# THE NORMALIZERS OF SOLVABLE SPHERICAL SUBGROUPS

ROMAN AVDEEV

Abstract. For an arbitrary connected solvable spherical subgroup $H$ of a connected semisimple algebraic group $G$ we compute the group $N_G(H)$, the normalizer of $H$ in $G$. Thereby we complete a classification of all (not necessarily connected) solvable spherical subgroups in semisimple algebraic groups.



## 1. Introduction

Let $G$ be a connected semisimple complex algebraic group. A closed subgroup $H \subset G$ (resp. a homogeneous space $G/H$) is said to be *spherical* if a Borel subgroup $B \subset G$ has an open orbit in $G/H$. The latter condition is equivalent to the relation $\mathfrak{b} + (\operatorname{Ad} g)\mathfrak{h} = \mathfrak{g}$ holding for some element $g \in G$, where $\mathfrak{b}, \mathfrak{h}, \mathfrak{g}$ are the tangent algebras of the groups $B$, $H$, $G$, respectively. In particular, this implies that the sphericity of $H$ (as well as of $G/H$) is a *local* property, that is, it depends only on the pair of algebras $(\mathfrak{g}, \mathfrak{h})$. Thus one may talk about spherical subalgebras in $\mathfrak{g}$. Equivalently, a subgroup $H$ is spherical if and only if its connected component of the identity $H^0$ is spherical.

As the sphericity is a local property, the classification of spherical subgroups in $G$ naturally divides into two stages. The first stage is the classification of *connected* spherical subgroups in $G$, which is equivalent to the classification of spherical subalgebras in $\mathfrak{g}$. At the second stage, for every connected spherical subgroup $H_0 \subset G$ one has to find all subgroups $H \subset G$ with $H^0 = H_0$.

By the present time, the first stage of the classification is accomplished for two 'opposite' classes of spherical subgroups, namely, reductive and solvable. For reductive spherical subgroups, in the case of simple $G$ this is done in [Krä] and in the case of non-simple semisimple $G$ this is done in [Mik] and, independently, in [Br1] (see also [Yak] for a more accurate formulation). For solvable spherical subgroups this is done in [Av2].

Let $H_0$ be a connected subgroup of $G$. Then every subgroup $H \subset G$ with $H^0 = H_0$ is contained in the subgroup $N_G(H_0)$, the normalizer of $H_0$ in $G$. Thereby one establishes a one-to-one correspondence between finite extensions of the subgroup $H_0$ and finite subgroups of the group $N_G(H_0)/H_0$. Thus the second stage of the classification reduces to the computation of the group $N_G(H_0)$ for every connected spherical subgroup $H_0$.

In [Av1], the normalizers of most of reductive spherical subgroups of simple groups are found. In the present paper, we find the normalizers of all connected solvable spherical subgroups of semisimple algebraic groups. For this purpose, we use the structure theory

*Date*: July 27, 2011.

2010 *Mathematics Subject Classification.* 20G07, 14M27.

*Key words and phrases.* Algebraic group, spherical subgroup, solvable subgroup, normalizer.

Partially supported by Russian Foundation for Basic Research (grant no. 09-01-00648) and by Alexander von Humboldt foundation.

of connected solvable spherical subgroups developed in [Av2]. The main results of this paper are Theorem 3 and Proposition 4 (see § 4).

It is known that for an arbitrary spherical subgroup $H \subset G$ the group $N_G(H)/H$ is commutative (and even diagonalizable, see [BP, Corollary 5.2]). Therefore:

(1) a spherical subgroup $H \subset G$ is solvable if and only if the subgroup $H^0$ is solvable; this means that the description of all finite extensions of connected solvable spherical subgroups provides a classification of all (not necessarily connected) solvable spherical subgroups in semisimple algebraic groups;

(2) the subgroup $N_G(H)$ coincides with the subgroup $N_G(H^0)$; in view of this the results of this paper enable one to compute the normalizers of arbitrary solvable spherical subgroups in semisimple algebraic groups.

*Remark* 1. By this moment, a program of classification of *all* spherical homogeneous spaces in terms of so-called *spherical homogeneous data* is completed. This program was initiated by Luna in [Lun] (more details see in [Tim, § 30.11]). However this classification is implicit in the following sense: so far there is no general procedure of constructing a spherical subgroup given by its set of combinatorial data.

This paper is organized as follows. In § 2 we list some notation and conventions used in the paper. In § 3 we collect initial facts from the structure theory of connected solvable spherical subgroups that are needed for formulation of the main results. In § 4 we state the main results of the paper (Theorem 3 and Proposition 4) and give some examples. In § 5 we collect further facts from the structure theory of connected solvable spherical subgroups that are needed for proving the main results. At last, the proofs of the main results are given in § 6.

The author expresses his gratitude to E. B. Vinberg for reading the paper and valuable comments.

## 2. Some notation and conventions

In this paper the base field is the field $\mathbb{C}$ of complex numbers. All groups are assumed to be algebraic and their subgroups closed in the Zarisky topology. The tangent algebras of groups denoted by capital Latin letters are denoted by the corresponding small German letters. For any group $L$ we denote by $\mathfrak{X}(L)$ the character lattice of $L$.

Some notation:

$G$ is an arbitrary connected semisimple algebraic group;

$B \subset G$ is a fixed Borel subgroup of $G$;

$T \subset B$ is a fixed maximal torus of $G$;

$U \subset B$ is the maximal unipotent subgroup of $G$ contained in $B$;

$N_G(T)$ is the normalizer of $T$ in $G$;

$W = N_G(T)/T$ is the Weyl group of $G$ with respect to $T$;

$\theta \colon N_G(T) \to W$ is the canonical homomorphism;

$Q = \mathfrak{X}(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ is the rational vector space spanned by $\mathfrak{X}(T)$;

$(\cdot\,,\cdot)$ is a fixed inner product in $Q$ invariant under $W$;

$\Delta \subset \mathfrak{X}(T)$ is the root system of $G$ with respect to $T$;

$\Delta_+ \subset \Delta$ is the subset of positive roots with respect to $B$;

$\Pi \subset \Delta_+$ is the set of simple roots;

$r_\alpha \in W$ is the simple reflection corresponding to a root $\alpha \in \Pi$;

$\mathfrak{g}_\alpha \subset \mathfrak{g}$ is the root subspace corresponding to a root $\alpha \in \Delta$;
$e$ is the identity element of an arbitrary group;
$|X|$ is the cardinality of a finite set $X$;
$\langle A \rangle$ is the linear span in $Q$ of a subset $A \subset \mathfrak{X}(T)$;
$N_L(K)$ is the normalizer of a subgroup $K$ in a group $L$;
$L^0$ is the connected component of the identity of a group $L$;
$\Sigma(\widetilde{\Pi})$ is the Dynkin diagram of a subset $\widetilde{\Pi} \subset \Pi$.

