by the last O-estimate and applying condition condition (M_3') to its first term, for $\delta_n/2\pi <| u |\le 1/2$, we obtain

$$\Re(F_{m,b}(e^{-\alpha_n+2\pi iu})) - F_{m,b}(e^{-\alpha_n}) \le -C_2\alpha_n^{-\epsilon} + O(\log^{\rho} \alpha_n^{-1}).$$

Now, we are ready to compare the growth of (3.3) with that of (3.5). So, if m satisfies (2.19) as $n \to \infty$, we have

$$\begin{aligned} &| J_2(m,n) |\le \exp(-n\alpha_n + F_{m,b}(e^{-\alpha_n})) \\ &\times \int_{\frac{\delta_n}{2\pi}<|u|\le\frac{1}{2}} | f_{m,b}(e^{-\alpha_n+2\pi iu})/f_{m,b}(e^{-\alpha_n}) | \, du \\ &= \exp(-n\alpha_n + F_{m,b}(e^{-\alpha_n})) \int_{\frac{\delta_n}{2\pi}<|u|\le\frac{1}{2}} (\Re(F_{m,b}(e^{-\alpha_n+2\pi iu}) - F_{m,b}(e^{-\alpha_n}))du \\ &= O(\exp(-n\alpha_n + F_{m,b}(e^{-\alpha_n}) - C_2\alpha_n^{-\epsilon} + O(\log^{\rho} \alpha_n^{-1}))) \\ &= O(e^{-C_2\alpha_n^{-\epsilon}} \sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})} J_1(m,n)) = o(J_1(m,n)). \end{aligned} \tag{3.7}$$

It is now clear that (3.1), (3.5) and (3.7) imply that

$$p_b(n)P(X_n \le m) \sim \frac{e^{-n\alpha_n}}{\sqrt{2\pi\mathcal{B}_b(e^{-\alpha_n})}} e^{F_{m,b}(e^{-\alpha_n})}, \quad n \to \infty.$$

Subsequent application of the asymptotic equivalence (2.17) from Lemma 5 implies that

$$P(X_n \le m) \sim \exp\{F_{m,b}(e^{-\alpha_n}) - F_b(e^{-\alpha_n})\}, \tag{3.8}$$

where α_n and m satisfy (2.9) and (2.19) as $n \to \infty$, respectively.

Further on we shall study the asymptotic behavior of the exponent in (3.8). Our analysis will be based on a generalization of Perron's formula that expresses partial sums of a Dirichlet series as complex integrals of the inverse Mellin type transforms applied to the Dirichlet series itself. We shall use it in the form given in the Supplement of [16; Sect. 3]. So, first we represent $F_{m,b}(e^{-\alpha_n})$ using representation (2.25) of Lemma 8 and then, we apply Perron's formula to the partial sum $D_m(s)$ of the Dirichlet series $D(s)$ that is included in the integrand of (2.25) (recall also (2.24) and (1.3)). In this way we arrive at the following complex integral representation: for any $\Delta > 1$, we have

$$\begin{aligned} &F_{m,b}(e^{-\alpha_n}) \\ &= \frac{1}{2\pi i} \int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha^{-s}\Gamma(s)\zeta(s+1) \left(\frac{A(m+1)^{\rho-s}}{\rho-s} + D(s) + o(1) \right) ds. \end{aligned} \tag{3.9}$$

Furthermore, (3.8) and (3.9) imply that

$$P(X_n \le m) \tag{3.10}$$
$$\sim \exp\left\{-\frac{A\alpha_n^{-\rho}}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty}((m+1)\alpha_n)^{-s}\Gamma(s+\rho)\zeta(s+\rho+1)\frac{ds}{s}\right\}.$$

The proofs of (3.9) and (3.10) contain some technical details that will be given in the Appendix.

