TECHNIQUES OF COMPUTATIONS OF DOLBEAULT COHOMOLOGY OF SOLVMANIFOLDS

HISASHI KASUYA

Abstract. We consider semi-direct products $\mathbb{C}^n \ltimes_\phi N$ of Lie groups with lattices Γ such that N are nilpotent Lie groups with left-invariant complex structures. We compute the Dolbeault cohomology of direct sums of holomorphic line bundles over G/Γ by using the Delbeaut cohomology of the Lie algebras of the direct product $\mathbb{C}^n \times N$. As a corollary of this computation, we can compute the Dolbeault cohomology $H^{p,q}(G/\Gamma)$ of G/Γ by using a finite dimensional cochain complexes. Computing some examples, we observe that the Dolbeault cohomology varies for choices of lattices Γ.

1. Introduction

For compact homogeneous spaces, in many cases we have useful techniques of computations of de Rham cohomology by using Lie algebras, for examples nilmanifolds([9]) and solvmanifolds with certain conditions([6], [8]) where solvmanifolds(resp. nilamnifolds) are compact quotients G/Γ of solvable(resp. nilpotent) Lie groups by lattices Γ. As similar to de Rham cohomology, we expect that we can compute the Dolbeault cohomology of compact complex homogeneous spaces by using Lie algebra. For nilmanifolds, the study motivated by this expectation gives some results. For examples in [2] and [3], the authors of these papers showed that for nilmanifolds M with left-invariant complex structure satisfying some conditions the Dolbeault cohomology $H^{p,q}_{\bar\partial}(M)$ is completely determined by Lie algebras. However, for solvmanifolds, techniques of computations of the Dolbeault cohomology are hardly known. The purpose of this paper is to give ideas of computations of the Dolbeault cohomology of certain class of solvmanifolds G/Γ. The importance of this idea is to consider the Dolbeault cohomology on certain direct sum of holomorphic line bundles. We will show that such cohomology isomorphic to the Dolbeault cohomology of certain nilpotent Lie algebra. Applying this computation to compute the ordinary Dolbeault cohomology $H^{p,q}_{\bar\partial}(G/\Gamma)$, we get useful techniques of computations of the Dolbeault cohomology. By this techniques, we actually compute the Dolbeault cohomology of some examples. As difference from nilmanifolds, we observe that in many cases the Dolbeault cohomology of solvmanifolds can not be completely computed by using only Lie algebras.

2. Holomorphic line bundles over complex tori

Lemma 2.1. *Consider a finitely generated free abelian group $\mathbb{Z}^n$. For a non-trivial character $\alpha : \mathbb{Z}^n \to \mathbb{C}^*$, we have $H^*(\mathbb{Z}^n, \mathbb{C}_\alpha) = 0$.*

Proof. We consider the case $n = 1$. Then we have

$$H^0(\mathbb{Z}, \mathbb{C}_\alpha) = \{\alpha(g)m = m | m \in \mathbb{C}_\alpha\} = 0.$$

As the de Rham cohomology on S^1, by the Poincaré duality we have

$$H^1(\mathbb{Z}, \mathbb{C}_\alpha) \cong H^0(\mathbb{Z}, \mathbb{C}_{\alpha^{-1}})^* = 0,$$

and obviously $H^p(\mathbb{Z}, \mathbb{C}_\alpha) = 0$ for $2 \geq p$. Hence the lemma holds if $n = 1$. In general case, we consider a decomposition $\mathbb{Z}^n = A \oplus B$ such that A is a rank 1 subgroup and the restriction of α on A is also non-trivial. Then we have the Hochshild-Serre spectral sequence E_r such that

$$E_2^{p,q} = H^p(\mathbb{Z}^n/A, H^q(A, \mathbb{C}_\alpha))$$

and this converges to $H^{p+q}(\mathbb{Z}^n, \mathbb{C}_\alpha)$. Since $H^q(A, \mathbb{C}_\alpha) = 0$ for any p, this sequence degenerates at $E_2 = 0$ and hence the lemma follows. □

We consider a complex vector space $\mathbb{C}^n$ with a lattice Γ. For a C^∞-character $\alpha : \mathbb{C}^n \to \mathbb{C}^*$, we have the holomorphic line bundle $L_\alpha = \mathbb{C}^n \times \mathbb{C}_\rho/\Gamma$ over the complex torus $\mathbb{C}^n/\Gamma$. If α is holomorphic, then L_α is trivial as a holomorphic line bundle. We define the equivalence relation on the space of C^∞-characters of $\mathbb{C}^n$ such that $\alpha \sim \beta$ if $\alpha\beta^{-1}$ is holomorphic. Then we have the correspondence $\alpha \mapsto L_\alpha$ between $\{C^\infty\text{−characters}\}/\sim$ and isomorphism classes of holomorphic line bundles over $\mathbb{C}^n/\Gamma$. But this is not injective.

Lemma 2.2. *For a character α of $\mathbb{C}^n$ we have a unique unitary character β such that $\alpha \sim \beta$.*

Proof. For a coordinate $(x_1 + \sqrt{-1}y_1, \dots, x_n + \sqrt{-1}y_n) \in \mathbb{C}^n$, a character α is written as

$$\alpha(x_1 + \sqrt{-1}y_1, \dots, x_n + \sqrt{-1}y_n) = \exp(\sum_{i=1}^{n}(a_i x_i + b_i y_i + \sqrt{-1}(c_i x_i + d_i y_i)))$$

for $a_i, b_i, c_i, d_i \in \mathbb{R}^n$. We consider the holomorphic character

$$\alpha'(x_1+\sqrt{-1}y_1, \dots, x_n+\sqrt{-1}y_n) = \exp(\sum_{i=1}^{n}(-a_i(x_i+\sqrt{-1}y_i)+\sqrt{-1}b_i(x_i+\sqrt{-1}y_i)).$$

Then the character $\beta = \alpha\alpha'$ is unitary. If a unitary character is holomorphic, then it is trivial. Hence such β is unique. □

Theorem 2.3. ([11]) *The correspondence*

$$\mathrm{Hom}(\Gamma, U(1)) \ni \alpha \mapsto L_\alpha \in \left\{ {}^{\textit{isomorphism classes of}}_{\textit{holomorphic line bundles over } \mathbb{C}^n/\Gamma} \right\}$$

is injective.