For every root $\alpha = \sum\limits_{\delta \in \Pi} k_\delta \delta \in \Delta_+$ we consider its *support* $\operatorname{Supp} \alpha = \{\delta \mid k_\delta > 0\}$ and *height* $\operatorname{ht} \alpha = \sum\limits_{\delta \in \Pi} k_\delta$. If $\alpha \in \Delta_+$, we put $\Delta(\alpha) = \Delta \cap \langle \operatorname{Supp} \alpha \rangle$ and $\Delta_+(\alpha) = \Delta_+ \cap \langle \operatorname{Supp} \alpha \rangle$. The set $\Delta(\alpha)$ is an indecomposable root system whose set of simple roots is $\operatorname{Supp} \alpha$; the set of positive roots of this root system coincides with $\Delta_+(\alpha)$.

Let $L$ be a group and let $L_1, L_2$ be subgroups of it. We write $L = L_1 \leftthreetimes L_2$ if $L$ is a semidirect product of $L_1, L_2$, that is, $L = L_1 L_2$, $L_1 \cap L_2 = \{e\}$, and $L_2$ is a normal subgroup of $L$.

By abuse of language, we identify roots in $\Pi$ and the corresponding nodes of the Dynkin diagram of $\Pi$.

When we say that two nodes of a Dynkin diagram are joined by and edge we mean that the edge may be multiple.

The enumeration of nodes (that is, of simple roots) of connected Dynkin diagrams is the same as in the book [OV].

## 3. Preliminaries on connected solvable spherical subgroups

In this subsection we collect basic definitions and initial facts of the structure theory of connected solvable spherical subgroups in semisimple algebraic groups (see [Av2]) needed for formulation of the main results of this paper.

Let $H \subset B$ be a connected solvable subgroup and $N \subset U$ its unipotent radical. We say that the subgroup $H$ is *standardly embedded in* $B$ (with respect to $T$) if the subgroup $S = H \cap T \subset T$ is a maximal torus of $H$. Evidently, in this situation we have $H = S \leftthreetimes N$. Every connected solvable subgroup of $G$ is conjugated to a subgroup that is standardly embedded in $B$.

Suppose that a connected solvable subgroup $H \subset G$ standardly embedded in $B$ is fixed. As above, we put $S = H \cap T$ and $N = H \cap U$ so that $H = S \leftthreetimes N$. We denote by $\tau \colon \mathfrak{X}(T) \to \mathfrak{X}(S)$ the character restriction map from $T$ to $S$. Let $\Phi = \tau(\Delta_+) \subset \mathfrak{X}(S)$ be the weight system of the action of $S$ on $\mathfrak{u}$ by means of the adjoint representation of $G$. We have $\mathfrak{u} = \bigoplus\limits_{\varphi \in \Phi} \mathfrak{u}_\varphi$, where $\mathfrak{u}_\varphi \subset \mathfrak{u}$ is the weight subspace of weight $\varphi$ with respect to $S$. For $\varphi \in \Phi$ we put $\mathfrak{n}_\varphi = \mathfrak{n} \cap \mathfrak{u}_\varphi$. Evidently, we have $\mathfrak{n} = \bigoplus\limits_{\varphi \in \Phi} \mathfrak{n}_\varphi$. For every $\varphi \in \Phi$ let $c_\varphi$ denote the codimension of the subspace $\mathfrak{n}_\varphi$ in the space $\mathfrak{u}_\varphi$.

In the notation introduced above, the following criterion of sphericity of $H$ takes place.

**Theorem 1** ([Av2, Theorem 1])**.** *The following conditions are equivalent:*
(1) *$H$ is spherical in $G$;*
(2) *$c_\varphi \leqslant 1$ for every $\varphi \in \Phi$, and all weights with $c_\varphi = 1$ are linearly independent in $\mathfrak{X}(S)$.*

Later on, we assume that $H \subset G$ is a connected solvable spherical subgroup standardly embedded in $B$ and preserve all the notations introduced above. Put $\Psi = \{\alpha \in \Delta_+ \mid \mathfrak{g}_\alpha \not\subset \mathfrak{n}\} \subset \Delta_+$.

**Definition 1.** The roots in the set $\Psi$ are said to be *active*.

Let $\varphi_1, \ldots, \varphi_m$ denote the weights $\varphi \in \Phi$ with $c_\varphi = 1$. For $i = 1, \ldots, m$ we put $\Psi_i = \{\alpha \in \Psi \mid \tau(\alpha) = \varphi_i\}$. It is clear that $\Psi = \Psi_1 \cup \Psi_2 \cup \ldots \cup \Psi_m$ and $\Psi_i \cap \Psi_j = \varnothing$ for $i \ne j$. A key role in the structure theory of connected solvable spherical subgroups is played by the following proposition.

**Proposition 1** ([Av2, Proposition 1]). *Suppose that $1 \leqslant i, j \leqslant m$ and different roots $\alpha \in \Psi_i$, $\beta \in \Psi_j$ are such that $\gamma = \beta - \alpha \in \Delta_+$. Then $\Psi_i + \gamma \subset \Psi_j$.*

In particular, Proposition 1 is used in the proof of the following theorem.

**Theorem 2** ([Av2, Theorem 2]). *Up to conjugation by an element of $T$, the subgroup $H$ is uniquely determined by the pair $(S, \Psi)$.*

The set of active roots has the following property (see [Av2, Lemma 4]): if $\alpha$ is an active root and $\alpha = \beta + \gamma$ for some roots $\beta, \gamma \in \Delta_+$, then exactly one of the two roots $\beta, \gamma$ is active. Taking this property into account, we say that an active root $\beta$ is *subordinate* to an active root $\alpha$ if $\alpha = \beta + \gamma$ for some root $\gamma \in \Delta_+$. For every active root $\alpha$ we denote by $F(\alpha)$ the set consisting of $\alpha$ and all active roots subordinate to $\alpha$. An active root $\alpha$ is said to be *maximal* if it is not subordinate to any other active root. Let $\mathrm{M}$ denote the set of maximal active roots.

Let us mention the following corollary from Proposition 1.