We continue with the computation of the complex integral in the exponent of (3.10). The exact value of $m = m(n)$ will be specified later. First, setting in the integral of (3.10)

$$u = u_n = (m+1)\alpha_n, \tag{3.11}$$

we consider it as inverse Mellin transform of the function $\Gamma(s+\rho)\zeta(s+\rho+1)/s$. For the sake of convenience, we also set

$$H(u) = \frac{1}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty} u^{-s}\Gamma(s+\rho)\zeta(s+\rho+1)\frac{ds}{s}. \tag{3.12}$$

Then, clearly $g_1(s) = 1/s$ is the Mellin transform of

$$H_1(u) = \begin{cases} 1 & \text{if} \quad u \le 1, \\ 0 & \text{if} \quad u < 1 \end{cases}$$

(see [5; formula 6.2.18]), while $g_2(s) = \Gamma(s)\zeta(s)$ is the Mellin transform of

$$H_2(u) = \sum_{j=1}^{\infty}\frac{e^{-ju}}{j} = -\log(1-e^{-u})$$

(see [7; p. 764]). Next, for $\Delta > 1$, we apply formula (6.1.14) from [5] with $\alpha = 0$ and $\beta = \rho - 1$. We obtain

$$H(u) = u^{\alpha}\int_0^{\infty}\xi^{\beta}H_1(u/\xi)H_2(\xi)d\xi = -\int_u^{\infty}y^{\rho-1}\log(1-e^{-y})dy$$
$$= \int_u^{\infty}y^{\rho-1}e^{-y}dy + R(u) = \Gamma(\rho,u) + R(u), \tag{3.13}$$

where $\Gamma(\rho, u)$ denotes the incomplete gamma function, while $R(u)$ is an error term given by

$$R(u) = \int_u^{\infty}y^{\rho-1}\left(\sum_{j=2}\frac{e^{-jy}}{j}\right)dy.$$

It is easily estimated as follows:

$$R(u) = O(\int_u^{\infty}y^{\rho-1}e^{-2y}dy) = O(e^{-u}\Gamma(\rho,u)), \quad u \to \infty.$$

Combining this estimate with (3.10)-(3.13) and applying the asymptotic expansion of the incomplete gamma function [1; Sect. 6.5], for $u = u_n \sim m\alpha_n \to \infty$, we obtain

$$\begin{aligned} &P(X_n \le m) = \exp\{-A\alpha_n^{-\rho}(u_n^{\rho-1}e^{-u_n} + O(u_n^{\rho-1}e^{-2u_n}))\} \\ &= \exp\{-A\alpha_n^{-1}m^{\rho-1}e^{-m\alpha_n}(1+O(e^{-m\alpha_n}))\}. \end{aligned} \tag{3.14}$$

It is now clear that $P(X_n \le m)$ converges to the distribution function $e^{-e^{-t}}, -\infty < t < \infty$, if $m = m(n)$ satisfies

$$-m\alpha_n + (\rho-1)\log m + \log(A\alpha_n^{-1}) = -t + o(1)$$

as $n \to \infty$. From this we deduce

$$m = \alpha_n^{-1}\log\alpha_n^{-1} + (\rho-1)\alpha_n^{-1}\log m + (\log A + t)\alpha_n^{-1} + o(\alpha_n^{-1}), \tag{3.15}$$

which in turn implies that

$$\begin{aligned} &\log m = \log(\alpha_n^{-1}\log\alpha_n^{-1}) \\ &+\log\left(1 + \frac{\log A + t}{\log\alpha_n^{-1}} + (\rho-1)\frac{\log m}{\log\alpha_n^{-1}}\right) \\ &= \log\alpha_n^{-1} + \log\log\alpha_n^{-1} \\ &+\log\left(1 + \frac{\log A + t}{\log\alpha_n^{-1}} + (\rho-1)\frac{\log\alpha_n^{-1} + \log\log\alpha_n^{-1} + O(1)}{\log\alpha_n^{-1}}\right) \\ &= \log\alpha_n^{-1} + \log\log\alpha_n^{-1} + \log\left(1 + (\rho-1) + O\left(\frac{\log\log\alpha_n^{-1}}{\log\alpha_n^{-1}}\right)\right) \\ &= \log\alpha_n^{-1} + \log\log\alpha_n^{-1} + \log\rho + O\left(\frac{\log\log\alpha_n^{-1}}{\log\alpha_n^{-1}}\right). \end{aligned}$$

