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TOPOLOGIES ON CENTRAL EXTENSIONS OF VON NEUMANN ALGEBRAS

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Abstract

Given a von Neumann algebra M we consider the central extension $E(M)$ of M. We introduce the topology $t_c(M)$ on $E(M)$ generated by a center-valued norm and prove that it coincides with the topology of convergence locally in measure on $E(M)$ if and only if M does not have direct summands of type II. We also show that $t_c(M)$ restricted on the set $E(M)_h$ of self-adjoint elements of $E(M)$ coincides with the order topology on $E(M)_h$ if and only if M is a σ-finite type I_{fin} von Neumann algebra.

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1 Introduction

In the series of paper [1]-[5] we have considered derivations on the algebra $LS(M)$ of locally measurable operators affiliated with a von Neumann algebra M, and on various subalgebras of $LS(M)$. A complete description of derivations has been obtained in the case of von Neumann algebras of type I and III. A comprehensive survey of recent results concerning derivations on various algebras of unbounded operators affiliated with von Neumann algebras is presented in [4]. A general form of automorphisms on the algebra $LS(M)$ in the case of von Neumann algebras of type I has been obtained in [5]. In proof of the main results of the above papers the crucial role is played by the co-called central extensions of von Neumann algebras and also by various topologies considered in [3].

Let M be an arbitrary von Neumann algebra with the center $Z(M)$ and let $LS(M)$ denote the algebra of all locally measurable operators with respect M. We consider the set $E(M)$ of all elements x from $LS(M)$ for which there exists a sequence of mutually orthogonal central projections $\{z_i\}_{i\in I}$ in M with $\bigvee\limits_{i\in I} z_i = \mathbf{1}$, such that $z_i x \in M$ for all $i \in I$. It is known [3] that $E(M)$ is a *-subalgebra in $LS(M)$ with the center $S(Z(M))$, where $S(Z(M))$ is the algebra of all measurable operators with respect to $Z(M)$, moreover, $LS(M) = E(M)$ if and only if M does not have direct summands of type II.

A similar notion (i.e. the algebra $E(\mathcal{A})$) for arbitrary *-subalgebras $\mathcal{A} \subset LS(M)$ was independently introduced by M.A. Muratov and V.I. Chilin [7]. The algebra $E(M)$ is called *the central extension of* M. It is known ([3], [7]) that an element $x \in LS(M)$ belongs to $E(M)$ if and only if there exists $f \in S(Z(M))$ such that $|x| \leq f$. Therefore for each $x \in E(M)$ one can define the following vector-valued norm $||x|| = \inf\{f \in S(Z(M)) : |x| \leq f\}$. This center-valued norm naturally generates a topology on $E(M)$ which denoted by $t_c(M)$.

In this paper we study the relationship between the topology $t_c(M)$ on $E(M)$ generated by the above center-valued norm, the topology $t(M)$ – of convergence locally in measure, and the order topology $t_o(M)$ on $E(M)_h$. We prove that $t_c(M)$ coincides with the topology $t(M)$ on $E(M)$ if and only if M does not have direct summands of type II. We show that $t_c(M)$ coincides with the order topology on $E(M)_h$ if and only if M is a σ-finite type I_{fin} algebra.

2 Central extensions of von Neumann algebras

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. Consider a von Neumann algebra M in $B(H)$ with the operator norm $\|\cdot\|_M$. Denote by $P(M)$ the lattice of projections in M.

A linear subspace $\mathcal{D}$ in H is said to be *affiliated* with M (denoted as $\mathcal{D}\eta M$), if

$u(\mathcal{D}) \subset \mathcal{D}$ for every unitary u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the von Neumann algebra M.

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be *affiliated* with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$.

A linear subspace $\mathcal{D}$ in H is said to be *strongly dense* in H with respect to the von Neumann algebra M, if

1) $\mathcal{D}\eta M$;

2) there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbf{1}$, $p_n(H) \subset \mathcal{D}$ and $p_n^{\perp} = \mathbf{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, where $\mathbf{1}$ is the identity in M.

A closed linear operator x acting in the Hilbert space H is said to be *measurable* with respect to the von Neumann algebra M, if $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H. Denote by $S(M)$ the set of all measurable operators with respect to M (see [10]).