**Corollary 1.** *For every $i = 1, \ldots, m$, we have either $\Psi_i \subset \mathrm{M}$ or $\Psi_i \cap \mathrm{M} = \varnothing$.*

**Proposition 2** ([Av2, Proposition 3]). *Let $\alpha$ be an active root. Then there exists a unique simple root $\pi(\alpha) \in \operatorname{Supp} \alpha$ with the following property: if $\alpha = \beta + \gamma$ for some roots $\beta, \gamma \in \Delta_+$, then the root $\beta$ is active if and only if $\pi(\alpha) \notin \operatorname{Supp} \beta$ (and so the root $\gamma$ is active if and only if $\pi(\alpha) \notin \operatorname{Supp} \gamma$).*

From Proposition 2 it follows that for every active root $\alpha$ the set $F(\alpha)$ is uniquely determined by the simple root $\pi(\alpha)$. Therefore the whole set $\Psi$ is uniquely determined by the subset $\mathrm{M}$ and the map $\pi\colon \mathrm{M} \to \Pi$.

We now introduce an equivalence relation on $\mathrm{M}$ as follows. For any two roots $\alpha, \beta \in \mathrm{M}$ we write $\alpha \sim \beta$ if and only if $\tau(\alpha) = \tau(\beta)$.

To the subgroup $H$ we assign the set $\Upsilon_0(H) = (\mathrm{M}, \pi, \sim)$, where $\pi$ is regarded as a map from $\mathrm{M}$ to $\Pi$.

**Proposition 3** ([Av2, Remark 4]). *Up to conjugation by an element in $T$, the unipotent radical $N$ of $H$ is uniquely determined by the set $\Upsilon_0(H) = (\mathrm{M}, \pi, \sim)$.*

## 4. Formulation of the main results

In this subsection we state the main results of this paper (Theorem 3 and Proposition 4) and give some examples.

Let $H \subset G$ be a connected solvable spherical subgroup standardly embedded in $B$. We preserve all the notations introduced in § 3.

Let $L$ denote the sublattice in $\mathfrak{X}(T)$ generated by all elements of the form $\alpha-\beta$, where $\alpha,\beta\in \mathrm{M}$ and $\tau(\alpha)=\tau(\beta)$. In view of Proposition 1 and Corollary 1, $L$ is also generated by all elements of the form $\alpha-\beta$, where $\alpha,\beta\in\Psi$ and $\tau(\alpha)=\tau(\beta)$. Further, we introduce the lattice $L_0=\langle L\rangle\cap\mathfrak{X}(T)\supset L$ and denote by $A$ (resp. by $A_0$) the subgrooup in $T$ defined by the condition that all characters in $L$ (resp. in $L_0$) are equal to one. We note that $A^0=A_0$ and $A/A_0\simeq L_0/L$. Clearly, the group $A$ (resp. $A_0$) is the largest subgroup (resp. the largest connected subgroup) in $T$ normalizing $N$.

**Definition 2.** An active root $\delta$ is said to be *regular* if the projection of the subspace $\mathfrak{n}\subset\mathfrak{u}$ to the root subspace $\mathfrak{g}_\delta$ is zero.

Let $\delta$ be an active root and choose $i\in\{1,\ldots,m\}$ such that $\delta\in\Psi_i$. Then from Definition 2 it follows that the root $\delta$ is regular if and only if $|\Psi_i|=1$, that is, $\Psi_i=\{\delta\}$.

Let $\Psi^{\mathrm{reg}}\subset\Psi$ be the set of regular active roots.

We denote by P the set of active roots $\alpha$ satisfying the following two conditions:

(1) $\alpha\in\Psi^{\mathrm{reg}}\cap\Pi$;

(2) $(\alpha,\beta)=0$ for every root $\beta\in\Psi\backslash\{\alpha\}$.

It is easy to see that for every root $\delta\in\mathrm{P}$ and every element $\rho\in\theta^{-1}(r_\delta)$ the subgroup $H'=\rho H\rho^{-1}$ is also standardly embedded in $B$, at that, $\Upsilon_0(H')=\Upsilon_0(H)$ in view of condition (2). By Proposition 3 we have $N'=tNt^{-1}$ for some element $t\in T$. This implies that $t^{-1}\rho\in N_G(N)$, whence $\theta^{-1}(r_\delta)\cap N_G(N)\neq\varnothing$.

For every root $\delta\in\mathrm{P}$ we fix an arbitrary element $\rho_\delta\in\theta^{-1}(r_\delta)\cap N_G(N)$. We put $\mathrm{P}_S=\{\delta\in\mathrm{P}\mid r_\delta(\operatorname{Ker}\tau)=\operatorname{Ker}\tau\}=\{\delta\in\mathrm{P}\mid\rho_\delta S\rho_\delta^{-1}=S\}$.

We now are able to state the main results of this paper.

**Theorem 3.** *The group $N_G(H)$ is generated by the groups $A$, $N$ and all elements $\rho_\delta$, where $\delta$ runs over the set $\mathrm{P}_S$. In particular, $N_G(H)^0=A_0\rightthreetimes N$.*

**Corollary 2.** *There are isomorphisms $N_G(H)/H\simeq A/S\times(\mathbb{Z}/2\mathbb{Z})^r$ and $N_G(H)/N_G(H)^0\simeq A/A_0\times(\mathbb{Z}/2\mathbb{Z})^r\simeq L_0/L\times(\mathbb{Z}/2\mathbb{Z})^r$, where $r=|\mathrm{P}_S|$.*

*Remark* 2. The group $N_G(H)^0$ was computed earlier in [Av2, § 5.5].

Let $\alpha$ be an active root. A simple root $\delta\in\operatorname{Supp}\alpha$ is said to be *terminal with respect to* $\operatorname{Supp}\alpha$ if in the diagram $\Sigma(\operatorname{Supp}\alpha)$ the node corresponding to $\delta$ is joined by an edge with exactly one node.

The following proposition enables one to find the set P explicitly given the set $\Upsilon_0(H)=(\mathrm{M},\pi,\sim)$.

**Proposition 4.** *A root $\alpha\in\Pi$ is contained in the set* P *in exactly one of the following two cases:*

*Case* 1. *The following conditions are fulfilled:*

(1) $\alpha\in\mathrm{M}$*;*

(2) $\alpha\not\sim\beta$ *for every root* $\beta\in\mathrm{M}\backslash\{\alpha\}$*;*

(3) *for every root* $\beta\in\mathrm{M}\backslash\{\alpha\}$ *the diagram* $\Sigma(\{\alpha\}\cup\operatorname{Supp}\beta)$ *is disconnected.*

*Case* 2. *There is a root* $\beta\in\mathrm{M}\backslash\{\alpha\}$ *with the following properties:*

(1) $\beta=\sum\limits_{\delta\in\operatorname{Supp}\beta}\delta$*;*

(2) $\alpha\in\operatorname{Supp}\beta$ *and* $\alpha$ *is terminal with respect to* $\operatorname{Supp}\beta$*;*

(3) $\pi(\beta)\neq\alpha$*;*

(4) *there is a root* $\alpha' \in \operatorname{Supp}\beta$ *such that in the diagram* $\Sigma(\operatorname{Supp}\beta)$ *the nodes* $\alpha$ *and* $\alpha'$ *are joined by a double edge with the arrow directed to* $\alpha$*;*

(5) *for every root* $\gamma \in \mathrm{M}\backslash\{\beta\}$ *the diagram* $\Sigma(\{\alpha\} \cup \operatorname{Supp}\gamma)$ *is disconnected.*

**Example 1.** Let $G$ be an adjoint group (that is, its center is trivial). It is well known that in this case the lattice $\mathfrak{X}(T)$ is generated by the set $\Pi$. In view of properties (A), (D), (E), (C) (see Theorem 5 in § 5) this implies that $L = L_0$ and thereby $A = A_0$. Hence $N_G(H)/N_G(H)^0 \simeq (\mathbb{Z}/2\mathbb{Z})^r$, where $r = |\mathrm{P}_S|$.