Proposition 2.4. *For a C^∞-character $\alpha : \mathbb{C}^n \to \mathbb{C}^*$, if L_α is a non-trivial holomorphic line bundle, then the Dolbeault cohomology $H^*_{Dol}(\mathbb{C}^n/\Gamma, L_\alpha)$ with values in the line bundle L_α is trivial.*

Proof. By Lemma 2.2, we can suppose α is unitary. Consider the flat connection D on L_α induced by α. We have decompositon $D = D' + D''$ where D'' is the Dolbeault operator on L_α. Since the image of α lies in a compact subgroup of $\mathbb{C}^*$, we have a Hermittian metric on L_α such that for a Kähler metric on $\mathbb{C}^n/\Gamma$ we have the standard identity of the Laplacians of D and D'' (see [4, Section 7]). Hence we have an isomorphism $H^*_{D''}(\mathbb{C}^n/\Gamma, L_\alpha) \cong H^*_D(\mathbb{C}^n/\Gamma, L_\alpha)$. If α is non-trivial, then $H^*_D(\mathbb{C}^n/\Gamma, L_\alpha) = 0$ by Lemma 2.1 and the proposition follows. □

3. Invariant complex structure

Let G be a simply connected solvabale Lie group and $\mathfrak{g}$ the Lie algebra of G. A left-invariant complex structure on G is a left-invariant almost complex structure J(i.e. $J\in \mathrm{End}(\mathfrak{g})$ with $J^2=-\mathrm{id}$) with the integrability (i.e. $[X,Y]-[JX,JY]+[JX,Y]+[X,JY]=0$ for all $X,Y\in\mathfrak{g}$). We assume G has a left-invariant complex structure J. Denote $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}\otimes\mathbb{C}$. Consider the decomposition $\mathfrak{g}_{\mathbb{C}}=\mathfrak{g}^{1,0}\oplus\mathfrak{g}^{0,1}$ into the $\pm\sqrt{-1}$-eigenspaces of the left invariant complex structure J on G. The exterior algebra of the dual space $\mathfrak{g}^{*}_{\mathbb{C}}$ decomposes as

$$\bigwedge^{k}\mathfrak{g}^{*}_{\mathbb{C}}=\bigoplus_{p+q=k}\bigwedge^{p}\mathfrak{g}^{*1,0}\otimes\bigwedge^{q}\mathfrak{g}^{*0,1}=\bigoplus_{p+q=k}\bigwedge^{p,q}\mathfrak{g}^{*},$$

and the differential which is the dual map of Lie-bracket decomposes as $d=\partial+\bar\partial$ such that ∂ and $\bar\partial$ are $(1,0)$ and $(0,1)$ components for the bi-grading $\bigoplus_{p+q=k}\bigwedge^{p,q}\mathfrak{g}^{*}$ respectively. We can identify $(\bigwedge^{p,q}\mathfrak{g}^{*},\bar\partial)$ with left-invariant (p,q)-forms on (G,J) with the Dolbeault operator.

Let $\alpha: G\to\mathbb{C}^{*}$ be a C^{∞} unitary character of G. We consider the bi-graded cochain complex $(A^{p,q}(G,\mathbb{C}_{\alpha}), bar\partial)=(A^{p,q}(G)\otimes\mathbb{C}_{\alpha},\bar\partial)$ of the differential forms on G with values in the representation space $\mathbb{C}_{\alpha}$ and the subcomplex $A^{p,q}(G,\mathbb{C}_{\alpha})^{G},\bar\partial$ of the elements of $A^{p,q}(G,\mathbb{C}_{\alpha})$ which are invariant of left G-action. Take a basis v_{α} of $\mathbb{C}_{\alpha}$ and denote $l_{\rho}=\alpha^{-1}v_{\alpha}$. We have an isomorphism

$$(\bigwedge^{p,q}\mathfrak{g}^{*}\otimes\langle l_{\alpha}\rangle,\bar\partial)\cong(A^{p,q}(G,\mathbb{C}_{\alpha})^{G},\bar\partial).$$

Suppose G has a lattice Γ. For the homogeneous space, we consider the Dolbeault complex $(A^{p,q}(G/\Gamma,L_{\alpha}),\bar\partial)=(A^{p,q}(G,\mathbb{C}_{\alpha})^{\Gamma},\bar\partial)$. Similarly we have an isomorphism

$$(A^{p,q}(G/\Gamma)\otimes\langle l_{\alpha}\rangle,\bar\partial)\cong(A^{p,q}(G/\Gamma,L_{\alpha}),\bar\partial).$$

Take a basis $X_1,\dots,X_n$ of $\mathfrak{g}^{1,0}$ and the dual basis $\theta_1,\dots,\theta_n$ of $\mathfrak{g}^{*1,0}$. Let g be the Hermittian metric on (G,J) defined by $X_1,\dots,X_n$ as a orthonormal basis. Consider the Hodge star operator $\bar{*}: A^{p,q}(G)\to A^{n-p,n-q}(G)$. Then we have

$$\bar{*}(b\theta_I\wedge\theta_{\bar J})=(\sqrt{-1})^{n}\epsilon(I\bar J I'\bar J')\bar b\theta_{I'}\wedge\theta_{\bar J'}$$

where $I=(i_1,\dots,i_p)$, $\bar J=(\bar j_1,\dots\bar j_q)$, $I'=(i_{p+1},\dots,i_n)$ and $\bar J'=(\bar j_{q+1},\dots,\bar j_n)$ are complements of I and $\bar J$ respectively and we wirte $\theta_I=\theta_{i_1}\wedge\dots\wedge\theta_{i_p}$, $\bar\theta_{\bar J}=\bar\theta_{\bar j_1}\wedge\dots\wedge\bar\theta_{\bar j_q}$ and $\epsilon(I\bar J I'\bar J')$ is the sign of the permutation $(1,\dots,n,\bar 1,\dots,\bar n)\to(I\bar J I'\bar J')$. For a unitary character α of G, we consider the Line bundle $L_{\alpha}=(G\times\mathbb{C}_{\alpha})/\Gamma$ over G/Γ. For a basis v_{α} of the trivial bundle $G\times\mathbb{C}_{\alpha}$, we define the Hermittian metric h_{α} as $h_{\alpha}(v_{\alpha},v_{\alpha})=1$. Since α is unitary, h_{α} induces the Hermittian metric on L_{α}. Consider the dual $(L_{\alpha})^{*}$ of L_{α}. We have $(L_{\alpha})^{*}=L_{\alpha^{-1}}$. Regarding h_{α} as a $\mathbb{C}$-anti linear isomorphism $h_{\alpha}: L_{\alpha}\to(L_{\alpha})^{*}=L_{\alpha^{-1}}$, we have $h_{\alpha}(v_{\alpha})=v_{\alpha^{-1}}$. Hence for the Hodge star operator $\bar{*}: A^{p,q}(G/\Gamma,L_{\alpha})\to A^{n-p,n-q}(G/\Gamma,L_{\alpha^{-1}})$, we have

$$\bar{*}(b\theta_I\wedge\theta_J\otimes l_{\alpha})=(\sqrt{-1})^{n}\epsilon(I\bar J I'\bar J')\bar b\theta_{I'}\wedge\theta_{J'}\otimes l_{\alpha^{-1}}.$$

Consider the adjoint operator $\bar\delta=-\bar{*}\bar\partial\bar{*}$ of d and $\bar\partial$-Laplacian $\Box=\bar\partial\bar\delta+\bar\delta\bar\partial$. Let $\mathcal{H}^{p,q}(G/\Gamma,L_{\alpha})$ be the space of $\bar\partial$-harmonic (p,q)-forms and

$$\mathcal{H}^{p,q}(\mathfrak{g}_{\mathbb{C}},\mathbb{C}_{\alpha})=\mathcal{H}^{p,q}(G/\Gamma,L_{\alpha})\cap(\bigwedge^{p,q}\mathfrak{g}^{*}\otimes\langle l_{\alpha}\rangle)$$

the space of $\bar\partial$-harmonic left-invariant p,q-forms. We have:

Theorem 3.1. ([13]) *We have isomorphisms*

$$H^{p,q}_{\bar\partial}((G/\Gamma, L_\alpha) \cong \mathcal{H}^{p,q}(G/\Gamma, L_\alpha)$$

and

$$H^{p,q}_{\bar\partial}(\bigwedge^{p,q}\mathfrak{g}^*\otimes\langle l_\alpha\rangle) \cong \mathcal{H}^{p,q}(\mathfrak{g}_{\mathbb{C}}, \mathbb{C}_\alpha)$$

induced by inclusions. Hence the inclusion $(\bigwedge^{p,q}\mathfrak{g}^*\otimes\langle l_\alpha\rangle, \bar\partial) \subset (A^{p,q}(G/\Gamma, L_\alpha), \bar\partial)$ *induces an injection* $H^{p,q}_{\bar\partial}(\bigwedge^{p,q}\mathfrak{g}^*\otimes\langle l_\alpha\rangle) \to H^{p,q}_{\bar\partial}((G/\Gamma, L_\alpha)$.