A closed linear operator x in H is said to be *locally measurable* with respect to the von Neumann algebra M, if $x\eta M$ and there exists a sequence $\{z_n\}_{n=1}^{\infty}$ of central projections in M such that $z_n \uparrow \mathbf{1}$ and $z_n x \in S(M)$ for all $n \in \mathbb{N}$ (see [11]).

It is well-known [6], [11] that the set $LS(M)$ of all locally measurable operators with respect to M is a unital *-algebra when equipped with the algebraic operations of strong addition and multiplication and taking the adjoint of an operator, and contains $S(M)$ as a solid *-subalgebra.

Let (Ω, Σ, μ) be a measure space and from now on suppose that the measure μ has the direct sum property, i. e. there is a family $\{\Omega_i\}_{i\in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup\limits_{i\in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0(\Omega, \Sigma, \mu)$ the algebra of all (equivalence classes of) complex measurable functions on (Ω, Σ, μ) equipped with the topology of convergence in measure.

Consider the algebra $S(Z(M))$ of operators which are measurable with respect to the center $Z(M)$ of the von Neumann algebra M. Since $Z(M)$ is an abelian von Neumann algebra it is *-isomorphic to $L^{\infty}(\Omega, \Sigma, \mu)$ for an appropriate measure space (Ω, Σ, μ). Therefore the algebra $S(Z(M))$ coincides with $Z(LS(M))$ and can be identified with the algebra $L^0(\Omega, \Sigma, \mu)$ of all measurable functions on (Ω, Σ, μ).

The basis of neighborhoods of zero in the topology of convergence locally in measure on $L^0(\Omega, \Sigma, \mu)$ consists of the sets

$$W(A, \varepsilon, \delta) = \{f \in L^0(\Omega, \Sigma, \mu) : \exists B \in \Sigma,\ B \subseteq A,\ \mu(A \setminus B) \leq \delta,$$

$$f \cdot \chi_B \in L^{\infty}(\Omega, \Sigma, \mu),\ ||f \cdot \chi_B||_{L^{\infty}(\Omega,\Sigma,\mu)} \leq \varepsilon\},$$

where $\varepsilon, \delta > 0$, $A \in \Sigma$, $\mu(A) < +\infty$, and χ_B is the characteric function of the set $B \in \Sigma$.

Recall the definition of the dimension functions on the lattice $P(M)$ of projection from M (see [6], [10]).

By L_+ we denote the set of all measurable functions $f : (\Omega, \Sigma, \mu) \to [0, \infty]$ (modulo functions equal to zero μ-almost everywhere).

Let M be an arbitrary von Neumann algebra with the center $Z(M) \equiv L^\infty(\Omega, \Sigma, \mu)$. Then there exists a map $D : P(M) \to L_+$ with the following properties:

(i) $d(e)$ is a finite function if only if the projection e is finite;

(ii) $d(e + q) = d(e) + d(q)$ for $p, q \in P(M)$, $eq = 0$;

(iii) $d(uu^*) = d(u^*u)$ for every partial isometry $u \in M$;

(iv) $d(ze) = zd(e)$ for all $z \in P(Z(M))$, $e \in P(M)$;

(v) if $\{e_\alpha\}_{\alpha \in J}$, $e \in P(M)$ and $e_\alpha \uparrow e$, then

$$d(e) = \sup_{\alpha \in J} d(e_\alpha).$$

This map $d : P(M) \to L_+$, is a called the *dimension functions* on $P(M)$.

Recall that for an element $x \in LS(M)$ the projection defined as

$$c(x) = \inf\{z \in P(Z(M)) : zx = x\}$$

is called *the central cover of* x.

Remark 2.1. Let M be a type I von Neumann algebra. If $p, q \in P(M)$ abelian projections are faithful (i.e. with $c(p) = c(q) = \mathbf{1}$,) then the property (iii) implies that $0 < d(p)(\omega) = d(q)(\omega) < \infty$ for μ-almost every $\omega \in \Omega$. Therefore replacing d by $d(p)^{-1}d$ we can assume that $d(p) = c(p)$ for every faithful abelian projection $p \in P(M)$. Thus for all $e \in P(M)$ we have that $d(e) \geq c(e)$.

