"SPECTRAL IMPLIES TILING" FOR THREE INTERVALS REVISITED

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ABSTRACT. In [2] it was shown that "Tiling implies Spectral" holds for a union of three intervals and the reverse implication was studied under certain restrictive hypotheses on the associated spectrum. In this paper, we reinvestigate the "Spectral implies Tiling" part of Fuglede's conjecture for the three interval case. We first show that the "Spectral implies Tiling" for two intervals follows from the simple fact that two distinct circles have at most two points of intersections. We then attempt this for the case of three intervals and except for one situation are able to prove "Spectral implies Tiling". Finally, for the exceptional case, we show a connection to a problem of generalized Vandermonde varieties.

1. **Introduction**

We begin with the standard definitions and the statement of Fuglede's conjecture.

Let Ω and T be Lebesgue measurable subsets of $\mathbb{R}^d$ with finite positive measure. For $\lambda \in \mathbb{R}^d$, let

$$e_\lambda(x) := |\Omega|^{-1/2} e^{2\pi i \lambda . x} \chi_\Omega(x), \;\; x \in \mathbb{R}^d.$$

Definition 1.1. *Ω is said to be a spectral set if there exists a subset $\Lambda \subset \mathbb{R}^d$ such that the set of exponential functions $E_\Lambda := \{e_\lambda : \lambda \in \Lambda\}$ is an orthonormal basis for the Hilbert space $L^2(\Omega)$. The set Λ is said to be a spectrum for Ω and the pair (Ω, Λ) is called a spectral pair.*

Definition 1.2. *T is said to be a prototile if T tiles $\mathbb{R}^d$ by translations; i.e., if there exists a subset $\mathcal{T} \subset \mathbb{R}^d$ such that $\{T + t : t \in \mathcal{T}\}$ forms a partition a.e. of $\mathbb{R}^d$, where $T + t = \{x + t : x \in T\}$. The set $\mathcal{T}$ is said to be a tiling set for T and the pair $(T, \mathcal{T})$ is called a tiling pair.*

The study of relationships between spectral and tiling properties of sets began with the work of B. Fuglede [5], who proved the following result:

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Theorem 1.3. *(Fuglede [5]) Let $\mathcal{L}$ be a full rank lattice in $\mathbb{R}^d$ and let $\mathcal{L}^*$ be the dual lattice. Then $(\Omega, \mathcal{L})$ is a tiling pair if and only if $(\Omega, \mathcal{L}^*)$ is a spectral pair.*

In the same paper, Fuglede made the following conjecture, which is also known as the Spectral Set conjecture.

Conjecture 1.4. *(Fuglede's conjecture) A set $\Omega \subset \mathbb{R}^d$ is a spectral set if and only if Ω tiles $\mathbb{R}^d$ by translation.*

This led to the study of spectral and tiling properties of sets. We refer the reader to [1] for a survey and the present status of this problem.

In one dimension, for the simplest case when Ω is a finite union of intervals, the problem is open in both directions and only the 2-interval case has been completely resolved by Laba in [7], where she proved that the conjecture holds true.

In [2] the case of three intervals was explored. It was shown there that the "Tiling implies Spectral" part of Fuglede's conjecture is true in this case, and the reverse implication was proved under some restrictive hypothesis on the associated spectrum.

Recently in [1], the authors have shown that any spectrum associated with a spectral set which is a finite union of intervals is periodic (see [6] for a simplification of the proof). One of the key ingredients in both proofs is an embedding of the spectrum in a suitable vector space, equipped with an indefinite conjugate linear form. In this note we develop these ideas to give another proof of the "Spectral implies Tiling" part of Fuglede's conjecture for two intervals and then we attempt this for the case of three intervals. With the exception of one case, we are able to conclude that the "Spectral implies Tiling" indeed holds. In the last section, we show a connection of the exceptional case to a question on the intersections of generalized Vandermonde varieties restricted to the 3-torus.

2. Embedding Λ in a vector space

In this section we recall the embedding of the spectrum in a vector space [1].