4. Main results

We consider a semi-direct product $G=\mathbb{C}^n\ltimes_\phi N$ such that:
(1) N is simply connected nilpotent Lie group with a left-invariant complex structure.
(2) For any $t\in\mathbb{C}^n$, $\phi(t)$ is a holomorphic automorphism of (N,J).
(3) ϕ induces a semi-simple action on the Lie algebra of N.
Denote $\mathfrak{a}$ and $\mathfrak{n}$ as the Lie subalgebras of $\mathfrak{g}$ corresponding to $\mathbb{C}^n$ and N respectively.

4.1. **left-Invariant forms on** G**.** We can write $\mathfrak{g}=\mathfrak{a}\ltimes_{d\phi}\mathfrak{n}$. Consider the decomposition $\mathfrak{n}_{\mathbb{C}}=\mathfrak{n}^{1,0}\oplus\mathfrak{n}^{0,1}$. By the condition (2), this decomposition is a direct sum of $\mathbb{C}^n$-submodules. By the condition (3) we have a basis $Y_1,\dots,Y_m$ of $\mathfrak{n}^{1,0}$ such that the action ϕ on $\mathfrak{n}^{1,0}$ is represented by $\phi(t)=diag(\alpha_1(t),\dots,\alpha_m(t))$. Let $X_1,\dots,X_n$ be a basis of $\mathfrak{a}^{1,0}$. As we regard $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ as (1,0)-invariant vector fields on $\mathbb{C}^n$ and N respectively, by $\phi(t)Y_i=\alpha_i(t)Y_i$ for $t\in\mathbb{C}$, we have the basis $X_1,\dots,X_n,\alpha_1Y_1,\dots,\alpha_mY_m$ of $\mathfrak{g}^{1,0}=(\mathfrak{a}\ltimes_{d\phi}\mathfrak{n})^{1,0}$. Take the $(1,0)$ left-invariant differential forms $x_1,\dots,x_n$ on $\mathbb{C}^n$ and $y_1,\dots,y_m$ on N which are the dual to $X_1,\dots,X_n$ and $Y_1,\dots,Y_m$ respectively. Then $x_1,\dots,x_n,\alpha_1^{-1}y_1,\dots,\alpha_m^{-1}y_m$ is a basis of the $(1,0)$ left-invariant forms $\mathfrak{g}^{*1,0}=(\mathfrak{a}\ltimes_{d\phi}\mathfrak{n})^{*1,0}$ on G. Hence we have

$$\bigwedge^{p,q}\mathfrak{g}^*=\bigwedge^{p}\langle x_1,\dots,x_n,\alpha_1^{-1}y_1,\dots,\alpha_m^{-1}y_m\rangle\otimes\bigwedge^{q}\langle\bar x_1,\dots,\bar x_n,\bar\alpha_1^{-1}\bar y_1,\dots,\bar\alpha_m^{-1}\bar y_m\rangle.$$

4.2. **Holomorphic fibration.** Suppose G has a lattice Γ. Then Γ can be written by $\Gamma=\Gamma'\ltimes_\phi\Gamma''$ such that Γ' and Γ'' are lattices of $\mathbb{C}^n$ and N respectively and for any $t\in\Gamma'$ $\phi(t)$ preserves Γ''.

Proposition 4.1. *By the projection* $p:\mathbb{C}^n\ltimes_\phi N\to\mathbb{C}^n$, G/Γ *is a holomorphic fiber bundle* $p: G/\Gamma\to\mathbb{C}^n/\Gamma'$ *with the fiber* N/Γ'' *such that the structure group of this fibration is discrete.*

Proof. Consider the covering $\mathbb{C}^n\times(N/\Gamma'')\to G/\Gamma$ such that the covering transformation is the action of Γ' on $\mathbb{C}^n\times(N/\Gamma'')$ given by $g\cdot(a,b)=(a+g,\phi(g)b)$. Hence we have the fiber bundle $G/\Gamma\to\mathbb{C}^n/\Gamma'$ with the fiber N/Γ'' and the discrete structure group $\phi(\Gamma')\subset \mathrm{Aut}(N)$. Since $\phi(g)$ is holomorphic automorphism, this fiber bundle is holomorphic. □

4.3. **Unitary characters and line bundles.** We consider the C^∞-unitary characters $\mathrm{Hom}(\mathbb{C}^n, U(1))$ of $\mathbb{C}^n$. For the projection $p:\mathbb{C}^n\ltimes_\phi N\to\mathbb{C}^n$, the pull-back $p^*:\mathrm{Hom}(\mathbb{C}^n,U(1))\to\mathrm{Hom}(G,U(1))$ is injective. Similarly we also have the injection $p^*:\mathrm{Hom}(\Gamma',U(1))\to\mathrm{Hom}(\Gamma,U(1))$. Taking the restriction on Γ, we have the map $R_\Gamma:\mathrm{Hom}(G,U(1))\ni\alpha\mapsto\alpha_{|_\Gamma}\in\mathrm{Hom}(\Gamma,U(1))$. Similarly we also have $R_{\Gamma'}:\mathrm{Hom}(\mathbb{C}^n,U(1))\ni\alpha\mapsto\alpha_{|_{\Gamma'}}\in\mathrm{Hom}(\Gamma',U(1))$. For $\alpha'\in R_\Gamma(\mathrm{Hom}(\mathbb{C}^n,U(1))$, we have the line bundle $L_{\alpha'}$ over the complex torus $\mathbb{C}^n/\Gamma'$. By the fibration

$p : G/\Gamma \to \mathbb{C}^n/\Gamma'$, we consider the pull-back $p^*L_{\alpha'}$. The holomorphic line bundle $p^*L_{\alpha'}$ is given by $p^*L_{\alpha'} = (G \times \mathbb{C}_{p^*\alpha'})/\Gamma$. By Theorem 2.3, the map $\alpha' \mapsto p^*L_{\alpha'}$ from $R_\Gamma(\mathrm{Hom}(\mathbb{C}^n, U(1))$ to the set of isomorphism classes of holomorphic line bundle over G/Γ is injective. Hence we have the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}(\mathbb{C}^n, U(1)) & \xrightarrow{R_{\Gamma'}} & \mathrm{Hom}(\Gamma', U(1)) & \longrightarrow & \left\{ \substack{\textit{isomorphism classes of} \\ \textit{holomorphic line bundles over } \mathbb{C}/\Gamma'} \right\} \\
\downarrow p^* & & \downarrow p^* & & \downarrow p^* \\
\mathrm{Hom}(G, U(1)) & \xrightarrow{R_{\Gamma}} & \mathrm{Hom}(\Gamma, U(1)) & \longrightarrow & \left\{ \substack{\textit{isomorphism classes of} \\ \textit{holomorphic line bundles over } G/\Gamma} \right\}
\end{array}$$

such that except R_Γ and $R_{\Gamma'}$ each homomorphism in this diagram is injective. In this paper, for $\alpha \in \mathrm{Hom}(\mathbb{C}, U(1))$ we also regard $\alpha \in \mathrm{Hom}(G, U(1))$ if it is not necessary to distinguish α from $p^*\alpha$.