*Remark* 3. The equality $A = A_0$ in case of an adjoint group $G$ implies that $A = Z(G)A_0$ in case of an arbitrary group $G$.

**Example 2.** Suppose that $S = A_0$. Then $H = N_G(H)^0$ and $\operatorname{Ker}\tau = L_0$. It is easy to see that for every root $\delta \in \mathrm{P}$ the reflection $r_\delta$ preserves the subspace $\langle L_0\rangle \subset Q$ and acts trivially on it. This implies that $\mathrm{P}_S = \mathrm{P}$. Therefore $N_G(H)/N_G(H)^0 \simeq L_0/L \times (\mathbb{Z}/2\mathbb{Z})^r$, where $r = |\mathrm{P}|$.

**Example 3.** Suppose that $G$ is simple and adjoint and $H$ satisfies the condition $\bigcup\limits_{\alpha\in\mathrm{M}} \operatorname{Supp}\alpha = \Pi$. Then we have $S = A_0$. In view of Proposition 4 there are the following possibilities in dependence of the type of the root system $\Delta$:

(1) if $\Delta$ is of type different from $\mathsf{A}_1$ or $\mathsf{B}_n$ ($n \geqslant 2$), then $\mathrm{P} = \varnothing$ and $N_G(H) = H$;

(2) if $\Delta$ is of type $\mathsf{A}_1$, then, evidently, $H = T$, $\mathrm{P} = \Pi$, and $N_G(H)/H \simeq \mathbb{Z}/2\mathbb{Z}$;

(3) if $\Delta$ is of type $\mathsf{B}_n$ ($n \geqslant 2$), then either $\mathrm{P} = \varnothing$ (which implies $N_G(H) = H$) or $|\mathrm{P}| = 1$ (which implies $N_G(H)/H \simeq \mathbb{Z}/2\mathbb{Z}$). Suppose that $\Pi = \{\alpha_1, \ldots, \alpha_n\}$. Then as an example of the first situation we can take $\mathrm{M} = \{\alpha\}$, where $\alpha = \alpha_1 + \ldots + \alpha_n$, and $\pi(\alpha) = \alpha_n$. As an example of the second situation we can take $\mathrm{M} = \{\alpha\}$, where $\alpha = \alpha_1 + \ldots + \alpha_n$, and $\pi(\alpha) \neq \alpha_n$.

**Example 4.** Suppose that $G = \mathrm{SL}_3$ and the groups $B, U, T$ consist of all upper-triangular, upper unitriangular, diagonal matrices, respectively, contained in $G$. For $t = \operatorname{diag}(t_1, t_2, t_3) \in T$ and $k = 1, 2$ we put $\alpha_k(t) = t_k t_{k+1}^{-1}$. Let $H$ be the group of matrices of the form

$$\begin{pmatrix} t & 0 & a \\ 0 & 1 & b \\ 0 & 0 & t^{-1} \end{pmatrix}, \tag{1}$$

where $a, b, t \neq 0$ are arbitrary numbers. Then $S = \{\operatorname{diag}(t, 1, t^{-1}) \mid t \neq 0\}$ and the subgroup $N$ consists of all matrices of the form (1) with $t = 1$. We have $A = A_0 = T$ and $\Psi = \mathrm{M} = \mathrm{P} = \{\alpha_1\}$. Besides, $\langle\operatorname{Ker}\tau\rangle = \langle\alpha_1 - \alpha_2\rangle$, whence $r_{\alpha_1}\langle\operatorname{Ker}\tau\rangle \neq \langle\operatorname{Ker}\tau\rangle$ and $\mathrm{P}_S = \varnothing$. Thus, $N_G(H) = T \leftthreetimes N$. Further, we have $N_G(N_G(H)) = N_G(H) \cup \rho N_G(H)$, where

$$\rho = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

In particular, we find that $N_G(N_G(H)) \neq N_G(H)$.

*Remark* 4. Example 4 disproves Theorem 4.3(iii) in [Br2] and Lemma 30.2 in [Tim], which claim that $N_G(N_G(H)) = N_G(H)$ for an arbitrary spherical subgroup $H \subset G$. This error does not influence the truth of other results of the general theory of spherical homogeneous spaces, though.

*Remark* 5. For a connected solvable spherical subgroup $H \subset G$ the equality $N_G(N_G(H)) = N_G(H)$ holds if and only if $\mathrm{P} = \mathrm{P}_S$.

## 5. Further results on connected solvable spherical subgroups

In this subsection we collect all results from the structure theory and classification of connected solvable spherical subgroups needed for proving Theorem 3 and Proposition 4.

Throughout this subsection we denote by $H$ and $H'$ two (not necessarily different) connected solvable spherical subgroups standardly embedded in $B$. The notations $S$, $N$, $\Psi$, $\pi$, ... (resp. $S'$, $N'$, $\Psi'$, $\pi'$, ...) refer to the subgroup $H$ (resp. to the subgroup $H'$).

**Proposition 5** ([Av2, Lemma 7(a–c)]). *Suppose that $\alpha \in \Psi$. Then:*
(a) $|F(\alpha)| = |\operatorname{Supp}\alpha|$;
(b) *all weights $\tau(\beta)$, where $\beta \in \operatorname{Supp}\alpha$, are linearly independent in $\mathfrak{X}(S)$;*
(c) $\langle F(\alpha)\rangle = \langle \operatorname{Supp}\alpha\rangle$.