Consider the $2n$-dimensional vector space $\mathbb{C}^n \times \mathbb{C}^n$. We write its elements as $\underline{v} = (v_1, v_2)$ with $v_1, v_2 \in \mathbb{C}^n$. We define a conjugate linear form $\odot$ on $\mathbb{C}^n \times \mathbb{C}^n$ as follows: for $\underline{v}, \underline{w} \in \mathbb{C}^n \times \mathbb{C}^n$, let

$$\underline{v} \odot \underline{w} := \langle v_1, w_1 \rangle - \langle v_2, w_2 \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product on $\mathbb{C}^n$. Note that this conjugate linear form is degenerate, i.e., there exists $\underline{v} \in \mathbb{C}^n \times \mathbb{C}^n$, $\underline{v} \neq 0$ such that $\underline{v} \odot \underline{v} = 0$. We call such a vector a *null-vector*. For example, every element of $\mathbb{T}^n \times \mathbb{T}^n$ is a null-vector.

A subset $S \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is called a set of *mutually null-vectors* if $\forall\ \underline{v}, \underline{w} \in S$, we have $\underline{v} \odot \underline{w} = 0$.

It is clear from the definition that elements of a set of mutually null-vectors are themselves necessarily null-vectors. Any linear subspace V spanned by a set of mutually null vectors is itself a set of mutually null-vectors and $dim(V) \leq n$.

Now, suppose $\Omega = \cup_{j=1}^{n} [a_j, a_j + r_j)$ is a union of n disjoint intervals with $a_1 = 0$ and $|\Omega| = \sum_1^n r_j = 1$. We define a map φ_Ω from $\mathbb{R}$ to $\mathbb{T}^n \times \mathbb{T}^n \subseteq \mathbb{C}^n \times \mathbb{C}^n$ by

$$x \to \varphi_\Omega(x) = (\varphi_1(x); \varphi_2(x)),$$

where

$$\varphi_1(x) = \left(e^{2\pi i(a_1+r_1)x}, e^{2\pi i(a_2+r_2)x}, \ldots, e^{2\pi i(a_n+r_n)x}\right)$$
$$\varphi_2(x) = \left(1, e^{2\pi i a_2 x}, \ldots, e^{2\pi i a_n x}\right).$$

For a set $\Lambda \subset \mathbb{R}$, the mutual orthogonality of the set of exponentials $E_\Lambda = \{e_\lambda : \lambda \in \Lambda\}$ is equivalent to saying that the set $\varphi_\Omega(\Lambda) = \{\varphi_\Omega(\lambda); \lambda \in \Lambda\}$ is a set of mutually null vectors, and so the vector space $V_\Omega(\Lambda)$ spanned by $\varphi_\Omega(\Lambda)$ has dimension at most n. Therefore if (Ω, Λ) is a spectral pair, we can say that Λ has a "local finiteness property", in the sense that there exists a finite subset $\mathcal{B} = \{y_1, \ldots, y_m\} \subseteq \Lambda$, $m \leq n$ which determines Λ uniquely. More precisely we have,

Lemma 2.1. *Let (Ω, Λ) be a spectral pair and let $\mathcal{B} \subseteq \Lambda$ be such that $\varphi_\Omega(\mathcal{B}) := \{\varphi_\Omega(y) : y \in \mathcal{B}\}$ forms a basis of $V_\Omega(\Lambda)$. Then $x \in \Lambda$ if and only if $\varphi_\Omega(x) \odot \varphi_\Omega(y) = 0,\ \forall\, y \in \mathcal{B}$.*

Next, we give a criterion for the periodicity of the spectrum.

Lemma 2.2. *Let (Ω, Λ) be a spectral pair. If $\exists\ \lambda_1, \lambda_2 \in \Lambda$ such that $\varphi_\Omega(\lambda_1) = \varphi_\Omega(\lambda_2)$, then $d = |\lambda_1 - \lambda_2| \in \mathbb{N}$ and Λ is d-periodic, i.e., $\Lambda = \{\lambda_1, \ldots, \lambda_d\} + d\mathbb{Z}$.*

Proof. Since $\varphi_\Omega(\lambda_1) = \varphi_\Omega(\lambda_2)$, we have $\varphi_\Omega(d) = (1, \ldots, 1; 1, \ldots, 1)$ and hence $\varphi_\Omega(x + d) = \varphi_\Omega(x), \forall\, x \in \mathbb{R}$. Let $\mathcal{B} \subseteq \Lambda$ be such that $\varphi_\Omega(\mathcal{B})$ is a basis of $V_\Omega(\Lambda)$. Then, whenever $\lambda \in \Lambda$, we have $\varphi_\Omega(\lambda + nd) \odot \varphi_\Omega(y) = \varphi_\Omega(\lambda) \odot \varphi_\Omega(y) = 0, \forall\, n \in \mathbb{Z}$ and $\forall\, y \in \mathcal{B}$. Thus $\lambda + d\mathbb{Z} \subseteq \Lambda$ and so Λ is d-periodic. By a simple application of Poisson summation formula we see that $d \in \mathbb{N}$. But Λ must have density 1 (by Landau's density theorem [8]), so we conclude that $\Lambda = \{\lambda_1, \ldots, \lambda_d\} + d\mathbb{Z}$. ☐

3. Spectral Implies Tiling for 2 intervals

We will now use the ideas developed in the previous section to give a simple proof of the "Spectral implies Tiling" part of Fuglede's conjecture for a set which is a union of two intervals. See [7] for the original proof.