4.4. **Direct sum of Dolbeault complexes.** We consider the direct sum

$$\bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n, U(1))} (A^{p,q}(G/\Gamma, p^*L_{\alpha'}), \bar\partial)$$

of Dolbeault complexes and the direct sum

$$\bigoplus_{\alpha \in \mathrm{Hom}(\mathbb{C}^n, U(1))} (A^{p,q}(G, \mathbb{C}_{p^*\alpha})^G, \bar\partial)$$

of left-invariant Dolbeault complexes. By the inclusion

$$(A^{p,q}(G, \mathbb{C}_{p^*\alpha})^G, \bar\partial) \subset (A^{p,q}(G, \mathbb{C}_{p^*\alpha})^\Gamma, \bar\partial) = (A^{p,q}(G/\Gamma, p^*L_{R_\Gamma(\alpha)}), \bar\partial),$$

We have the cochain map

$$I : \bigoplus_{\alpha \in \mathrm{Hom}(\mathbb{C}^n, U(1))} (A^{p,q}(G, \mathbb{C}_{p^*\alpha})^G, \bar\partial) \to \bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n, U(1))} (A^{p,q}(G/\Gamma, p^*L_{\alpha'}), \bar\partial).$$

If distinct characters $\alpha \neq \beta \in \mathrm{Hom}(\mathbb{C}^n, U(1))$ satisfy $R_{\Gamma'}(\alpha) = R_{\Gamma'}(\beta)$, then $I(A^{p,q}(G, \mathbb{C}_{p^*\alpha})^G)$ and $I(A^{p,q}(G, \mathbb{C}_{p^*\beta})^G)$ are contained in $A^{p,q}(G/\Gamma, p^*L_{R_{\Gamma'}(\alpha)}) = A^{p,q}(G/\Gamma, p^*L_{R_{\Gamma'}(\beta)})$. Otherwise in this case we have

$$\begin{aligned}
&I(A^{p,q}(G, \mathbb{C}_{p^*\alpha})^G) \cap I(A^{p,q}(G, \mathbb{C}_{p^*\alpha})^G) \\
&= ((A^{p,q}(G)^G \otimes (p^*\alpha^{-1}\mathbb{C}_{R_{\Gamma'}(\alpha)=R_{\Gamma'}(\beta)})) \cap ((A^{p,q}(G)^G \otimes (p^*\beta^{-1}\mathbb{C}_{R_{\Gamma'}(\alpha)=R_{\Gamma'}(\beta)})) = \emptyset.
\end{aligned}$$

Hence the map

$$I : \bigoplus_{\alpha \in \mathrm{Hom}(\mathbb{C}^n, U(1))} (A^{p,q}(G, \mathbb{C}_{p^*\alpha})^G, \bar\partial) \to \bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n, U(1))} (A^{p,q}(G/\Gamma, p^*L_{\alpha'}), \bar\partial)$$

is injective.

4.5. **The subcomplex $A^{p,q}$.** Since the action ϕ on $\bigwedge \mathfrak{n}_\mathbb{C}$ is semi-simple, we have the weight decomposition

$$\bigwedge \mathfrak{n}^*_\mathbb{C} = \bigoplus_I W_{\alpha_I}$$

for the action ϕ. Then as similar to subsection 4.1, we have

$$\bigwedge \mathfrak{g}_\mathbb{C} = \bigoplus_I \bigwedge \mathfrak{a}^*_\mathbb{C} \otimes \alpha_I^{-1} W_{\alpha_I}.$$

By Lemma 2.2, we have $\beta_I \in \mathrm{Hom}(\mathbb{C}^n, U(1))$ such that $\alpha_I \sim \beta_I$ for each α_I. We define the subspace A^* of $\bigoplus_{\alpha\in\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G,\mathbb{C}_{p^*\alpha})^G,\bar\partial)$ as

$$A^* = \bigoplus_I \bigwedge \mathfrak{a}^*_{\mathbb{C}} \otimes \alpha_I^{-1} W_{\alpha_I} \otimes \langle l_{\beta_I^{-1}}\rangle$$

where we consider the identification $(\bigwedge^{p,q}\mathfrak{g}^*\otimes\langle l_{\beta_I^{-1}}\rangle,\bar\partial)\cong(A^{p,q}(G,\mathbb{C}_{\beta_I^{-1}})^G,\bar\partial)$ as Section 3.

Lemma 4.2. *$(A^{p,q},\bar\partial)$ is a subcomplex of $\bigoplus_{\alpha\in\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G,\mathbb{C}_{p^*\alpha})^G,\bar\partial)$ and we have an isomorphism*

$$(A^{p,q},\bar\partial)\cong(\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*,\bar\partial)$$

of bi-graded cochain complexes.

Proof. For $\theta\wedge\alpha_I^{-1}w_{\alpha_I}\otimes l_{\beta_I^{-1}}\in\bigwedge\mathfrak{a}^*\otimes\alpha_I^{-1}W_{\alpha_I}\otimes\langle l_{\beta_I^{-1}}\rangle$, since $\alpha_I^{-1}\beta_I$ is holomorphic, we have

$$\begin{aligned}&\bar\partial(\theta\wedge\alpha_I^{-1}w_{\alpha_I}\otimes l_{\beta_I^{-1}})\\ &\qquad=(-1)^{\deg\theta}\theta\wedge\bar\partial(\alpha_I^{-1}\beta_I w_{\alpha_I})\otimes v_{\beta^{-1}}=(-1)^{\deg\theta}\theta\wedge\bar\alpha_I^{-1}\bar\partial w_{\alpha_I}\otimes l_{\beta^{-1}}.\end{aligned}$$

Since $\phi(t)$ is holomorphic for any $t\in\mathbb{C}^n$, we have $\bar\partial W_I\subset W_I$. Thus $\bar\partial$ preserves A^* and so $(A^{p,q},\bar\partial)$ is a subcomplex of $\bigoplus_{\alpha\in\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G,\mathbb{C}_{p^*\alpha})^G,\bar\partial)$. Consider the linear map $A^*\to\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*$ given by

$$\begin{aligned}A^*=\bigoplus_I\bigwedge\mathfrak{a}^*_{\mathbb{C}}\otimes\alpha_I^{-1}W_{\alpha_I}\otimes\langle l_{\beta_I^{-1}}\rangle\ni\theta\wedge\alpha_I^{-1}w_{\alpha_I}\otimes l_{\beta_I^{-1}}\\ \mapsto\theta\wedge w_{\alpha_I}\in\bigoplus_I\bigwedge\mathfrak{a}^*_{\mathbb{C}}\otimes W_{\alpha_I}=\bigwedge(\mathfrak{a}\oplus\mathfrak{n})^*_{\mathbb{C}}.\end{aligned}$$

Then this map is an isomorphism of vector spaces and by the above computation this map is a homomorphism of cochain complexes and hence the lemma follows. □