**Proposition 6** ([Av2, Corollary 7]). *Suppose that $\alpha \in \Psi$. Then the map $\pi\colon F(\alpha) \to \operatorname{Supp}\alpha$ is a bijection.*

**Proposition 7** ([Av2, Lemma 10]). *Suppose that roots $\alpha, \beta \in \Psi$ are such that $\pi(\alpha) = \pi(\beta)$. Then $\tau(\alpha) = \tau(\beta)$.*

**Proposition 8.** *Suppose that different roots $\alpha, \beta \in \Psi$ are such that $\tau(\alpha) = \tau(\beta)$ and $\pi(\alpha) = \pi(\beta)$. Then there are roots $\widetilde{\alpha} \in F(\alpha)$ and $\widetilde{\beta} \in F(\beta)$ such that $\tau(\widetilde{\alpha}) = \tau(\widetilde{\beta})$ and $\pi(\widetilde{\alpha}) \notin \operatorname{Supp}\beta$.*

This proposition follows from Lemma 13 in [Av2] whose proof is based on the classification of all possibilities for a pair of active roots (see Theorem 5 below). Further we give a proof of Proposition 8 that does not use this classification.

*Proof of Proposition* 8. We first introduce the notation $I = \operatorname{Supp}\alpha \cup \operatorname{Supp}\beta$. In view of Propositions 6 and 7, for every root $\gamma \in I$ we have a well-defined index $i(\gamma) \in \{1, \ldots, m\}$ such that there exists a root $\widetilde{\gamma} \in \Psi_{i(\gamma)}$ with $\pi(\widetilde{\gamma}) = \gamma$. If the map $i\colon I \to \{1, \ldots, m\}$ is injective, then the set $\tau(F(\alpha) \cup F(\beta))$ contains at least $|I|$ different weights, which are linearly independent by Theorem 1. In view of Propositions 5(a) and 6 this implies that all weights $\tau(\gamma)$, where $\gamma \in I$, are linearly independent. Hence we have $\tau(\alpha) \ne \tau(\beta)$, which is false. Therefore there are two different roots $\gamma_1, \gamma_2 \in I$ with $i(\gamma_1) = i(\gamma_2)$. In view of Propositions 6 and 5 the roots $\gamma_1, \gamma_2$ cannot lie simultaneously in the set $\operatorname{Supp}\alpha$; by the same reason they cannot lie simultaneously in the set $\operatorname{Supp}\beta$. Hence one of these two roots lies in $\operatorname{Supp}\alpha \backslash \operatorname{Supp}\beta$ and the other one lies in $\operatorname{Supp}\beta \backslash \operatorname{Supp}\alpha$. Without loss of generality we may assume that $\gamma_1 \in \operatorname{Supp}\alpha \backslash \operatorname{Supp}\beta$ and $\gamma_2 \in \operatorname{Supp}\beta \backslash \operatorname{Supp}\alpha$. Then the unique root $\widetilde{\alpha} \in F(\alpha)$ with $\pi(\widetilde{\alpha}) = \gamma_1$ and the unique root $\widetilde{\beta} \in F(\beta)$ with $\pi(\widetilde{\beta})$ are the desired roots. ☐

**Theorem 4** ([Av2, Theorem 3]). *For every root $\alpha \in \Psi$ the pair $(\alpha, \pi(\alpha))$ is contained in Table* 1.

TABLE 1

| No. | Type of $\Delta(\alpha)$ | $\alpha$ | $\pi(\alpha)$ |
|---|---|---|---|
| 1 | any of rank $n$ | $\alpha_1+\alpha_2+\ldots+\alpha_n$ | $\alpha_1,\alpha_2,\ldots,\alpha_n$ |
| 2 | $\mathsf{B}_n$ | $\alpha_1+\alpha_2+\ldots+\alpha_{n-1}+2\alpha_n$ | $\alpha_1,\alpha_2,\ldots,\alpha_{n-1}$ |
| 3 | $\mathsf{C}_n$ | $2\alpha_1+2\alpha_2+\ldots+2\alpha_{n-1}+\alpha_n$ | $\alpha_n$ |
| 4 | $\mathsf{F}_4$ | $2\alpha_1+2\alpha_2+\alpha_3+\alpha_4$ | $\alpha_3,\alpha_4$ |
| 5 | $\mathsf{G}_2$ | $2\alpha_1+\alpha_2$ | $\alpha_2$ |
| 6 | $\mathsf{G}_2$ | $3\alpha_1+\alpha_2$ | $\alpha_2$ |

Let us explain the notation in Table 1. In the column '$\alpha$' the expression of $\alpha$ as the sum of simple roots in $\operatorname{Supp}\alpha$ is given. At that, the $i$th simple root in the diagram $\Sigma(\operatorname{Supp}\alpha)$ is denoted by $\alpha_i$. In the column '$\pi(\alpha)$' we list all possibilities for $\pi(\alpha)$ for a given active root $\alpha$.

**Corollary 3** ([Av2, Corollary 8])**.** *If $\alpha\in\Psi$, $|\operatorname{Supp}\alpha|\geqslant 2$, and $\delta\in\operatorname{Supp}\alpha\cap\Psi$, then $\delta$ is terminal with respect to* $\operatorname{Supp}\alpha$.

Further we give a list of several conditions on a pair $\alpha,\beta$ of active roots. These conditions will be used below when we formulate Theorem 5.

(D0) $\operatorname{Supp}\alpha\cap\operatorname{Supp}\beta=\varnothing$;

(D1) $\operatorname{Supp}\alpha\cap\operatorname{Supp}\beta=\{\delta\}$, where $\pi(\alpha)\neq\delta$, $\pi(\beta)\neq\delta$, and $\delta$ is terminal with respect to both $\operatorname{Supp}\alpha$ and $\operatorname{Supp}\beta$;

(E1) $\operatorname{Supp}\alpha\cap\operatorname{Supp}\beta=\{\delta\}$, where $\delta=\pi(\alpha)=\pi(\beta)$, $\alpha-\delta\in\Delta_+$, $\beta-\delta\in\Delta_+$, and $\delta$ is terminal with respect to both $\operatorname{Supp}\alpha$ and $\operatorname{Supp}\beta$;

(D2) the diagram $\Sigma(\operatorname{Supp}\alpha\cup\operatorname{Supp}\beta)$ has the form shown on Figure 1 (for some $p,q,r\geqslant 1$), $\alpha=\alpha_1+\ldots+\alpha_p+\gamma_0+\gamma_1+\ldots+\gamma_r$, $\beta=\beta_1+\ldots+\beta_q+\gamma_0+\gamma_1+\ldots+\gamma_r$, $\pi(\alpha)\notin\operatorname{Supp}\alpha\cap\operatorname{Supp}\beta$, and $\pi(\beta)\notin\operatorname{Supp}\alpha\cap\operatorname{Supp}\beta$;

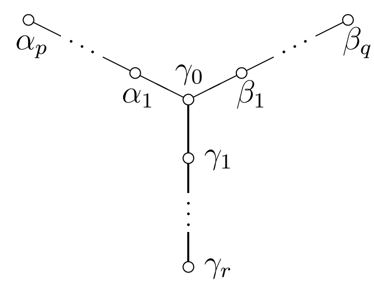


FIGURE 1

(E2) the diagram $\Sigma(\operatorname{Supp}\alpha\cup\operatorname{Supp}\beta)$ has the form shown on Figure 1 (for some $p,q,r\geqslant 1$), $\alpha=\alpha_1+\ldots+\alpha_p+\gamma_0+\gamma_1+\ldots+\gamma_r$, $\beta=\beta_1+\ldots+\beta_q+\gamma_0+\gamma_1+\ldots+\gamma_r$, and $\pi(\alpha)=\pi(\beta)\in\operatorname{Supp}\alpha\cap\operatorname{Supp}\beta$.