Let $\Omega = [0, r] \cup [a, a + 1 - r]$, where $0 < r < 1$, $r < a$, and let (Ω, Λ) be a spectral pair. Without loss of generality, we may assume that $0 \in \Lambda$.

Consider the map $\varphi_\Omega : \Lambda \to \mathbb{C}^2 \times \mathbb{C}^2$ given by

$$\varphi_\Omega(\lambda) := (e^{2\pi i\lambda r}, e^{2\pi i\lambda(a+1-r)}; 1, e^{2\pi i\lambda a})$$

and let $V_\Omega(\Lambda)$ be the subspace spanned by $\varphi_\Omega(\Lambda) = \{\varphi_\Omega(\lambda) : \lambda \in \Lambda\}$. Then we have $dim(V_\Omega(\Lambda)) \leq 2$. We will now show that in fact $dim V_\Omega(\Lambda) = 2$, unless Ω is degenerate, i.e., Ω consists of a single interval of length 1.

First, observe that for $\lambda, \lambda' \in \mathbb{R}$, the vectors $\varphi_\Omega(\lambda)$ and $\varphi_\Omega(\lambda')$ are linearly dependent if and only if $\varphi_\Omega(\lambda) = \varphi_\Omega(\lambda')$ (since the third coordinate in $\varphi_\Omega(x)$ is $1 \,\forall\, x \in \mathbb{R}$). Now if $dim(V_\Omega(\Lambda)) = 1$, then $\forall\, \lambda \in \Lambda$, $\varphi_\Omega(\lambda) = \varphi_\Omega(0) = (1, 1; 1, 1)$ and so $\chi_\Omega(n\lambda) = 0 \,\forall n \in \mathbb{Z}$ and thus Poisson summation implies that $\lambda \in \mathbb{Z}$. Further, observe that Λ is actually a subgroup of $\mathbb{Z}$, therefore using Landau's density criteria we get $\Lambda = \mathbb{Z}$. In particular, $1 \in \Lambda$, and so $e^{2\pi i r} = 1$, which implies that $r = 0$ or 1, and thus this is a degenerate case.

Now let $\lambda_1 = 0, \lambda_2, \lambda_3$ be the first three elements of $\Lambda \cap [0, \infty)$. We claim that $\varphi_\Omega(\lambda_2) \neq \varphi_\Omega(0)$. For if $\varphi_\Omega(\lambda_2) = \varphi_\Omega(0)$ then by Lemma 2.2, Λ is λ_2-periodic, and since by our assumption λ_2 is the smallest positive element of Λ, we get $\Lambda = \lambda_2\mathbb{Z}$ and $dim(V_\Omega(\Lambda)) = 1$, a contradiction. A similar argument shows that $\varphi_\Omega(\lambda_2) \neq \varphi_\Omega(\lambda_3)$. Hence, we have two possible cases to consider:

(1) $\varphi_\Omega(0) = \varphi_\Omega(\lambda_3)$,
(2) $\varphi_\Omega(0), \varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3)$ are all distinct.

Case(1). By Lemma 2.2, $\lambda_3 = d \in \mathbb{N}$ and $\Lambda = d\mathbb{Z} \cup (\lambda_2 + d\mathbb{Z})$. But Λ must have density 1, so $d = 2$. Next, $\varphi_\Omega(2) = \varphi_\Omega(0)$ implies that $e^{2\pi i 2a} = e^{2\pi i 2r} = e^{2\pi i 2(a+1-r)} = 1$, and so $a \in \mathbb{Z}/2$ and $r = 1/2$. That such an Ω tiles $\mathbb{R}$ is now easy to see.

Case(2). Suppose that $\varphi_\Omega(0)$, $\varphi_\Omega(\lambda_2)$, $\varphi_\Omega(\lambda_3)$ are all distinct. Then any two of these are linearly independent and form a basis of $V_\Omega(\Lambda)$.