Consider the basis $x_1,\dots,x_n,\alpha_1^{-1}y_1,\dots,\alpha_m^{-1}y_m$ of $\mathfrak{g}^{1,0}$ as Subsection 4.1. By Lemma 2.2, we take unique unitary characters $\beta_i,\gamma_i\in\mathrm{Hom}(\mathbb{C}^n,U(1))$ such that $\alpha_i\sim\beta_i$ and $\bar\alpha_i\sim\gamma_i$. We have

$$\begin{aligned}A^{p,q}=\bigwedge^p\langle x_1,\dots,x_n,\alpha_1^{-1}y_1\otimes l_{\beta_1^{-1}},\dots,\alpha_m^{-1}y_m\otimes l_{\beta_m^{-1}}\rangle\\ \otimes\bigwedge^q\langle\bar x_1,\dots,\bar x_n,\bar\alpha_1^{-1}\bar y_1\otimes l_{\gamma_1^{-1}},\dots,\bar\alpha_m^{-1}\bar y_m\otimes l_{\gamma_m^{-1}}\rangle\end{aligned}$$

and the isomorphism $(A^{p,q},\bar\partial)\cong(\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*,\bar\partial)$ as above is given by

$$x_I\wedge\bar x_J\wedge\alpha_K y_K\wedge\bar\alpha_L y_L\otimes l_{\beta_K\gamma_L}\mapsto x_I\wedge\bar x_J\wedge y_K\wedge\bar y_L\in\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*.$$

where $I=\{i_1,\dots,i_{p_1}\}$, $J=\{j_1,\dots,j_{q_1}\}$, $K=\{k_1,\dots,k_{p_2}\}$, and $L=\{l_1,\dots,l_{q_2}\}$ with $p_1+p_2=p$, $q_1+q_2=q$ and I', J', K' and L' are complements and we write $x_I=x_{i_1}\wedge\cdots\wedge x_{i_{p_1}}$, $\alpha_K=\alpha_{k_1}\cdots\alpha_{k_{p_2}}$, etc.

4.6. **Harmonic forms and Dolbeault cohomology.** Let g be the Hermittian metric such that $x_1, \dots, x_n, \alpha_1 y_1, \dots, \alpha_m y_m$ is an orthonormal basis. Consider the Hermittian metric h_α on L_α for $\alpha \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n, U(1))$ as Section 3. Then for the Hodge star operator $\bar{*}$ on $\bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G/\Gamma, p^* L_{\alpha'}), \bar\partial)$ we have

$$\begin{aligned}&\bar{*}(x_I \wedge \bar{x}_J \wedge \alpha_K y_K \wedge \bar\alpha_L y_L \otimes l_{\beta_K \gamma_L})\\ &\qquad = (\sqrt{-1})^{n^2} \epsilon(IJKLI'J'K'L') x_{I'} \wedge \bar{x}_{J'} \wedge \alpha_{K'} y_{K'} \wedge \bar\alpha_{L'} y_{L'} \otimes l_{\beta_{K'} \gamma_{L'}}.\end{aligned}$$

Hence the operator $\bar{*}$ preserves the subcomplex $A^{p,q}$. Consider the Hermittian metric on $\mathbb{C}^n \times N$ such that $X_1, \dots, X_n, Y_1, \dots, Y_m$ is an orthonormal basis. Then by this computation, for the isomorphism $\Phi : (A^{p,q}, \bar\partial) \cong (\bigwedge^{p,q} (\mathfrak{a} \oplus \mathfrak{n})^*, \bar\partial)$ in the proof of Lemma 4.2, we have $\Phi \circ \bar{*} = \bar{*} \circ \Phi$. Thus by the inclusion

$$A^{p,q} \subset \bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))} A^{p,q}(G/\Gamma, p^* L_{\alpha'}),$$

we have an injection

$$\mathcal{H}^{p,q}((\mathfrak{a} \oplus \mathfrak{n})_{\mathbb{C}}, \mathbb{C}) \to \bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1)))} \mathcal{H}^{p,q}((G/\Gamma, p^* L_{\alpha'})$$

and hence by Theorem 3.1 we have:

Lemma 4.3. *There exists an injection*

$$H^{p,q}_{\bar\partial}(\bigwedge^{p,q}(\mathfrak{a} \oplus \mathfrak{n})^*) \to H^{p,q}_{\bar\partial}(\bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))} (A^{p,q}(G/\Gamma, p^* L_{\alpha'})).$$

4.7. **Main theorem.**

Theorem 4.4. *Let $G = \mathbb{C}^n \ltimes_\phi N$ such that:*
(1) N is simply connected nilpotent Lie group with a left-invariant complex structure.
(2) For any $t \in \mathbb{C}^n$, $\phi(t)$ is a holomorphic automorphism of (N, J).
(3) ϕ induces a semi-simple action on the Lie algebra of N.
Denote $\mathfrak{a}$ and $\mathfrak{n}$ as the Lie subalgebras of $\mathfrak{g}$ corresponding to $\mathbb{C}^n$ and N respectively. Suppose G has a lattice $\Gamma = \Gamma' \ltimes_\phi \Gamma''$ such that Γ' and Γ'' are lattices of $\mathbb{C}^n$ and N respectively. Moreover we suppose that the inclusion $\bigwedge^{p,q} \mathfrak{n}^* \subset A^{p,q}(N/\Gamma'')$ induces an isomorphism $H^{p,q}_{\bar\partial}(\bigwedge^{p,q} \mathfrak{n}^*) \cong H^{p,q}_{\bar\partial}(N/\Gamma'')$. Then we have an isomorphism

$$H^{p,q}_{\bar\partial}(\bigwedge^{p,q}(\mathfrak{a} \oplus \mathfrak{n})^*) \cong H^{p,q}_{\bar\partial}(\bigoplus_{\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))} A^{p,q}(G/\Gamma, p^* L_{\alpha'})).$$

Proof. For $\alpha' \in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n, U(1))$, by Borel's results in [7, Appendix 2], we have the spectral sequence (E_r, d_r) of the filtration of $A^{p,q}(G/\Gamma, p^* L_{\alpha'})$ induced by the holomorphic fiber bundle $p : G/\Gamma \to \mathbb{C}^n/\Gamma'$ in Proposition 4.1 such that :
(1)E_r is 4-graded, by the fiber-degree, the base-degree and the type. Let ${}^{p,q}E_r^{s,t}$ be the subspace of elements of E_r of type (p,q), fiber-degree s and base-degree t. We have ${}^{p,q}E_r^{s,t} = 0$ if $p + q = s + t$ or if one of p, q, s, t is negative.
(2) If $p + q = s + t$, then we have

$${}^{p,q}E_2^{s,t} \cong \sum_{i \geq 0} H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma', L_{\alpha,} \otimes \mathbf{H}^{p-i,q-s+i}(N/\Gamma''))$$

where $\mathbf{H}^{p-i,q-s+i}(N/\Gamma'')$ is the holomorphic fiber bundle $\bigcup_{b\in\mathbb{C}^n/\Gamma'} H^{p,q}_{\bar\partial}(p^{-1}(b))$.
(3)The spectral sequence converges to $H_{\bar\partial}(G/\Gamma, L_{\alpha'})$.
By the assumption $H^{p,q}_{\bar\partial}(\bigwedge^{p,q}\mathfrak{n}^*)\cong H^{p,q}_{\bar\partial}(N/\Gamma'')$, the fiber bundle $\mathbf{H}^{p-i,q-s+i}(N/\Gamma'')$ is the holomorphic vector bundle with the fiber $H^{p-i,q-s+i}_{\bar\partial}(\bigwedge^{p-i,q-s+i}\mathfrak{n}^*)$ induced by the action ϕ of Γ on $H^{p-i,q-s+i}_{\bar\partial}(\bigwedge^{p-i,q-s+i}\mathfrak{n}^*)$. Since the action ϕ on $\mathfrak{n}$ is semi-simple, diagonalizing the action ϕ, we have the weight decomposition