The basis of neighborhoods of zero in *the topology* $t(M)$ *of convergence locally in measure* on $LS(M)$ consists (in the above notations) of the following sets

$$V(A, \varepsilon, \delta) = \{x \in LS(M) : \exists p \in P(M), \exists z \in P(Z(M)), xp \in M,$$

$$||xp||_M \leq \varepsilon,\ z^\perp \in W(A, \varepsilon, \delta),\ d(zp^\perp) \leq \varepsilon z\},$$

where $\varepsilon, \delta > 0$, $A \in \Sigma$, $\mu(A) < +\infty$.

The topology $t(M)$ is metrizable if and only if the center $Z(M)$ is σ-finite (see [6]).

Given an arbitrary family $\{z_i\}_{i \in I}$ of mutually orthogonal central projections in M with $\bigvee_{i \in I} z_i = \mathbf{1}$ and a family of elements $\{x_i\}_{i \in I}$ in $LS(M)$ there exists a unique element $x \in LS(M)$ such that $z_i x = z_i x_i$ for all $i \in I$. This element is denoted by $x = \sum_{i \in I} z_i x_i$.

We denote by $E(M)$ the set of all elements x from $LS(M)$ for which there exists a sequence of mutually orthogonal central projections $\{z_i\}_{i\in I}$ in M with $\bigvee\limits_{i\in I} z_i = \mathbf{1}$, such that $z_i x \in M$ for all $i \in I$, i.e.

$$E(M) = \{x \in LS(M) : \exists z_i \in P(Z(M)), z_i z_j = 0, i \neq j, \bigvee_{i\in I} z_i = \mathbf{1}, z_i x \in M, i \in I\},$$

where $Z(M)$ is the center of M.

It is known [3] that $E(M)$ is *-subalgebras in $LS(M)$ with the center $S(Z(M))$, where $S(Z(M))$ is the algebra of all measurable operators with respect to $Z(M)$, moreover, $LS(M) = E(M)$ if and only if M does not have direct summands of type II.

A similar notion (i.e. the algebra $E(\mathcal{A})$) for arbitrary *-subalgebras $\mathcal{A} \subset LS(M)$ was independently introduced recently by M.A. Muratov and V.I. Chilin [7]. The algebra $E(M)$ is called *the central extension of* M.

It is known ([3], [7]) that an element $x \in LS(M)$ belongs to $E(M)$ if and only if there exists $f \in S(Z(M))$ such that $|x| \leq f$. Therefore for each $x \in E(M)$ one can define the following vector-valued norm

$$||x|| = \inf\{f \in S(Z(M)) : |x| \leq f\} \tag{2.1}$$

and this norm satisfies the following conditions:

1)$\|x\| \geq 0; \|x\| = 0 \Longleftrightarrow x = 0;$

2)$\|fx\| = |f|\|x\|;$

3)$\|x + y\| \leq \|x\| + \|y\|;$

4)$||xy|| \leq ||x||||y||;$

5)$||xx^*|| = ||x||^2$

for all $x, y \in E(M), f \in S(Z(M))$.

3 Topologies on the central extensions of von Neumann algebras

Let M be an arbitrary von Neumann algebra with the center $Z(M) \equiv L^\infty(\Omega, \Sigma, \mu)$. On the space $E(M)$ we consider the following sets:

$$O(A, \varepsilon, \delta) = \{x \in E(M) : ||x|| \in W(A, \varepsilon, \delta)\}, \tag{3.1}$$

where $\varepsilon, \delta > 0,\ \ A \in \sum,\ \ \mu\,(A) < +\infty$.

The following proposition gives elementary properties of the sets $O(A, \varepsilon, \delta)$, which immediately follow from the corresponding properties of the sets $V(A, \varepsilon, \delta)$ (see [6, Proposition 3.5.1]).

Proposition 3.1. *Let* $\varepsilon, \varepsilon_j > 0$ *and* $\delta, \delta_j > 0, j = 1, 2, A \in \Sigma, \mu(A) < \infty$. *Then*

i) $\lambda O(A,\varepsilon,\delta) = O(A,|\lambda|\varepsilon,\delta) \quad \lambda \in \mathbb{C}, \lambda \neq 0$;
ii) $O(A,\varepsilon_1,\delta_1) \subseteq O(A,\varepsilon_2,\delta_2) \quad \varepsilon_1 \leq \varepsilon_2, \delta_1 \leq \delta_2$;
iii) $O(A,\varepsilon_1,\delta_1) + O(A,\varepsilon_2,\delta_2) \subseteq O(A,\varepsilon_1+\varepsilon_2,\delta_1+\delta_2)$;
iv) $O(A,\varepsilon_1,\delta_1)O(A,\varepsilon_2,\delta_2) \subseteq O(A,\varepsilon_1\varepsilon_2,\delta_1+\delta_2)$;
v) $O^*(A,\varepsilon,\delta) = O(A,\varepsilon,\delta)$, *where* $O^*(A,\varepsilon,\delta) = \{x^* : x \in O(A,\varepsilon,\delta)\}$;
vi) $\bigcap\{O(A,\varepsilon,\delta) : \varepsilon > 0, \delta > 0, A \in \Sigma, \mu(A) < \infty\} = \{0\}$.