Let,

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{2\pi i\lambda_2 a} & e^{2\pi i\lambda_2 r} & e^{2\pi i\lambda_2(a+1-r)} \\ 1 & e^{2\pi i\lambda_3 a} & e^{2\pi i\lambda_3 r} & e^{2\pi i\lambda_3(a+1-r)} \end{pmatrix} \tag{1}$$

Then we have $Rank(A) = 2$, and in particular

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i\lambda_2 a} & e^{2\pi i\lambda_2 r} \\ 1 & e^{2\pi i\lambda_3 a} & e^{2\pi i\lambda_3 r} \end{vmatrix} = 0 \tag{2}$$

Therefore,

$$\begin{vmatrix} e^{2\pi i\lambda_2 a} - 1 & e^{2\pi i\lambda_2 r} - 1 \\ e^{2\pi i\lambda_3 a} - 1 & e^{2\pi i\lambda_3 r} - 1 \end{vmatrix} = 0 \tag{3}$$

So finally we get,

$$(e^{2\pi i\lambda_2 a} - 1)(e^{2\pi i\lambda_3 r} - 1) = (e^{2\pi i\lambda_2 r} - 1)(e^{2\pi i\lambda_3 a} - 1). \tag{4}$$

Put $e^{2\pi i\lambda_2 a} - 1 = \alpha$, and $e^{2\pi i\lambda_2 r} - 1 = \beta$ in equation (4).

If $\alpha = 0$, then $e^{2\pi i\lambda_2 r} = e^{2\pi i\lambda_2(a+1-r)} = 1$, and so $\varphi_\Omega(\lambda_2) = \varphi_\Omega(0)$ which is a contradiction. If $\beta = 0$, we have $e^{2\pi i\lambda_2 a} = e^{2\pi i\lambda_2(a+1-r)}$ and thus $\varphi_\Omega(\lambda_2)$ is of the form $(1, c; 1, c)$. Since the set $\{\varphi_\Omega(0), \varphi_\Omega(\lambda_2)\}$ generates $V_\Omega(\Lambda)$, all elements of $V_\Omega(\Lambda)$ are of this form. So that $\Lambda \subseteq \mathbb{Z}$ and is a subgroup of $\mathbb{Z}$. Thus $\Lambda = \mathbb{Z}$, and Ω tiles $\mathbb{R}$ by $\mathbb{Z}$.

So without loss of generality, let $\alpha, \beta \neq 0$, so that we have

$$\alpha(e^{2\pi i\lambda_3 r} - 1) = \beta(e^{2\pi i\lambda_3 a} - 1).$$

Consider now two circles given by $C_1(t) = \alpha(e^{2\pi it} - 1)$, and $C_2(s) = \beta(e^{2\pi is} - 1)$, t, $s \in [0, 1]$. Both circles pass through 0. Further note that

$$C_1(\lambda_3 r) = C_2(\lambda_3 a), \tag{5}$$

$$C_1(\lambda_2 r) = C_2(\lambda_2 a). \tag{6}$$

We consider the various possibilities.

First, if the two circles coincide, then they have the same radius and center i.e., $\alpha = \beta$ and so $e^{2\pi i\lambda_2 a} = e^{2\pi i\lambda_2 r}$. Thus $\varphi_\Omega(\lambda_2)$ is of the form $(1, c; c, 1)$, and we conclude that $\Lambda = \mathbb{Z}$ as before.

Next, we consider the case when the two circles $C_1(t), C_2(t)$ are distinct. As mentioned above, both the circles pass through 0 and now there are two more points of intersection given by equations (5) and (6). But two distinct circles can have at most two distinct points of intersection. If $C_1(\lambda_2 r) = C_2(\lambda_2 a) = 0$ then $\varphi_\Omega(0) = \varphi_\Omega(\lambda_2)$ and similarly if $C_1(\lambda_3 r) = C_2(\lambda_3 a) = 0$, then $\varphi_\Omega(0) = \varphi_\Omega(\lambda_3)$. By our assumption these cases are not possible. Thus the only possibility is that $C_1(\lambda_2 r) = C_2(\lambda_3 a) = C_1(\lambda_3 r) = C_2(\lambda_2 a) = \alpha\beta$. Then $e^{2\pi i\lambda_2 a} = e^{2\pi i\lambda_3 a}$ and $e^{2\pi i\lambda_2 r} = e^{2\pi i\lambda_3 r}$, i.e., $\varphi_\Omega(\lambda_2) = \varphi_\Omega(\lambda_3)$ which is again not possible. This completes the proof.