$$H^{p-i,q-s+i}_{\bar\partial}(\bigwedge^{p-i,q-s+i}\mathfrak{n}^*)=\oplus V_{\beta_j}$$

as $\mathbb{C}^n$-modules and hence the fiber bundle splits as $\mathbf{H}^{p-i,q-s+i}(N/\Gamma'')=\oplus L_{R_{\Gamma'}(\beta_j)}$.
Hence we have

$$H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma', L_{\alpha'}\otimes \mathbf{H}^{p-i,q-s+i}(N/\Gamma''))=H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma', \bigoplus_{\beta_j} L_{\alpha'}\otimes L_{R_{\Gamma'}(\beta_j)}).$$

By Proposition 2.4, we have $H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma', L_{\alpha'}\otimes L_{R_{\Gamma'}(\beta_j)})\cong H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma')$ if $\alpha'^{-1}=R_{\Gamma'}(\beta_j)$ and $H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma', L_{\alpha'}\otimes L_{\beta_j})=0$ if $\alpha'^{-1}\neq R_{\Gamma'}(\beta_j)$. Hence we have

$$\begin{aligned}&H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma', \bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}L_{\alpha'}\otimes \mathbf{H}^{p-i,q-s+i}(N/\Gamma''))\\ &\qquad\cong H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma')\otimes H^{p-i,q-s+i}_{\bar\partial}(\bigwedge^{p-i,q-s+i}\mathfrak{n}^*).\end{aligned}$$

For the direct sum $\bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G/\Gamma, p^*L_{\alpha'})$, we consider this spectral sequence E_r. Then we have

$$\begin{aligned}{}_{p,q}E_2^{s,t}&\cong\sum_{i\ge 0}H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma', \bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}L_{\alpha'}\otimes \mathbf{H}^{p-i,q-s+i}(N/\Gamma''))\\ &\cong\sum_{i\ge 0}H^{i,i-s}_{\bar\partial}(\mathbb{C}^n/\Gamma')\otimes H^{p-i,q-s+i}_{\bar\partial}(\bigwedge^{p-i,q-s+i}\mathfrak{n}^*)\end{aligned}$$

By this we have an isomorphism $E_2\cong\bigoplus_{p,q}H^{p,q}_{\bar\partial}(\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*)$. Otherwise by Lemma 4.3, we have

$$E_\infty\cong\bigoplus_{p,q}H^{p,q}_{\bar\partial}(\bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}A^{p,q}(G/\Gamma, p^*L_{\alpha'}))\supset\bigoplus_{p,q}H^{p,q}_{\bar\partial}(\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*).$$

Hence the spectral sequence degenerates at E_2 and the theorem follows. □

Remark 1. It is not known whether an isomorphism $H^{p,q}_{\bar\partial}(\bigwedge^{p,q}\mathfrak{n}^*)\cong H^{p,q}_{\bar\partial}(N/\Gamma'')$ holds for any simply connected nilpotent Lie group N with a lattice Γ'' and a left invariant complex structure J. But this holds under the one of the following conditions:
(N) J is a nilpotent complex structure i.e. there exists a basis $y_1,\dots,y_m$ such that

$$dy_i\in\bigwedge^{2}\langle y_1,\dots,y_{i-1}\rangle\oplus\bigwedge^{1,1}\langle y_1,\dots,y_{i-1}\rangle\otimes\langle\bar y_1,\dots,\bar y_{i-1}\rangle$$

for each i (see [3]).
(Q) J is a rational complex structure i.e. for the rational structure $\mathfrak{n}_{\mathbb{Q}}\subset\mathfrak{n}$ of the Lie algebra $\mathfrak{n}$ induced by a lattice Γ(see [12, Section 2]) we have $J(\mathfrak{n}_{\mathbb{Q}})\subset\mathfrak{n}_{\mathbb{Q}}$ (see

[2]).
(C) (N, J) is a complex Lie group (see [14]).

Remark 2. By the wedge products and the tensor products, the direct sum of cochain complexes $\bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G/\Gamma, p^*L_{\alpha'}), \bar\partial)$ is a differential bi-graded algebra(DBA). Since the subcomplex $A^{p,q}$ is a subalgebra of this DBA, the isomorphism in Theorem 4.4 is an isomorphism of graded algebra. If N has a nilpotent complex structure as (N) in Remark 1, then $(\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*, \bar\partial)$ is the minimal model of the DBA $\bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G/\Gamma, p^*L_{\alpha'}), \bar\partial)$(see [10])

Consider the injection $A^{p,q} \to \bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}(A^{p,q}(G/\Gamma, p^*L_{\alpha'}))$. For the Dolbeault complex $A^{p,q}(G/\Gamma)$ on the trivial holomorphic line bundle, by the above theorem the injection $I : I^{-1}(A^{p,q}(G/\Gamma)) \to A^{p,q}(G/\Gamma)$ induces a cohomology isomorphism. Consider

$$\bigwedge^{p,q}\mathfrak{g}^* = \bigwedge^{p}\langle x_1,\dots,x_n,\alpha_1^{-1}y_1,\dots,\alpha_m^{-1}y_m\rangle \otimes \bigwedge^{q}\langle \bar x_1,\dots,\bar x_n,\bar\alpha_1^{-1}\bar y_1,\dots,\bar\alpha_m^{-1}\bar y_m\rangle$$

and $\beta_i, \gamma_i \in \mathrm{Hom}(\mathbb{C}^n, U(1))$ such that $\alpha_i \sim \beta_i$ and $\bar\alpha_i \sim \gamma_i$. Then we have

$$A^{p,q} = \bigoplus_{\substack{|I|+|K|=p,\\ |J|+|L|=q}} \langle x_I \wedge \bar x_J \wedge \alpha_K^{-1} y_K \wedge \bar\alpha_L^{-1} y_L \otimes l_{\beta_K^{-1}\gamma_L^{-1}}\rangle.$$

Then $I(x_I \wedge \bar x_J \wedge \alpha_K^{-1} y_K \wedge \bar\alpha_L^{-1} y_L \otimes l_{\beta_K\gamma_L}) \in A^{p,q}(G/\Gamma)$ if and only if $R_{\Gamma'}(\beta_K\gamma_L) = 1$. Thus we have:

Corollary 4.5. *Consider the subcomplex $B^{p,q} \subset A^{p,q}(G/\Gamma)$ given by*

$$B^{p,q} = \bigoplus_{\substack{|I|+|K|=p,\\ |J|+|L|=q,\\ R_\Gamma(\beta_K\gamma_L)=1}} \langle x_I \wedge \bar x_J \wedge \alpha_K^{-1}\beta_K y_K \wedge \bar\alpha_L^{-1}\gamma_L y_L\rangle.$$