From Proposition 3.1 it follows that the system of sets

$$\{x + O(A,\varepsilon,\delta)\}, \tag{3.2}$$

where $x \in E(M), \varepsilon > 0, \delta > 0, A \in \Sigma, \mu(A) < \infty$, defines on $E(M)$, a Hausdorff vector topology $t_c(M)$, for which the sets (3.2) form the base of neighborhoods of the element $x \in E(M)$. Moreover in this topology the involution is continuous and the multiplication is jointly continuous, i.e. $(E(M), t_c(M))$ is a topological *-algebra. From [4, Proposition 5.3] it follows that $(E(M), t_c(M))$ is complete.

Thus we obtain the following result.

Proposition 3.2. *i)* $(E(M), t_c(M))$ *is a complete topological *-algebra;*
ii) M *is a* $t_c(M)$*-dense in* $E(M)$.

Proof. i) is proved above.
ii). Let $x \in E(M)$ and $A \in \Sigma$, $\mu(A) < \infty$, $\varepsilon, \delta > 0$. For $n \in \mathbb{N}$ put

$$B_n = \{\omega \in A : ||x||(\omega) \leq n\}.$$

Since $\mu(A \setminus B_n) \to 0$ as $n \to \infty$ there is $k \in \mathbb{N}$ such that $\mu(A \setminus B_k) \leq \varepsilon$. Put $x_k = \chi_{B_k} x$. Then

$$||x_k|| \leq k\mathbf{1}$$

and

$$||x - x_k||\chi_{B_k} = ||x\chi_{B_k} - x_k\chi_{B_k}|| = ||x\chi_{B_k} - x\chi_{B_k}|| = 0.$$

Thus $x_k \in x + O(A,\varepsilon,\delta)$. This means that $\overline{M}^{t_c(M)} = E(M)$. The proof is complete. □

Remark 3.3. Note that if M is a commutative von Neumann algebra then $||x|| = |x|$ for each $x \in E(M)$, and therefore $O(A,\varepsilon,\delta) = W(A,\varepsilon,\delta)$ for all $\varepsilon, \delta > 0$, $A \in \sum$, $\mu(A) < +\infty$. Hence the topology $t_c(M)$ on $E(M)$ coincides with the topology of convergence locally in measure $t(M)$.

If M is a factor, then $E(M) = M$ and $t_c(M) = t_{\|\cdot\|_M}$, where $t_{\|\cdot\|_M}$ uniform topology on M.

Proposition 3.4. *i) A net* $\{p_\alpha\} \subset P(M)$ *converges to zero with respect to the topology* $t_c(M)$ *if and only if* $c(p_\alpha) \stackrel{t(Z(M))}{\longrightarrow} 0$, *where* $t(Z(M))$ *is the topology of convergence locally in measure on* $Z(M)$.

ii) A net $\{x_\alpha\} \subset E(M)$ converges to zero with respect to the topology $t_c(M)$ if and only if $e_\lambda^\perp(|x_\alpha|) \xrightarrow{t_c(M)} 0$ for any $\lambda > 0$, where $\{e_\lambda(|x_\alpha|)\}$ is a spectral projections family for the operator x_α.