4. On 3 intervals

In this section, we investigate the "Spectral implies Tiling" part of Fuglede's conjecture for three intervals, in the same spirit as in the previous section.

Let $\Omega = [0, r] \cup [a, a+s] \cup [b, b+1-r-s]$ where $0 < r, s, r+s < 1$ and let (Ω, Λ) be a spectral pair. We will assume here that $0 \in \Lambda$. Again we define the map $\varphi_\Omega : \Lambda \to \mathbb{C}^3 \times \mathbb{C}^3$ by

$$\varphi_\Omega(\lambda) := (e^{2\pi i\lambda r}, e^{2\pi i\lambda(a+s)}, e^{2\pi i\lambda(b+1-r-s)}; 1, e^{2\pi i\lambda a}, e^{2\pi i\lambda b})$$

and let $V_\Omega(\Lambda)$ be the subspace spanned by $\varphi_\Omega(\Lambda) = \{\varphi_\Omega(\lambda) : \lambda \in \Lambda\}$. We know that $dim(V_\Omega(\Lambda)) \leq 3$. As in the 2-interval case we will first show that if Ω is non-degenerate, in the sense that the three intervals are disjoint and have non zero length, then $dim(V_\Omega(\Lambda)) = 3$.

Proposition 4.1. *Let Ω, as above, be a spectral set and let Λ be a associated spectrum. Then $dim(V_\Omega(\Lambda)) = 3$.*

Proof. If $dim(V_\Omega(\Lambda)) = 1$, we again get $\Lambda = \mathbb{Z},\ r, s, 1-r-s \in \mathbb{Z}$, i.e., Ω consists of a single interval of length 1 and this is a degenerate case.

So let, if possible, $dim(V_\Omega(\Lambda)) = 2$. Suppose $0 < \lambda_2 < \lambda_3$ are the first three elements of $\Lambda \cap [0, \infty)$. In the two cases $\varphi_\Omega(0) = \varphi_\Omega(\lambda_2)$ or $\varphi_\Omega(\lambda_2) = \varphi_\Omega(\lambda_3)$ we conclude that $dim(V_\Omega(\Lambda)) = 1$, and if $\varphi_\Omega(0) = \varphi_\Omega(\lambda_3)$ we see easily that $\Lambda = 2\mathbb{Z} \cup (2\mathbb{Z} + \alpha)$ and $r, s \in \mathbb{Z}/2$, i.e., Ω is a union of 2 intervals of length $1/2$ or is a single interval of length 1, and this case too is degenerate.

Finally, suppose that $\varphi_\Omega(0), \varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3)$ are all distinct. Define

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{2\pi i\lambda_2 r} & e^{2\pi i\lambda_2(a+s)} & e^{2\pi i\lambda_2(b+1-r-s)} & 1 & e^{2\pi i\lambda_2 a} & e^{2\pi i\lambda_2 b} \\ e^{2\pi i\lambda_3 r} & e^{2\pi i\lambda_3(a+s)} & e^{2\pi i\lambda_3(b+1-r-s)} & 1 & e^{2\pi i\lambda_3 a} & e^{2\pi i\lambda_3 b} \end{pmatrix} \tag{7}$$

Then, by our assumption, $Rank(A) = 2$, and the rows of A are distinct. Since $\varphi_\Omega(0) \neq \varphi_\Omega(\lambda_2)$ and $\varphi_\Omega(0) \odot \varphi_\Omega(\lambda_2) = 0$, at least two entries in the second row of A are different from 1, i.e., $\exists\ i_1, i_2$ such that $A(2, i_1), A(2, i_2) \neq 1$. Consider the 3×3 matrix constructed out of the 1st, i_1th and i_2th column of A, since $Rank(A) = 2$ it is singular. Hence we have,

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & A(2,i_1) & A(2,i_2) \\ 1 & A(3,i_1) & A(3,i_2) \end{vmatrix} = 0 \tag{8}$$