Then $B^{p,q}$ is a subcomplex of $\bigwedge^{p,q}(\mathfrak{a}\oplus\mathfrak{n})^*$ and the inclusion $B^{p,q} \subset A^{p,q}(G/\Gamma)$ induces a cohomology isomorphism

$$H^{p,q}_{\bar\partial}(B^{p,q}) \cong H^{p,q}_{\bar\partial}(G/\Gamma).$$

Remark 3. We suppose the following condition:
($\star$) For any K, L, if $\beta_K\gamma_L) \neq 1$, then $R_\Gamma(\beta_K\gamma_L) \neq 1$.
Then we have $B^{p,q} \subset \bigwedge^{p,q}\mathfrak{g}^*$ and hence in this condition we have an isomorphism

$$H^{p,q}_{\bar\partial}(\mathfrak{g}_{\mathbb{C}}) \cong H^{p,q}_{\bar\partial}(G/\Gamma).$$

Remark 4. Suppose $\phi : \mathbb{C}^n \to \mathrm{Aut}(\mathfrak{n}^{1,0})$ is a holomorphic map. Then each α_i is holomorphic and hence

$$B^{1,0} = \langle x_1,\dots,x_n,\alpha_1^{-1}y_1,\dots,\alpha_m^{-1}y_m\rangle = \mathfrak{g}^{1,0}.$$

Moreover if N is a complex nilpotent Lie group, then $G = \mathbb{C}^n \ltimes_\phi N$ is also a complex Lie group and any element of $B^{1,0} = \mathfrak{g}^{1,0}$ is holomorphic and hence $\bar\partial B^{p,0} = 0$. Thus by the above corollary, in this case we have an isomorphism

$$H^{p,q}(G/\Gamma) \cong \bigwedge^{p}\mathfrak{g}^{1,0} \otimes H^{q}_{\bar\partial}(\bigwedge B^{0,q}).$$

5. Examples

5.1. **Example 1.** Let $G=\mathbb{C}\ltimes_{\phi}\mathbb{C}^2$ such that $\phi(x+\sqrt{-1}y)=\left(\begin{array}{cc} e^{x} & 0\\ 0 & e^{-x}\end{array}\right)$. Then for some $a\in\mathbb{R}$ the matrix $\left(\begin{array}{cc} e^{x} & 0\\ 0 & e^{-x}\end{array}\right)$ is conjugate to an element of $SL(2,\mathbb{Z})$. Hence for any $0\neq b\in\mathbb{R}$ we have a lattice $\Gamma=(a\mathbb{Z}+b\sqrt{-1}\mathbb{Z})\ltimes\Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$. Then for a coordinate $(z_1=x+\sqrt{-1}y,z_2,z_3)\in\mathbb{C}\ltimes_{\phi}\mathbb{C}^2$ we have

$$\bigwedge^{p,q}\mathfrak{g}^*=\bigwedge^{p,q}\langle dz_1,e^{-x}dz_2,e^{x}dz_3\rangle\otimes\langle d\bar z_1,e^{-x}d\bar z_2,e^{x}d\bar z_3\rangle.$$

Since we have $e^{x}\sim e^{-\sqrt{-1}y}$, the subcomplex

$$A^{p,q}\subset\bigoplus_{\alpha'\in R_{\Gamma'}(\mathrm{Hom}(\mathbb{C}^n,U(1))}A^{p,q}(G/\Gamma,p^*L_{\alpha'})$$

as Section 4 is given by

$$\begin{aligned}A^{p,q}=\bigwedge^{p,q}&\langle dz_1,e^{-x}dz_2\otimes l_{e^{\sqrt{-1}y}},e^{x}dz_3\otimes l_{e^{-\sqrt{-1}y}}\rangle\\ &\otimes\langle d\bar z_1,e^{-x}d\bar z_2\otimes l_{e^{\sqrt{-1}y}},e^{x}d\bar z_3\otimes l_{e^{-\sqrt{-1}y}}\rangle.\end{aligned}$$

$B^{p,q}\subset A^{p,q}(G/\Gamma)$ varies for a choice of $b\in\mathbb{R}$ as the following.
(A) If $b=2n\pi$ for $n\in\mathbb{Z}$, then $R_{\Gamma}(e^{\sqrt{-1}y})=1$ and we have:

$$B^{p,q}=\bigoplus\bigwedge^{p,q}\langle dz_1,e^{-x-\sqrt{-1}y}dz_2,e^{x+\sqrt{-1}y}dz_3\rangle\otimes\langle d\bar z_1,e^{-x-\sqrt{-1}y}d\bar z_2,e^{x+\sqrt{-1}y}d\bar z_3\rangle.$$

(B) If $b=(2n-1)\pi$ for $n\in\mathbb{Z}$, then $R_{\Gamma}(e^{\sqrt{-1}y})\neq1$ but $R_{\Gamma}(e^{2\sqrt{-1}y})=1$ and so we have:

$$B^{1,0}=\langle dz_1\rangle,\ B^{0,1}=\langle d\bar z_1\rangle,$$
$$B^{2,0}=\langle dz_2\wedge dz_3\rangle,\ B^{0,2}=\langle d\bar z_2\wedge d\bar z_3\rangle,$$
$$B^{1,1}=\langle dz_1\wedge d\bar z_1,e^{-2x-2\sqrt{-1}y}dz_2\wedge d\bar z_2,e^{2x+2\sqrt{-1}y}dz_3\wedge d\bar z_3,dz_2\wedge d\bar z_3,dz_3\wedge d\bar z_2\rangle,$$
$$B^{3,0}=\langle dz_1\wedge dz_2\wedge dz_3\rangle,$$

$$\begin{aligned}B^{2,1}=&\langle dz_2\wedge dz_3\wedge d\bar z_1,e^{-2x-2\sqrt{-1}y}dz_1\wedge dz_2\wedge d\bar z_2,\\ &e^{2x+2\sqrt{-1}y}dz_1\wedge dz_3\wedge d\bar z_3,dz_1\wedge dz_2\wedge d\bar z_3,dz_1\wedge dz_3\wedge d\bar z_2\rangle,\end{aligned}$$

$$\begin{aligned}B^{1,2}=&\langle dz_1\wedge d\bar z_2\wedge d\bar z_3,e^{-2x-2\sqrt{-1}y}dz_2\wedge d\bar z_1\wedge d\bar z_2,\\ &e^{2x+2\sqrt{-1}y}dz_3\wedge d\bar z_1\wedge d\bar z_3,dz_2\wedge d\bar z_3\wedge d\bar z_1,dz_3\wedge d\bar z_2\wedge d\bar z_1\rangle,\end{aligned}$$

$$B^{0,3}=\langle d\bar z_1\wedge d\bar z_2\wedge d\bar z_3\rangle,$$
$$B^{3,1}=\langle dz_1\wedge dz_2\wedge dz_3\wedge d\bar z_1\rangle,B^{1,3}=\langle d\bar z_1\wedge d\bar z_2\wedge d\bar z_3\wedge d\bar z_1\rangle,$$

$$\begin{aligned}B^{2,2}=&\langle dz_1\wedge dz_2\wedge d\bar z_1\wedge d\bar z_3,\\ &e^{-2x-2\sqrt{-1}y}dz_1\wedge dz_2\wedge d\bar z_1\wedge d\bar z_2,e^{2x+2\sqrt{-1}y}dz_1\wedge dz_3\wedge d\bar z_1\wedge d\bar z_3,\\ &dz_2\wedge dz_3\wedge d\bar z_2\wedge d\bar z_3,dz_1\wedge dz_3\wedge d\bar z_1\wedge d\bar z_2\rangle,\end{aligned}$$

$$B^{3,2}=\langle dz_2\wedge dz_3\wedge d\bar z_1\wedge d\bar z_2\wedge d\bar z_3\rangle,B^{2,3}=\langle dz_1\wedge dz_2\wedge dz_3\wedge d\bar z_2\wedge d\bar z_3\rangle,$$
$$B^{3,3}=\langle dz_1\wedge dz_2\wedge dz_3\wedge d\bar z_1\wedge d\bar z_2\wedge d\bar z_3\rangle.$$