Now let us introduce the set $\Pi_0=\bigcup\limits_{\delta\in\mathrm{M}}\operatorname{Supp}\delta\subset\Pi$.

In addition to the set $\Upsilon_0(H)=(\mathrm{M},\pi,\sim)$, to the subgroup $H$ we also assign the set $\Upsilon(H)=(S,\mathrm{M},\pi,\sim)$, where $\pi$ is regarded as a map from $\mathrm{M}$ to $\Pi$.

The following theorem gives a classification of all connected solvable spherical subgroups in $G$ standardly embedded in $B$.

**Theorem 5.** (a) [Av2, Theorem 4] *Up to conjugation by an element in $T$, $H$ is uniquely determined by the set $\Upsilon(H)=(S,\mathrm{M},\pi,\sim)$, and this set satisfies the following conditions:*

(A) $\pi(\alpha)\in\operatorname{Supp}\alpha$ *for every root $\alpha\in\mathrm{M}$, and the pair $(\alpha,\pi(\alpha))$ is contained in Table* 1*;*

(D) *if $\alpha,\beta\in\mathrm{M}$ and $\alpha\nsim\beta$, then one of possibilities* (D0)*,* (D1)*,* (D2) *is realized;*

(E) *if $\alpha,\beta\in\mathrm{M}$ and $\alpha\sim\beta$, then one of possibilities* (D0)*,* (D1)*,* (E1)*,* (D2)*,* (E2) *is realized;*

(C) *if $\alpha\in\mathrm{M}$, then* $\operatorname{Supp}\alpha\not\subset\bigcup\limits_{\delta\in\mathrm{M}\setminus\{\alpha\}}\operatorname{Supp}\delta$*;*

(T) $\langle\operatorname{Ker}\tau\rangle\cap\langle\Pi_0\rangle=\langle\alpha-\beta\mid\alpha,\beta\in\mathrm{M},\alpha\sim\beta\rangle$.

(b) [Av2, Theorem 5] *Suppose that a subtorus $S \subset T$, a subset $\mathrm{M} \subset \Delta_+$, a map $\pi \colon \mathrm{M} \to \Pi$, and an equivalence relation $\sim$ on $\mathrm{M}$ satisfy conditions* (A), (D), (E), (C), *and* (T). *Then there exists a connected solvable spherical subgroup $H \subset G$ standardly embedded in $B$ such that $\Upsilon(H) = (S, \mathrm{M}, \pi, \sim)$.*

*Remark* 6. Generally speaking, it may happen that $\Upsilon(H) \ne \Upsilon(H')$, but the subgroups $H, H'$ are conjugated in $G$. Concerning the problem of when two connected solvable spherical subgroups standardly embedded in $B$ are conjugated in $G$, see Proposition 9 and Theorem 6 below (a more detailed discussion of this problem see also in [Av2, § 5]).

**Proposition 9** ([Av2, Proposition 13])**.** *Suppose that $H$, $H'$ are such that $H' = gHg^{-1}$ for some $g \in G$. Then $g \in N' \cdot N_G(T) \cdot N$, where $N' = H' \cap U$.*

**Definition 3.** Suppose that $\delta \in \Psi^{\mathrm{reg}} \cap \Pi$. An *elementary transformation with center $\delta$* (or simply an *elementary transformation*) is a transformation $H \mapsto \rho_\delta H \rho_\delta^{-1}$, where $\rho_\delta \in \theta^{-1}(r_\delta)$.

As can be easily seen, in this definition the subgroup $\rho_\delta H \rho_\delta^{-1}$ is also standardly embedded in $B$.

The argument used in the proof of Theorem 6 in [Av2] actually proves the following theorem.

**Theorem 6.** *Suppose that $H$, $H'$ are such that $H' = \sigma H \sigma^{-1}$ for some $\sigma \in N_G(T)$. Then there are elements $\rho_1, \ldots, \rho_k \in N_G(T)$ with the following properties:*

(1) *$\theta(\rho_1), \ldots, \theta(\rho_k)$ are simple reflections;*

(2) *$\sigma = \rho_k \ldots \rho_1$;*

(3) *the chain $H \mapsto \rho_1 H \rho_1^{-1} \mapsto \rho_2\rho_1 H \rho_1^{-1}\rho_2^{-1} \mapsto \ldots \mapsto \sigma H \sigma^{-1} = H'$ is a chain of elementary transformations.*

**Proposition 10** ([Av2, Lemma 30(a,b)])**.** *Suppose that $H \mapsto H'$ is an elementary transformation with center $\delta$. Then:*

(a) *$\Psi' = r_\delta(\Psi \backslash \{\delta\}) \cup \{\delta\}$;*

(b) *$\pi'(r_\delta(\alpha)) = \pi(\alpha)$ for $\alpha \in \Psi \backslash \{\delta\}$;*

**Corollary 4.** *Suppose that $H \mapsto H'$ is an elementary transformation with center $\delta$ and $\alpha \in \Psi$ is an arbitrary root. Then for the unique positive root $\widetilde{\alpha}$ in the set $\{r_\delta(\alpha), -r_\delta(\alpha)\}$ we have $\widetilde{\alpha} \in \Psi'$ and $\pi'(\widetilde{\alpha}) = \pi(\alpha)$.*

Theorem 6 and Corollary 4 imply the following corollary.

**Corollary 5.** *Suppose that $\sigma \in N_G(H) \cap N_G(T)$ and $\alpha \in \Psi$ is an arbitrary root. Put $s = \theta(\sigma) \in W$. Then for the unique positive root $\beta$ in the set $\{s(\alpha), -s(\alpha)\}$ we have $\beta \in \Psi$ and $\pi(\beta) = \pi(\alpha)$.*

## 6. Proof of the main results

In this subsection we prove Theorem 3 and Proposition 4. We preserve the notations introduced in § 3–5.