Proof. i) The proof immediately follows from the definition of the topology $t_c(M)$ and the equality $||p|| = c(p)$, $p \in P(M)$.

ii) Let $x_\alpha \xrightarrow{t_c(M)} 0$ and $\lambda > 0$. Take any $A \in \Sigma, \mu(A) < \infty, 0 < \varepsilon < \lambda/2, \delta > 0$. Since $||x_\alpha|| \xrightarrow{t(M)} 0$, then there exists α_0 such that $||x_\alpha|| \in W(A, \varepsilon, \delta)$ for each $\alpha \geq \alpha_0$. Therefore there exists $B_\alpha \in \Sigma, B_\alpha \subseteq A$ such that $\mu(A \setminus B_\alpha) \leq \delta$, $||||x_\alpha||\chi_{B_\alpha}||_M \leq \varepsilon$. Thus $|||x_\alpha|\chi_{B_\alpha}||_M \leq \varepsilon$, i.e. $|x_\alpha|\chi_{B_\alpha} \leq \varepsilon\chi_{B_\alpha}$. Since $\varepsilon < \dfrac{\lambda}{2}$ then from the last inequality we have that $c(e_\lambda^\perp(|x_\alpha|))\chi_{B_\alpha} = 0$. The inequality $\mu(A \setminus B_\alpha) \leq \delta$ implies that $c(e_\lambda^\perp(|x_\alpha|)) \in W(A, \varepsilon, \delta)$, i.e. $c(e_\lambda^\perp(|x_\alpha|)) \xrightarrow{t(Z(M))} 0$. Thus $e_\lambda^\perp(|x_\alpha|) \xrightarrow{t_c(M)} 0$.

Now let $e_\varepsilon^\perp(|x_\alpha|) \xrightarrow{t_c(M)} 0$ and $0 < \varepsilon < 1, \delta > 0$. Then $c(e_\varepsilon^\perp(|x_\alpha|)) \xrightarrow{t(M)} 0$. Therefore there exists α_0 such that $c(e_\varepsilon^\perp(|x_\alpha|)) \in W(A, \varepsilon, \delta)$ for all $\alpha \geq \alpha_0$. Hence there exists $B_\alpha \in \Sigma, B_\alpha \subseteq A$ such that $\mu(A \setminus B_\alpha) \leq \delta$, $||c(e_\varepsilon^\perp(|x_\alpha|))\chi_{B_\alpha}||_M \leq \varepsilon < 1$. Thus $c(e_\varepsilon^\perp(|x_\alpha|))\chi_{B_\alpha} = 0$, i.e. $|x_\alpha|\chi_{B_\alpha} \leq \varepsilon\chi_{B_\alpha}$. Therefore

$$||||x_\alpha||\chi_{B_\alpha}||_M \leq \varepsilon$$

and

$$\mu(A \setminus B_\alpha) \leq \delta.$$

Thus $||x_\alpha|| \in W(A, \varepsilon, \delta)$, i.e. $||x_\alpha|| \xrightarrow{t(Z(M))} 0$. Therefore $x_\alpha \xrightarrow{t_c(M)} 0$. The proof is complete. $\square$

Let $t(M)$ denote the topology on $E(M)$ induced by the topology $t(M)$ from $LS(M)$.

Proposition 3.5. *The topology $t_c(M)$ is stronger than the topology $t(M)$ of convergence locally in measure.*

Proof. It is sufficient to show that

$$O(A, \varepsilon, \delta) \subset V(A, \varepsilon, \delta). \tag{3.3}$$

Let $x \in O(A, \varepsilon, \delta)$, i.e. $||x|| \in W(A, \varepsilon, \delta)$. Then there exists $B \in \Sigma$ such that

$$B \subseteq A, \ \ \mu(A \setminus B) \leq \delta,$$

and

$$||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu), \ \ ||||x\|\chi_B||_M \leq \varepsilon.$$

Put $z = p = \chi_B$. Then $||xp|| = ||x\chi_B|| = ||x||\chi_B \in L^\infty(\Omega, \Sigma, \mu)$, i.e. $xp \in M$ and moreover $||xp||_M \leq \varepsilon$. Since $\mu(A \setminus B) \leq \delta$ and $z^\perp\chi_B = \chi_B^\perp\chi_B = 0$, one has $z^\perp \in W(A, \varepsilon, \delta)$. Therefore

$$||xp||_M \leq \varepsilon, \ \ z^\perp \in W(A, \varepsilon, \delta), \ \ zp^\perp = \chi_B\chi_B^\perp = 0$$

and hence $x \in V(A,\varepsilon,\delta)$, i.e. $O(A,\varepsilon,\delta) \subset V(A,\varepsilon,\delta)$. The proof is complete. $\square$

Proposition 3.6. *If M is a type I or III von Neumann algebra and $0 < \varepsilon < 1$, then*

$$O(A,\varepsilon,\delta) = V(A,\varepsilon,\delta).$$

Proof. From above (3.3) we have that $O(A,\varepsilon,\delta) \subset V(A,\varepsilon,\delta)$. Therefore it is sufficient to show that $V(A,\varepsilon,\delta) \subset O(A,\varepsilon,\delta)$.