Using the fact that $Rank(A) = 2$, and $A(2, i_1), A(2, i_2) \neq 1$ we argue as in the two interval case to conclude that the circles $C_1(t) = (A(2, i_1) - 1)(e^{2\pi it} - 1)$ and $C_2(t) = (A(2, i_2) - 1)(e^{2\pi it} - 1)$ coincide. Therefore, $A(2, i_2) = A(2, i_1) = \alpha$, say. By choosing other columns of A, we see that the coordinates of $\varphi_\Omega(\lambda_2)$ are either 1 or α. But since $\varphi_\Omega(0) \odot \varphi_\Omega(\lambda_2) = 0$, $\varphi_\Omega(\lambda_2)$ is either of the form $(1, 1, \alpha; 1, 1, \alpha)$ or $(1, \alpha, \alpha; 1, \alpha, \alpha)$ (up to suitable permutations). Now $\{\varphi_\Omega(0), \varphi_\Omega(\lambda_2)\}$ forms a basis of $V_\Omega(\Lambda)$, so as before, we see that $\Lambda = \mathbb{Z}$. But then one of the intervals has length 0 or 1, and this is a degenerate case. □

So $Rank(A) = 3$. Let $\lambda_1 = 0, \lambda_2, \lambda_3, \lambda_4$ be the first 4 elements of $\Lambda \cap [0, \infty)$. Each of the cases $\varphi_\Omega(\lambda_i) = \varphi_\Omega(\lambda_{i+1})$ or $\varphi_\Omega(\lambda_i) = \varphi_\Omega(\lambda_{i+2})$ will imply Ω is degenerate, i.e., one of the intervals has length 0. Now if $\varphi_\Omega(0) = \varphi_\Omega(\lambda_4)$, then $\lambda_4 = d$ and the spectrum is d-periodic, and by a density argument we conclude $d = 3$. It follows then, that this is the case of three equal intervals i.e., $r = s = 1/3$ and spectral implies tiling follows by the result of [9] (see [2] for a proof). So, now it remains to consider the case that $\varphi_\Omega(0), \varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3), \varphi_\Omega(\lambda_4)$ are all distinct and $Rank(A) = 3$.

To proceed further, we will use the result that the spectrum is periodic [1]. Let d be the smallest positive integer such that $d\mathbb{Z} \subseteq \Lambda$. Let $V_\Omega(d\mathbb{Z})$ denote the linear space spanned by the image of the arithmetic projection $d\mathbb{Z}$ under the map ϕ_Ω. Now if $dim(V_\Omega(d\mathbb{Z})) = 3$ or 2, then by the results of [2], Section 5, we get Spectral implies Tiling.

So without loss of generality, we may assume that $dim(V_\Omega(d\mathbb{Z})) = 1$, and that Λ is d-periodic. There are now two possible cases to consider:

(1) $dim(V_\Omega(\Lambda \setminus d\mathbb{Z})) = 2$
(2) $dim(V_\Omega(\Lambda \setminus d\mathbb{Z})) = 3$.

In the first case we are able to show that $d = 3$, thus Ω is a union of three equal interval and hence Spectral implies Tiling as before. It is the second case that remains inconclusive.

Case(1). We show that in this case $d = 3$. Suppose not, and $d > 3$. Let $\Lambda \cap (0, d) = \{\lambda_2, \lambda_3, \ldots, \lambda_d\}$. Since d is the minimal period, $\varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3), \varphi_\Omega(\lambda_4)$ are all distinct and since $dim(V_\Omega(\Lambda \setminus d\mathbb{Z})) = 2$, $\{\varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3), \varphi_\Omega(\lambda_4)\}$ is a linearly dependent set. Let

$$\begin{aligned}\varphi_\Omega(\lambda_2) &= (\xi_1, \xi_2, \xi_3; 1, \xi_5, \xi_6)\\ \varphi_\Omega(\lambda_3) &= (\rho_1, \rho_2, \rho_3; 1, \rho_5, \rho_6)\\ \varphi_\Omega(\lambda_4) &= (\eta_1, \eta_2, \eta_3; 1, \eta_5, \eta_6)\end{aligned}$$

Now there exists i, j such that $\xi_i \neq \rho_i$ and $\xi_j \neq \rho_j$. By our assumption

$$\begin{vmatrix} 1 & \xi_i & \xi_j \\ 1 & \rho_i & \rho_j \\ 1 & \eta_i & \eta_j \end{vmatrix} = 0 \tag{9}$$

and so,

$$\begin{vmatrix} 1 & \xi_i & \xi_j \\ 0 & \rho_i - \xi_i & \rho_j - \xi_j \\ 0 & \eta_i - \xi_i & \eta_j - \xi_j \end{vmatrix} = 0 \tag{10}$$

Thus we obtain,

$$(\rho_i - \xi_i)(\eta_j - \xi_j) = (\rho_j - \xi_j)(\eta_i - \xi_i) \tag{11}$$

which we rewrite as

$$(\rho_i \bar{\xi_i} - 1)(\eta_j \bar{\xi_j} - 1) = (\rho_j \bar{\xi_j} - 1)(\eta_i \bar{\xi_i} - 1) \tag{12}$$