*Proof of Theorem* 3. In view of Proposition 9 we have $N_G(H) \subset N \cdot N_G(T) \cdot N$. Since $N \subset H \subset N_G(H)$, it remains to find the group $N_G(H) \cap N_G(T)$.

Suppose that $\sigma \in N_G(T)$ is such that $\sigma H \sigma^{-1} = H$. Put $s = \theta(\sigma) \in W$. Theorem 6 and Proposition 10 imply that $s$ is contained in the subgroup $W_0$ of $W$ generated by simple reflections corresponding to the roots in $\Pi_0$.

If $s = e$, then $\sigma \in N_G(H) \cap T$. It was shown in § 4 that $N_G(H) \cap T = A$.

Further we assume that $s \neq e$.

**Lemma 1.** *If $\alpha \in \Delta_+ \backslash \Psi^{\mathrm{reg}}$, then $s(\alpha) \in \Delta_+$.*

*Proof.* From the definition of the set $\Psi^{\mathrm{reg}}$ it follows that the projection of the subspace $\mathfrak{h}$ to the root subspace $\mathfrak{g}_\alpha$ is not zero. Since $(\operatorname{Ad}\sigma)\mathfrak{g}_\gamma = \mathfrak{g}_{s(\gamma)}$ for every $\gamma \in \Delta$, the condition $(\operatorname{Ad}\sigma)\mathfrak{h} = \mathfrak{h}$ implies that the projection of the subspace $\mathfrak{h}$ to the root subspace $\mathfrak{g}_{s(\alpha)}$ is not zero as well. Hence $s(\alpha) \in \Delta_+$. □

**Proposition 11.** *For every root $\alpha \in \Psi$ we have $s(\alpha) \in \{\alpha, -\alpha\}$.*

*Proof.* Let $\widetilde{\alpha}$ be the unique positive root in the set $\{s(\alpha), -s(\alpha)\}$. Our aim is to prove that $\widetilde{\alpha} = \alpha$. In view of Corollary 5 we have $\widetilde{\alpha} \in \Psi$ and $\pi(\widetilde{\alpha}) = \pi(\alpha)$. If the set $\pi^{-1}(\pi(\alpha))$ consists of one element, then $\widetilde{\alpha} = \alpha$ and the assertion is proved. Further we assume that the set $\pi^{-1}(\pi(\alpha))$ contains more than one element. Let $\beta \in \pi^{-1}(\pi(\alpha)) \backslash \{\alpha\}$ be an arbitrary root. Then by Proposition 7 we have $\tau(\alpha) = \tau(\beta)$. Further, in view of Proposition 8 there are roots $\alpha' \in F(\alpha)$ and $\beta' \in F(\beta)$ such that $\tau(\alpha') = \tau(\beta')$ and $\pi(\alpha') \notin \operatorname{Supp}\beta$. All the four roots $\alpha$, $\beta$, $\alpha'$, $\beta'$ are active and pairwise different, therefore none of them is regular. In particular, each of the roots $\alpha$, $\alpha'$ is contained in the set $\Delta_+ \backslash \Psi^{\mathrm{reg}}$. Besides, for the root $\alpha'' = \alpha - \alpha' \in \Delta_+$ we also have $\alpha'' \in \Delta_+ \backslash \Psi^{\mathrm{reg}}$, whence by Lemma 1 all the three roots $s(\alpha)$, $s(\alpha')$, and $s(\alpha'')$ are positive. In particular, $\widetilde{\alpha} = s(\alpha)$. Applying Corollary 5, we obtain $s(\alpha), s(\alpha') \in \Psi$ and $\pi(s(\alpha')) = \pi(\alpha')$. Now the equality $s(\alpha) = s(\alpha') + s(\alpha'')$ implies that $\pi(\alpha') \in \operatorname{Supp} s(\alpha)$. The latter means that $\widetilde{\alpha} \neq \beta$, which completes the proof. □

Proposition 11 and Lemma 1 imply the following corollary.

**Corollary 6.** *If a root $\alpha \in \Psi$ is not regular, then $s(\alpha) = \alpha$.*

**Proposition 12.** *Suppose that $\alpha \in \Psi$ and $\operatorname{ht}\alpha \geqslant 2$. Then $s(\alpha) = \alpha$.*

*Proof.* The condition $\operatorname{ht}\alpha \geqslant 2$ implies that $|\operatorname{Supp}\alpha| \geqslant 2$, whence $|F(\alpha)| \geqslant 2$ by Proposition 5(a). Let $\alpha' \in F(\alpha) \backslash \{\alpha\}$ be an arbitrary root. Since $\alpha - \alpha' \in \Delta_+ \backslash \Psi \subset \Delta_+ \backslash \Psi^{\mathrm{reg}}$, by Lemma 1 we obtain that $s(\alpha) - s(\alpha') \in \Delta_+$. Further, in view of Proposition 11 we have $s(\alpha) \in \{\alpha, -\alpha\}$ and $s(\alpha') \in \{\alpha', -\alpha'\}$. If $s(\alpha) = -\alpha$, then the root $s(\alpha) - s(\alpha')$ lies in the set $\{-\alpha + \alpha', -\alpha - \alpha'\}$, which evidently does not contain positive roots. Hence $s(\alpha) = \alpha$. □

In view of Corollary 6, Proposition 12 implies the following corollary.

**Corollary 7.** *If a root $\alpha \in \Psi$ has the property $s(\alpha) = -\alpha$, then $\alpha \in \Psi^{\mathrm{reg}} \cap \Pi$.*

**Proposition 13.** *Suppose that a root $\alpha \in \Psi^{\mathrm{reg}} \cap \Pi$ has the property $s(\alpha) = -\alpha$. Then $(\alpha, \beta) = 0$ for every root $\beta \in \Psi \backslash \{\alpha\}$.*

*Proof.* Since the group $W$ acts on the space $Q$ by orthogonal transformations, we have $(\alpha, \beta) = (s(\alpha), s(\beta))$. Assume that $(\alpha, \beta) \neq 0$. Then the hypothesis and Proposition 11 imply that $s(\beta) = -\beta$. By Corollary 7 we obtain that $\beta \in \Pi$, whence $(\alpha, \beta) < 0$. Therefore

$\alpha+\beta \in \Delta_+$ and $s(\alpha+\beta) = s(\alpha)+s(\beta) = -(\alpha+\beta) \notin \Delta_+$. On the other hand, it is easy to see that $\alpha+\beta \notin \Psi$, whence by Lemma 1 we obtain that $s(\alpha+\beta) \in \Delta_+$, a contradiction. □

Corollary 7 and Proposition 13 imply the following corollary.