Let $x \in V(A,\varepsilon,\delta)$. Then there exist $p \in P(M)$ and $z \in P(Z(M))$ such that

$$xp \in M, \quad ||xp||_M \leq \varepsilon, \quad z^\perp \in W(A,\varepsilon,\delta), \quad d(zp^\perp) \leq \varepsilon z.$$

If M is of type I then Remark 2.1 implies that $d(zp^\perp) \geq c(zp^\perp)$. Now from $d(zp^\perp) \leq \varepsilon z$ it follows that $c(zp^\perp) \leq \varepsilon z$. From $0 < \varepsilon < 1$ we obtain that $zp^\perp = 0$.

If M is of type III then the finiteness of the projection $zp^\perp$ implies that $zp^\perp = 0$.

Thus $z = zp$. Put $z = \chi_E$ for an appropriate $E \in \Sigma$. Since $z^\perp \in W(A,\varepsilon,\delta)$ one has that $\chi_{\Omega\setminus E} \in W(A,\varepsilon,\delta)$. Thus there exists $B \in \Sigma$ such that $B \subseteq A$, $\mu(A \setminus B) \leq \delta$, $|\chi_{\Omega\setminus E}\chi_B| \leq \varepsilon < 1$. Hence $\chi_B \leq \chi_E$. So we obtain

$$||x||\chi_B \leq ||x||\chi_E = ||x||z = ||xz|| = ||xzp|| = ||xp|| \leq \varepsilon.$$

This means that $x \in O(A,\varepsilon,\delta)$. The proof is complete. $\square$

Proposition 3.6 implies that following

Theorem 3.7. *If M is a type I or III von Neumann algebra then the topologies $t(M)$ and $t_c(M)$ coincide.*

Proposition 3.8. *If M is of type II then $t(M) < t_c(M)$.*

Proof. Since M is a type II then there exists a decreasing sequence of projections $\{p_n\}$ in M such that $c(p_n) = \mathbf{1}$ and $d(p_n) = \frac{1}{2^n}$ for all $n \in \mathbb{N}$. Then $\{p_n\}$ converges to zero with respect to the topology locally in measure. Indeed take any neighborhood of zero $V(A,\varepsilon,\delta)$ in the topology $t(M)$. Put $z = \mathbf{1}, p = p_k^\perp$, where the number k is such that $\frac{1}{2^k} < \varepsilon$. For $n \geq k$ we have that

$$p_n p = p_n p_k^\perp = (p_n p_k) p_k^\perp = 0,$$

$$z^\perp \in W(A,\varepsilon,\delta)$$

and

$$d(zp^\perp) = d(p_k) = \frac{1}{2^k}\mathbf{1} \leq \varepsilon z.$$

This means that $p_n \in V(A,\varepsilon,\delta)$ for all $n \geq k$, i.e. $\{p_n\}$ converges to zero with respect to the topology locally in measure.

On the other hand the equality $c(p_n) = \mathbf{1}$ implies that $||p_n|| = \mathbf{1}$. Thus a sequence $\{p_n\}$ does not converges to zero in the topology $t_c(M)$. Hence $t(M) < t_c(M)$. The proof is complete. $\square$

Theorem 3.7 and Proposition 3.8 imply the following result which describes the class of von Neumann algebras M for which the topologies $t(M)$ and $t_c(M)$ coincide.

Theorem 3.9. *The following conditions on a given von Neumann algebra M are equivalent:*

i) $t(M) = t_c(M)$;

ii) M does not have direct summands of type II.

By $E(M)_h$ we denote the set of all selfadjoint elements in $E(M)$. A net $\{x_\alpha\}_{\alpha\in I} \subset E(M)_h$ is called (o)-convergent to $x \in E(M)_h$ (denoted $x_\alpha \xrightarrow{(o)} x$), if there exist nets $\{a_\alpha\}_{\alpha\in I}$ and $\{b_\alpha\}_{\alpha\in I}$ in $E(M)_h$, such that $a_\alpha \leq x_\alpha \leq b_\alpha$ for each $\alpha \in I$ and $a_\alpha \uparrow x$, $b_\alpha \downarrow x$. The strongest topology on $E(M)_h$ for which (o)-convergence of nets implies their convergence in the topology is called *the order topology*, or *the (o)-topology*, and is denoted by $t_o(M)$.