(C) If $b \neq n\pi$ for any $n \in \mathbb{Z}$, then $R_\Gamma(e^{\sqrt{-1}y}) \neq 1$ and $R_\Gamma(e^{2\sqrt{-1}y}) \neq 1$ and so we have:

$$B^{1,0} = \langle dz_1 \rangle,\ B^{0,1} = \langle d\bar{z}_1 \rangle,$$
$$B^{2,0} = \langle dz_2 \wedge dz_3 \rangle,\ B^{0,2} = \langle d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$
$$B^{1,1} = \langle dz_1 \wedge d\bar{z}_1, dz_2 \wedge d\bar{z}_3, dz_3 \wedge d\bar{z}_2 \rangle,$$
$$B^{3,0} = \langle dz_1 \wedge dz_2 \wedge dz_3 \rangle, B^{2,1} = \langle dz_2 \wedge dz_3 \wedge d\bar{z}_1 dz_1 \wedge dz_2 \wedge d\bar{z}_3, dz_1 \wedge dz_3 \wedge d\bar{z}_2 \rangle,$$
$$B^{1,2} = \langle dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3, dz_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1, dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_1 \rangle, B^{0,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$
$$B^{3,1} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \rangle, B^{1,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_1 \rangle,$$
$$B^{2,2} = \langle dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_3, dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3, dz_1 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \rangle,$$
$$B^{3,2} = \langle dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle, B^{2,3} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle,$$
$$B^{3,3} = \langle dz_1 \wedge dz_2 \wedge dz_3 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \rangle.$$

By Corollary 4.5, for each case we have an isomorphism $H^{p,q}_{\bar\partial}(G/\Gamma) \cong B^{p,q}$. Moreover considering the left-invariant Hermittian metric $g = dz_1 d\bar{z}_1 + e^{-2x} dz_2 d\bar{z}_2 + e^{2x} dz_3 d\bar{z}_3$, we have $\mathcal{H}^{p,q}(G/\Gamma) \cong B^{p,q}$.

Remark 5. In the case (A), the Dolbeault cohomology $H^{p,q}_{\bar\partial}(G/\Gamma)$ is isomorphic to the Dolbeault cohomology of complex 3-torus. But G/Γ is not isomorphic to a complex 3-torus. Moreover considering the metric g, the space of the harmonic forms does not satisfy Hodge symmetry(i.e. $\bar{\mathcal{H}}^{p,q}(G/\Gamma) \neq \mathcal{H}^{q,p}(G/\Gamma)$).

Remark 6. By Hattori's result in [6], we have an isomorphism $H^*(G/\Gamma) \cong H^*(\mathfrak{g})$ of de Rham cohomology of G/Γ and the Lie algebra cohomology. Hence considering the space $\mathcal{H}^k_d(\mathfrak{g})$ of left-invariant d-harmonic forms of the left-invariant Hermittian metric g, we have an isomorphism $\mathcal{H}^k_d(\mathfrak{g}) \cong H^k_d(G/\Gamma)$. By simple computations, we have the Hodge decomposition $\mathcal{H}^k_d(G/\Gamma) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(G/\Gamma)$. Hence G/Γ has cohomological properties(for example the Frölicher spectral sequence degenerates at E_1) of compact Kähler manifolds. But by Arapura's result (solving Benson-Gordon's conjecture) in [1], G/Γ admits no Kähler structure.

Remark 7. In the case (C), an isomorphism $H^{p,q}_{\bar\partial}(\mathfrak{g}_{\mathbb{C}}) \cong H^{p,q}_{\bar\partial}(G/\Gamma)$ holds. But in the other cases, this isomorphism does not hold.

5.2. **Example 2.** Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^2$ such that

$$\phi(x + \sqrt{-1}y) = \begin{pmatrix} e^{x+\sqrt{-1}y} & 0 \\ 0 & e^{-x-\sqrt{-1}y} \end{pmatrix}.$$

Then we have $a + \sqrt{-1}b, c + \sqrt{-1}d \in \mathbb{C}$ such that $\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)$ is a lattice in $\mathbb{C}$ and $\phi(a + \sqrt{-1}b)$ and $\phi(c + \sqrt{-1}d)$ are conjugate to elements of $SL(4, \mathbb{Z})$ where we regard $SL(2, \mathbb{C}) \subset SL(4, \mathbb{R})$ (see [5]). Hence we have a lattice $\Gamma = (\mathbb{Z}(a + \sqrt{-1}b) + \mathbb{Z}(c + \sqrt{-1}d)) \ltimes_\phi \Gamma''$ such that Γ'' is a lattice of $\mathbb{C}^2$. For a coordinate $(z_1, z_2, z_3) \in \mathbb{C} \ltimes \mathbb{C}^2$, we have

$$\bigwedge^{p,q} \mathfrak{g}^* = \bigwedge^{p,q} \langle dz_1, e^{-z_1} dz_2, e^{z_1} dz_3 \rangle \otimes \langle d\bar{z}_1, e^{-\bar{z}_1} d\bar{z}_2, e^{\bar{z}_1} d\bar{z}_3 \rangle.$$

We have

$$A^{p,q} = \bigwedge^{p,q} \langle dz_1, e^{-z_1} dz_2, e^{z_1} dz_3 \rangle \otimes \langle d\bar{z}_1, e^{-\bar{z}_1} d\bar{z}_2 \otimes l_{e^{2\sqrt{-1}y_1}}, e^{\bar{z}_1} d\bar{z}_3 \otimes l_{e^{-2\sqrt{-1}y_1}} \rangle$$

for $z_1 = x_1 + \sqrt{-1}y_1$.

If $b, d \in \pi\mathbb{Z}$, then we have

$$H^{p,q}(G/\Gamma) \cong B^{p,q} = \bigwedge^{p,q} \langle dz_1, e^{-z_1}dz_2, e^{z_1}dz_3\rangle \otimes \langle d\bar{z}_1, e^{-z_1}d\bar{z}_2, e^{z_1}d\bar{z}_3\rangle.$$

In this case a solvmanifold in the case (A) of Example 1 is diffeomorphic to this G/Γ or a double covering of this G/Γ.

If $b \notin \pi\mathbb{Z}$ or $c \notin \pi\mathbb{Z}$, then we have

$$B^{0,1} = \langle d\bar{z}_1\rangle,\ B^{0,2} = \langle d\bar{z}_2 \wedge d\bar{z}_3\rangle,\ B^{0,3} = \langle d\bar{z}_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3\rangle$$

and

$$H^{p,q}(G/\Gamma) \cong B^{p,q} = \bigwedge^{p} \langle dz_1, e^{-z_1}dz_2, e^{z_1}dz_3\rangle \otimes B^{0,q}.$$

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(H.kasuya) Graduate school of mathematical science university of tokyo japan
E-mail address: khsc@ms.u-tokyo.ac.jp