Since $\xi_i \neq \rho_i$ and $\xi_j \neq \rho_j$ and $dim(V_\Omega(\Lambda \setminus d\mathbb{Z})) = 2$ we can exclude the two possibilities that $\{\eta_i = \rho_i, \eta_j = \rho_j\}$ or that $\{\eta_i = \xi_i, \eta_j = \xi_j\}$. Then, by the same argument with two circles as at the end of section 3, we see that $\rho_i \overline{\xi_i} = \rho_j \overline{\xi_j} = \alpha$. In particular this would hold for any other index j' such that $\xi_{j'} \neq \rho_{j'}$. This implies that $\varphi_\Omega(\lambda_3 - \lambda_2) = (1, 1, \alpha; 1, 1, \alpha)$ or $(1, \alpha, \alpha; 1, \alpha, \alpha)$ (or some suitable permutation). But since $\varphi_\Omega(\lambda) := (e^{2\pi i \lambda r}, e^{2\pi i \lambda(a+s)}, e^{2\pi i \lambda(b+1-r-s)}; 1, e^{2\pi i \lambda a}, e^{2\pi i \lambda b})$, we see that $\lambda_3 - \lambda_2 \in \mathbb{Z}$. We write $\lambda_3 - \lambda_2 = k$, and show that $k\mathbb{Z} \subset \Lambda$.

Consider the first situation, namely $\varphi_\Omega(k) = (1, 1, \alpha; 1, 1, \alpha)$. Now $\varphi_\Omega(0), \varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3)$ is a basis of $V_\Omega(\Lambda)$, and we are in the case where $\varphi_\Omega(\lambda_3) = (\xi_1, \xi_2, \alpha\xi_3; 1, \xi_5, \alpha\xi_6)$. But both $\varphi_\Omega(\lambda_2)$ and $\varphi_\Omega(\lambda_3)$ are null vectors, so $\xi_3 = \xi_6$. Then, $\varphi_\Omega(kn) \odot \varphi_\Omega(\lambda_i) = 0,\ i = 1, 2, 3$ for every $n \in \mathbb{Z}$. Thus by Lemma 2.1, $k\mathbb{Z} \subseteq \Lambda$, so that Λ and $k < d$. But d is the smallest positive integer with this property, which is a contradiction. A similar argument works for all other cases. Thus $d = 3$, and Ω is a union of 3 equal intervals, and Spectral implies Tiling follows.

5. Generalized Vandermonde Matrix

It remains now to consider the case when $dim(V_\Omega(d\mathbb{Z})) = 1$ and $dim(V_\Omega(\Lambda \setminus d\mathbb{Z})) = 3$ where d is the smallest integer such that $d\mathbb{Z}$ is in the spectrum Λ. Note that $dim(V_\Omega(d\mathbb{Z})) = 1$ implies that Ω can be written as

$$\Omega = [0, k_1/d] \cup [l_2/d, (l_2 + k_2)/d] \cup [l_3/d, (l_3 + k_3)/d],$$

where $l_i, k_i \in \mathbb{N}$ and $k_1 + k_2 + k_3 = d$.

If $d = 3$ then it is the case of three equal intervals in which case we know that Fuglede's conjecture holds. By known results it is possible to rule out the cases $d = 4$ and 5 as well. Hence the problem will be resolved if we can show that $d < 6$. In any case finding a bound on d is desirable.

Now if $d > 3$, let $0 = \lambda_1, \lambda_2, \lambda_3 < d$ be three elements of Λ such that $\{\varphi_\Omega(0), \varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3)\}$ forms a basis of $V_\Omega(\Lambda)$. By our assumption there exists $\lambda_4 < d$ in Λ such that $\varphi_\Omega(\lambda_2), \varphi_\Omega(\lambda_3), \varphi_\Omega(\lambda_4)$ are linearly independent. We construct the matrix