Hence, (3.15) becomes

$$\begin{aligned} &m = \rho\alpha_n^{-1}\log\alpha_n^{-1} + (\rho-1)\alpha_n^{-1}\log\log\alpha_n^{-1} \\ &+\alpha_n^{-1}(\rho-1)\log\rho + (\log A + t)\alpha_n^{-1} + O\left(\frac{\log\log\alpha_n^{-1}}{\log\alpha_n^{-1}}\right). \end{aligned}$$

Replacing now this value of m into (3.14) and using the continuity of the distribution function $e^{-e^{-t}}, -\infty < t < \infty$, we obtain

$$\begin{aligned} &P(X_n \le m) \\ &= P(\alpha_n X_n - \rho\log\alpha_n^{-1} - (\rho-1)\log\log\alpha_n^{-1} - (\rho-1)\log\rho - \log A \le t) + o(1) \\ &\to e^{-e^{-t}}, \quad n \to \infty. \end{aligned} \tag{3.16}$$

To complete the proof of the limit theorem (1.8), it remains to justify the appropriate normalization replacing the sequence $\alpha_n, n \ge 1$, in (3.16) by $a_n =$

$a_n(\rho, A), n \geq 1$, given by (1.7). To show this we recall (2.9) and notice that after taking logarithms from its both sides, we easily obtain

$$\log \alpha_n = \log a_n + O(n^{-\frac{\rho}{\rho+1}}). \tag{3.17}$$

Furthermore, (2.9) implies that

$$a_n X_n = \alpha_n X_n - Y_n, \tag{3.18}$$

where $Y_n = \frac{D(0)}{(\rho+1)n} X_n$. From (3.16) it is not difficult to conclude that, for every $\eta > 0$, $\lim_{n\to\infty} P(Y_n > \eta) = 0$, that is, the sequence of the random variables $Y_n, n \geq 1$, tends to 0 in probability, as $n \to \infty$. In other words, (3.18) represents $a_n X_n$ as a sum of two random variables: the first one, $\alpha_n X_n$, converges in distribution, as $n \to \infty$, to a random variable with a distribution function $e^{-e^{-t}}, -\infty < t < \infty$, while the second one, Y_n, tends to 0 in probability. Using this fact, (3.17) and the continuity of the distribution function $e^{-e^{-t}}$, it is an easy probabilistic exercise to show that the limit theorem given by (1.8) holds.

4 Appendix

Proof of (3.9). First, we recall eq. (2.25) of Lemma 8. Our goal is to represent the mth partial sum $D_m(s)$, defined by (2.24), using the inversion formula given by Thm. 3.1 in the Supplement of [16] (see also formula (3.4) there). Instead of $D(s)$, we shall consider now the Dirichelet series

$$D(s + \rho - 1) = \sum_{k=1}^{\infty} b_k k^{-s-\rho+1}, \quad s = \sigma + iy,$$

which converges absolutely for $\sigma > 1$. Lemma 6 implies that the coefficients of $D(s + \rho - 1)$ satisfy

$$b_k k^{-\rho+1} = o(k^\rho) k^{-\rho+1} = o(k) < \tilde{c}k$$

for any constant $\tilde{c} > 0$ and all k (in other words, the function $\Phi(x)$, introduced in [16] satisfies $\Phi(x) = x$ in this case). Furthermore, from Meinardus condition (M_1) it follows that

$$\sum_{k=1}^{\infty} b_k k^{-\rho+1} k^{-\sigma} = \frac{A}{\sigma - 1} + \phi(s),$$

where $\phi(s)$ denotes a function which is analytic for $\sigma \geq -C_0$. Hence

$$\sum_{k=1}^{\infty} b_k k^{-\rho+1} k^{-\sigma} = O((\sigma - 1)^{-1}), \quad \sigma \to 1^+.$$