**Corollary 8.** *Every root $\alpha \in \Psi$ with $s(\alpha) = -\alpha$ is contained in* P*.*

Let $\delta_1, \ldots, \delta_k$ be all the roots in the set $\{\alpha \in \Psi \mid s(\alpha) = -\alpha\}$. By Corollary 8 all of them are contained in P. Further, it is easy to see that every active root is invariant under the action of the element $r_{\delta_k} \ldots r_{\delta_1} s \in W_0$. It follows from Proposition 5(b) that $\Pi_0 \subset \langle \Psi \rangle$, whence $r_{\delta_k} \ldots r_{\delta_1} s = e$. Therefore $s = r_{\delta_1} \ldots r_{\delta_k}$ and $\sigma = \rho_{\delta_1} \ldots \rho_{\delta_k} t$ for some $t \in A$.

Each of the roots $\delta_1, \ldots, \delta_k$ is contained in the space $\langle \Pi_0 \rangle$ and is orthogonal to the subspace $\langle L \rangle$. Further, from condition (T) (see Theorem 5) we have $\langle \operatorname{Ker} \tau \rangle \cap \langle \Pi_0 \rangle = \langle L \rangle$. In view of what we have said above the condition $s\langle \operatorname{Ker} \tau \rangle = \langle \operatorname{Ker} \tau \rangle$ implies that $s$ acts trivially on $\langle \operatorname{Ker} \tau \rangle$, hence the same holds true for every element $r_{\delta_i}$, $i = 1, \ldots, k$. The latter means that $\delta_1, \ldots, \delta_k \in \mathrm{P}_S$.

Thus, we have obtained that $\sigma = \rho_{\delta_1} \ldots \rho_{\delta_k} t$ for some roots $\delta_1, \ldots, \delta_k \in \mathrm{P}_S$ and some element $t \in A$. On the other hand, as can be easily seen, for every root $\delta \in \mathrm{P}_S$ we have $\rho_\delta \in N_G(H)$. This completes the proof of the theorem. □

*Proof of Proposition* 4. Suppose that $\alpha \in \Pi$. First assume that $\alpha \in \mathrm{P}$. Then there are two possibilities.

1°. $\alpha \in \mathrm{M}$. Then it is easy to check that all the conditions of Case 1 are fulfilled.

2°. $\alpha \notin \mathrm{M}$. Then $\alpha \in \operatorname{Supp} \beta$ for some root $\beta \in \mathrm{M}$. From the condition $\alpha \in \Psi$ and Corollary 3 we obtain that $\alpha$ is terminal with respect to $\operatorname{Supp} \beta$. Having performed a case-by-case consideration of all the possibilities in Table 1, we find that the condition $(\alpha, \beta) = 0$ holds if and only if the diagram $\Sigma(\operatorname{Supp} \beta)$ is of type $\mathsf{B}_n$ ($n \geqslant 2$) and conditions (1)–(4) of Case 2 are fulfilled. Since a node of a Dynkin diagram cannot be incident to two double edges, in view of conditions (D) and (E) (see Theorem 5) for every root $\beta' \in \mathrm{M} \backslash \{\beta\}$ we have $\alpha \notin \operatorname{Supp} \beta'$, hence condition (5) of Case 2.

Now let us prove the converse implication of the proposition.

If Case 1 takes place, then in view of Corollary 1 we obtain $\alpha \in \mathrm{P}$.

Suppose that Case 2 takes place. From conditions (1)–(3) it follows that $\alpha \in \Psi$. Conditions (1), (4), and (5) imply that $(\alpha, \delta) = 0$ for every root $\delta \in \Psi \backslash \{\alpha\}$. It remains to prove that $\alpha \in \Psi^{\mathrm{reg}}$. If this is not the case then there is a root $\alpha' \in \Psi \backslash \{\alpha\}$ such that $\tau(\alpha) = \tau(\alpha')$. In view of Proposition 1 and Corollary 1 this implies that $\beta' = \alpha' + (\beta - \alpha) \in \mathrm{M}$. Then the diagram $\Sigma(\{\alpha\} \cup \operatorname{Supp} \beta')$ is connected, and we have a contradiction with condition (5). Thus, $\alpha \in \mathrm{P}$. □

## References


[Av1] R. S. Avdeev, *Excellent affine spherical homogeneous spaces of semisimple algebraic groups*, Trudy Mosk. Mat. Obshch. **71** (2010), 235–269 (in Russian); English transl.: Trans. Mosc. Mat. Soc. **71** (2010), 209–240.

[Av2] R. S. Avdeev, *On solvable spherical subgroups of semisimple algebraic groups*, Trudy Mosk. Mat. Obshch., **72** (2011), to appear (in Russian); see also `arXiv:math.GR/1102.4595`.

[Br1] M. Brion, *Classification des espaces homogènes sphériques*, Compositio Math. **63** (1987), no. 2, 189–208.

[Br2] M. Brion, *Variétés sphériques*, Notes de la session de la S. M. F. "Opérations hamiltoniennes et opérations de groupes algb́riques", Grenoble, 1997, `http://www-fourier.ujf-grenoble.fr/~mbrion/spheriques.pdf`.

[BP] M. Brion, F. Pauer, *Valuations des espaces homogènes sphériques*

[Krä] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Math., **38** (1979), no. 2, 129–153.

[Lun] D. Luna, *Variétés sphériques de type A*, IHÉS Publ. Math. **94** (2001), 161–226.

[Mik] I. V. Mikityuk, *On the integrability of invariant Hamiltonian systems with homogeneous configuration spaces*, Mat. Sb. **129(171)** (1986), no. 4, 514–534 (in Russian); English transl.: Math. USSR-Sb. **57** (1987), no. 2, 527–546.

[OV] E. B. Vinberg, A. L. Onishchik, *Seminar on Lie groups and algebraic groups*, 2nd ed., URSS, Moscow 1995 (in Russian); English transl. of 1st ed.: A. L. Onishchik and E. B. Vinberg, *Lie groups and algebraic groups*, Springer Ser. Soviet Math., Springer-Verlag, Berlin 1990.

[Tim] D. A. Timashev, *Homogeneous spaces and equivariant embeddings*, Encyclopaedia of Mathematical Sciences, vol. **138**, Springer-Verlag, Berlin Heidelberg 2011.

[Yak] O. S. Yakimova, *Weakly symmetric spaces of semisimple Lie groups*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **2** (2002), 57–60 (in Russian); English transl.: Moscow Univ. Math. Bull. **57** (2002), no. 2, 37–40.



Chair of Higher Algebra, Department of Mechanics and Mathematics, Moscow State University, 1, Leninskie Gory, Moscow, 119992, Russia

*E-mail address*: `suselr@yandex.ru`