(13)
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ e^{2\pi i \frac{\lambda_2 k_1}{d}} & e^{2\pi i \frac{\lambda_2 (l_2+k_2)}{d}} & e^{2\pi i \frac{\lambda_2 (l_3+k_3)}{d}} & 1 & e^{2\pi i \frac{\lambda_2 l_2}{d}} & e^{2\pi i \frac{\lambda_2 l_3}{d}} \\ e^{2\pi i \frac{\lambda_3 k_1}{d}} & e^{2\pi i \frac{\lambda_3 (l_2+k_2)}{d}} & e^{2\pi i \frac{\lambda_3 (l_3+k_3)}{d}} & 1 & e^{2\pi i \frac{\lambda_3 l_2}{d}} & e^{2\pi i \frac{\lambda_3 l_3}{d}} \\ e^{2\pi i \frac{\lambda_4 k_1}{d}} & e^{2\pi i \frac{\lambda_4 (l_2+k_2)}{d}} & e^{2\pi i \frac{\lambda_4 (l_3+k_3)}{d}} & 1 & e^{2\pi i \frac{\lambda_4 l_2}{d}} & e^{2\pi i \frac{\lambda_4 l_3}{d}} \end{pmatrix}$$

The rank of this matrix is 3 i.e., the rows are linearly dependent, hence each of its 4×4 minors are zero.

Observe that the 4×4 minors are of the form

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ X_1^i & X_1^j & X_1^k & X_1^l \\ X_2^i & X_2^j & X_2^k & X_2^l \\ X_3^i & X_3^j & X_3^k & X_3^l \end{vmatrix} \tag{14}$$

Thus we get after reductions, equations of the form

$$
\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & X_1^j & X_1^k & X_1^l \\ 1 & X_2^j & X_2^k & X_2^l \\ 1 & X_3^j & X_3^k & X_3^l \end{vmatrix} = 0 \tag{15}
$$

These are the determinants of generalized Vandermonde matrices in the variables (X_1, X_2, X_3) and exponents (j, k, l). We write (15) as

$$
R_{(j,k,l)}(X_1, X_2, X_3) = 0.
$$

We are interested in the common zero solution set of these Vandermonde varieties intersected with the set $1 \times \mathbb{T}^3$.

In particular, let us consider the generalized Vandermonde matrix which we get by taking those minors where the first three columns correspond to the left end-points of the set Ω and the 4th column is one of the right end point i.e., a minor obtained by choosing the 4th, 5th, and 6th columns of the matrix A and one of the first three columns. Thus we consider $R_{(i_5,i_6,i_1)}(X_1, X_2, X_3)$, $R_{(i_5,i_6,i_2)}(X_1, X_2, X_3)$, and $R_{(i_5,i_6,i_3)}(X_1, X_2, X_3)$.

In [4] (Theorem 3.1) it is proved that the polynomials

$$
T_{(j,k,l)}(X_1, X_2, X_3) = \frac{R_{(j,k,l)}(X_1, X_2, X_3)}{V(X_1^g, X_2^g, X_3^g)},
$$

are either irreducible or constant. Here $g = gcd(j, k, l)$ and V denotes the standard Vandermonde determinant, thus $V(X_1^g, X_2^g, X_3^g) = R_{(1,2,3)}(X_1^g, X_2^g, X_3^g)$.

Consider next, the Schur Polynomials given by

$$
S_{(j,k.l)}(X_1, X_2, X_3) = \frac{R_{(j,k,l)}(X_1, X_2, X_3)}{V(X_1, X_2, X_3)},
$$

Let $g_1 = gcd(i_5, i_6, i_1)$, $g_2 = gcd(i_5, i_6, i_2)$ and $g_3 = gcd(i_5, i_6, i_3)$. We know that $gcd(g_1, g_2, g_3) = 1$ by our choice of d. Theorem 4.1 in [4] regarding intersection of Fermat hypersurfaces seems to suggest that there can not be many solutions.

In the particular case when $gcd(g_1, g_2) = 1$ the analysis in [3] tells us that $S_{(i_5,i_6,i_1)}$ and $S_{(i_5,i_6,i_2)}$ are coprime and each hypersurface defined by $S_{(i_5,i_6,i_1)} = 0$ and $S_{(i_5,i_6,i_2)} = 0$ in $\mathbb{C}^3$ has distinct reduced irreducible components of dimension 2. Then their intersection W has dimension 1. (Note that with respect to the setting of [3], we have fixed the first coordinate, hence we get one dimension less).

In our case we need only those solutions such that $|X_j| = 1, \forall j$. In other words we need the set $W \cap \mathbb{T}^3$. This condition in itself is very restrictive. In the previous sections, where we used the two-circles argument along with mutual orthogonality, we saw that this set can be finite. If an analysis as in [3] can be carried through to get that $W \cap \mathbb{T}^3$ is indeed finite, we immediately get a bound on the period d. Then along with orthogonality, one may be able to resolve the remaining case of the 3-intervals!

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