So, the conditions of Thm. 3.1 (the Supplement of [16]) are satisfied and by its second part we conclude that, for large enough $T > 0$ and any pair of fixed constants $\Delta > 1$ and $d > 0$, we have

$$D_m(w+\rho-1) + \frac{1}{2}b_{m+1}(m+1)^{-\rho+1}(m+1)^{-w}$$
$$= \frac{1}{2\pi i}\int_{d-iT}^{d+iT} D(w+z+\rho-1)\frac{(m+1)^z}{z}dz$$
$$+O\left(\frac{m^d}{T}\right) + O\left(\frac{m^{1-\Delta}\log m}{T}\right), w = 1+\Delta+iv, -\infty < v < \infty. \quad (4.1)$$

Lemma 6 implies that the second term in the left-hand side of (4.1) is $o(m^{1-\rho-\Delta})$ as $m \to \infty$. To compute the integral in the right-hand side of (4.1), we use a contour integral around the rectangle $d-iT, d+iT, -C_0-\rho-\Delta+iT, -C_0-\rho-\Delta-iT$. Using Meinardus condition (M_2), we estimate the integral over the end segment $(-C_0-\rho-\Delta+iT, -C_0-\rho-\Delta-iT)$ by $O(T^{C_1}m^{-C_0-\rho-\Delta})$. Hence, it tends to 0 as $m, T \to \infty$, provided

$$T = o(m^{(C_0+\rho+\Delta)/C_1}). \quad (4.2)$$

An estimate for the integrals on the segments $(-C_0-\rho-\Delta+iT, d+iT)$ and $(-C_0-\rho-\Delta-iT, d-iT)$ is given by

$$O\left(\frac{m^d}{T\log m}\right) = o\left(\frac{m^d}{T}\right).$$

Taking into account the first O-estimate in the right-hand side of (4.1), we conclude that T and d should also satisfy

$$m^d = o(T), \quad m, T \to \infty, \quad (4.3)$$

Therefore, further we shall assume that m and T satisfy (4.2) and (4.3) with certain constants $\Delta > 1$ and $d > 0$. Thus, all integrals on the end segments except the integral on $(d-iT, d+iT)$ are negligible for such m and T as $m, T \to \infty$. The same obviously holds for both O-estimates in the right-hand side of (4.1). So, we can compute $D_m(w+\rho-1)$ summing up the residues of the integrand in (4.1). We obtain

$$D_m(w+\rho-1) = \frac{A(m+1)^{1-w}}{1-w} + D(w+\rho-1) + o(1), w = 1+\Delta+iv, -\infty < v < \infty.$$

Setting $w = s-\rho+1$ and substituting this expression into (2.25), we arrive at (3.9).

Proof of (3.10). (3.9) and (2.26) imply that

$$F_m(e^{-\alpha_n}) = \frac{A}{2\pi i}\int_{\rho+\Delta-i\infty}^{\rho+\Delta+i\infty} \alpha_n^{-s}\Gamma(s)\zeta(s+1)\frac{(m+1)^{\rho-s}}{\rho-s}ds$$
$$+F(e^{-\alpha_n}) + o(1)$$
$$= -\frac{A\alpha_n^{-\rho}}{2\pi i}\int_{\Delta-i\infty}^{\Delta+i\infty}((m+1)\alpha_n)^{-s}\Gamma(s+\rho)\zeta(s+\rho+1)\frac{ds}{s} + F(e^{-\alpha_n}) + o(1).$$

Replacing this into (3.8), we obtain (3.10).

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