Let $t_{ch}(M)$ (respectively $t_h(M)$) denote the topology on $E(M)_h$ induced by the topology $t_c(M)$ (respectively $t(M)$) from $E(M)$.

We now describe class of von Neumann algebras M for which the topologies $t_c(M)$ and $t_o(M)$ coincide.

Theorem 3.10. *(i) $t_{ch}(M) \leq t_o(M)$ if and only if M is of type I_{fin};*

(ii) $t_{ch}(M) = t_o(M)$ if and only if M is a σ-finite type I_{fin} algebra.

Proof. (i) Let $t_{ch}(M) \leq t_o(M)$. If the algebra M does not has type I_{fin} then there exists a nonzero projection $z \in P(Z(M))$ and a sequence of mutually orthogonal projections $\{p_n\}_{n=1}^{\infty}$ in M with $c(p_n) = z, n \in \mathbb{N}$. Then $p_n \xrightarrow{(o)} 0$, and therefore $p_n \xrightarrow{t_{ch}(M)} 0$. Hence $||p_n|| \xrightarrow{t(Z(M))} 0$. Since $||p_n|| = c(p_n) = z$ it follows that $z = 0$, this is a contradiction with $z \neq 0$. Hence M is a type I_{fin} algebra.

Conversely let M be a type I_{fin} algebra. Then by [3, Proposition 1.1] we have that $LS(M) = E(M)$. Thus theorem 3.9 implies that $t_{ch}(M) = t_h(M)$. Since $t_h(M) \leq t_o(M)$ (see [8, Theorem 1 (i)]) then $t_{ch}(M) \leq t_o(M)$.

(ii) If $t_{ch}(M) = t_o(M)$ then M is a type I_{fin} algebra (see (i)). Again using the theorem 3.9 we have that $t_{ch}(M) = t_h(M)$. Thus $t_h(M) = t_o(M)$. Now by [8, Theorem 1 (ii)] follows that M is a σ-finite algebra.

Conversely let M be a σ-finite type I_{fin} algebra. Then by theorem 3.9 we have that $t_{ch}(M) = t_h(M)$ and by [8, Theorem 1 (ii)] we obtain that $t_h(M) = t_o(M)$. Hence

$$t_{ch}(M) = t_h(M) = t_o(M).$$

The proof is complete. □

Theorem 3.10 yields the following corollary.

Corollary 3.11. *The following assertions are true:*

(i) If M is a σ-finite von Neumann algebra but is not type I_{fin}, then $t_o(M) < t_h(M)$;

(ii) If M is not a σ-finite von Neumann algebra but is type I_{fin}, then $t_h(M) < t_o(M)$.

Proposition 3.12. *The topology $t_c(M)$ is locally convex if and only if M is *-isomorphic to the C^*-product $\bigoplus\limits_{j\in J} M_j$, where M_j are factors.*

Proof. Let $t_c(M)$ be a locally convex topology on $E(M)$. Since $t_c(M)$ induces the topology $t(Z(M))$ on $Z(E(M)) = S(Z(M))$, we have that $(S(Z(M)), t(Z(M)))$ is a locally convex space. It follows from [9, 12, Ch. V, §3] that $Z(M)$ is an atomic von Neumann algebra. Hence, the algebra M is *-isomorphic to the C^*-product $\bigoplus\limits_{j\in J} M_j$, where M_j are factors for all $j \in J$.

Conversely, let $M = \bigoplus\limits_{j\in J} M_j$, where M_j are factors. Then

$$E(M_j) = M_j, t_c(M) = t_{\|\cdot\|_{M_j}}, E(M) = \prod_{j\in J} M_j$$

and, hence the topology $t_c(M)$ is a Tychonoff product of the normed topologies $t_{\|\cdot\|_{M_j}}$, that is, $t_c(M)$ is a locally convex topology. The proof is complete. □

Similarly, we obtain the following

Proposition 3.13. *The topology $t_c(M)$ can be normed if and only if $M = \bigoplus\limits_{j=1}^{n} M_j$, where M_j are factors, $j = \overline{1, n}, n \in \mathbb{N}$.*

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