A generalized Young inequality and some new results on fractal space

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Abstract: Starting with real line number system based on the theory of the Yang's fractional set, the generalized Young inequality is established. By using it some results on the generalized inequality in fractal space are investigated in detail.

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1 Introduction

The classical Young inequality [1–4] is not only interesting in itself but also very useful. The purpose of this work is to establish a generalized Young inequality on fractional set and other inequality based on it. Start with, we review the Yang's fractional set and Yang's geometric expressions for the real line number system.

1.1 Theory of the Yang's fractional set

Recently, the theory of Yang's fractional sets of element sets [6-12] was introduced as follows:

For $0 < \alpha \le 1$, we have the following α-type set of element sets:

N_0^α The α-type set of the natural numbers are defined as the set $\{0^\alpha, 1^\alpha, 2^\alpha, \ldots, n^\alpha, \ldots\}$;

N_+^α The α-type set of the natural numbers are defined as the set $\{1^\alpha, 2^\alpha, \ldots, n^\alpha, \ldots\}$;

Z_0^α The α-type set of the integers are defined as the set $\{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \ldots, \pm n^\alpha, \ldots\}$;

Z^α The α-type set of the integers are defined as the set $\{\pm 1^\alpha, \pm 2^\alpha, \ldots, \pm n^\alpha, \ldots\}$;

Q^α The α-type set of the rational numbers are defined as the set $\{m^\alpha = (p/q)^\alpha : p, q \in \mathrm{Z}, q \neq 0\}$;

$\Im^\alpha$ The α-type set of the irrational numbers are defined as the set $\{m^\alpha \neq (p/q)^\alpha : p, q \in \mathrm{Z}, q \neq 0\}$;

R^α The α-type set of the real line numbers are defined as the set $\mathrm{R}^\alpha = \mathrm{Q}^\alpha \cup \Im^\alpha$.

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1.2 Yang's geometric expressions for the real line number system

Geometric representation of real line numbers on a fractional set as points on a real line called the real line axis. For each real line number there correspond one and only one point on the real line[6].

For example, $1^\alpha + 2^\alpha = 3^\alpha$. That is, the geometric representation is that cantor set $[0, 3]$ is equivalent to the sum of cantor set $[0, 1]$ and cantor set $[1, 3]$. The dimension of cantor set is α , for $0 < \alpha \leq 1$ and, 1^α , 2^α and 3^α are real line numbers on a fractional set. If $a^\alpha, b^\alpha, c^\alpha$ belong to the set R^α of real line numbers, then we have the following operation:

(1) $a^\alpha + b^\alpha$ and $a^\alpha b^\alpha$ belong to the set R^α

(2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha = (a+b)^\alpha = (b+a)^\alpha$;

(3) $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$;

(4) $a^\alpha b^\alpha = b^\alpha a^\alpha = (ab)^\alpha = (ba)^\alpha$;

(5) $a^\alpha (b^\alpha c^\alpha) = (a^\alpha b^\alpha) c^\alpha$;

(6) $a^\alpha (b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$;

(7) $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$ and $a^\alpha \cdot 1^\alpha = 1^\alpha \cdot a^\alpha = a^\alpha$.

If $a^\alpha - b^\alpha$ is a nonnegative number, we say that a^α is greater than or equal to b^α or b^α is less than or equal to a^α , and write, respectively, $a^\alpha \geq b^\alpha$ or $b^\alpha \leq a^\alpha$. If there is no possibility that $a^\alpha = b^\alpha$, we write

$a^\alpha > b^\alpha$ or $b^\alpha < a^\alpha$.

Geometrically, $a^\alpha > b^\alpha$ if the point on the real line axis corresponding to a^α lies to the left of the point corresponding to b^α .

Suppose that a^α , b^α and c^α are any given real line numbers, then we have the following relations:

(1) Either $a^\alpha > b^\alpha$, $a^\alpha = b^\alpha$ or $a^\alpha < b^\alpha$ (Law of trichotomy);

(2) If $a^\alpha > b^\alpha$ and $b^\alpha > c^\alpha$, then $a^\alpha > c^\alpha$ (Law of transitivity);

(3) If $a^\alpha > b^\alpha$, then $a^\alpha + c^\alpha > b^\alpha + c^\alpha$;

(4) If $a^\alpha > b^\alpha$ and $c^\alpha > 0^\alpha$, then $a^\alpha c^\alpha > b^\alpha c^\alpha$;

(5) If $a^\alpha > b^\alpha$ and $c^\alpha < 0^\alpha$, then $a^\alpha c^\alpha < b^\alpha c^\alpha$.

The formula is similar to classical one in case of $\alpha = 1$. As direct results, the following inequalities are valid:

If $a^\alpha > b^\alpha$, then $a > b$;

If $a^\alpha = b^\alpha$, then $a = b$;

If $a^\alpha < b^\alpha$, then $a < b$.

2 The generalized Young inequality

In the section, we give the proof of the generalized Young inequality. Here we first start with the generalized Bernoulli's inequality.

Lemma 2.1 *(Generalized Bernoulli's inequality) Let* $y > 0$

(1) for $0 < m < 1$*, then*

$$y^{\alpha m} - 1^{\alpha} \leq m(y-1)^{\alpha}. \tag{2.1}$$

(2) for $m > 1$*, then*

$$y^{\alpha m} - 1^{\alpha} \geq m(y-1)^{\alpha}. \tag{2.2}$$

Remark 1. This is classical Bernoulli's inequality in case of fractal dimension $\alpha = 1$[5].

Theorem 2.2 . *Let* a*,* $b \geq 0$*,* $p > 1$*,* $\frac{1}{p} + \frac{1}{q} = 1$*,then*

$$a^{\alpha} b^{\alpha} \leq \frac{a^{p\alpha}}{p} + \frac{b^{q\alpha}}{q}, \tag{2.3}$$

which is equality holding if and only if $a^p = b^q$*.*

Proof. Setting $y = a/b$, by (2.1) we have

$$(a/b)^{\alpha m} - 1^{\alpha} \leq m(a/b - 1)^{\alpha}. \tag{2.4}$$

Multiplying both sides by b^{α} in (2.4) gives

$$a^{\alpha m} b^{\alpha(1-m)} - b^{\alpha} \leq m(a-b)^{\alpha}. \tag{2.5}$$

Then, we obtain that

$$a^{\alpha m} b^{\alpha(1-m)} \leq m a^{\alpha} + (1-m) b^{\alpha}. \tag{2.6}$$

Let $m = \frac{1}{p} < 1$, then we directly deduce (2.3).

Remark 2. (2.3) was proposed[6], here we give its proof.it is classical Young inequality in case of fractal dimension $\alpha = 1$.

Theorem 2.3 *Let a,b* ≥ 0*,* $0 < p < 1$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$a^{\alpha} b^{\alpha} \geq \frac{a^{p\alpha}}{p} + \frac{b^{q\alpha}}{q}, \tag{2.7}$$

where equality holds if and only if $a^p = b^q$*.*

Proof. For x, $y \geq 0$, $0 < p < 1$, by (2.3), we have

$$x^{\alpha} y^{\alpha} \leq p x^{\frac{\alpha}{p}} + (1-p) y^{\frac{\alpha}{q}}. \tag{2.8}$$

Set $a = \frac{1}{p^p} x^p y^p$, $b = \frac{1}{p^{1-p}} y^{-p}$. By (2.8) we obtain

$$a^{\alpha} b^{\alpha} \geq \frac{a^{p\alpha}}{p} + \frac{b^{q\alpha}}{q}.$$

Hence we complete the proof of Theorem 2.2.

As a direct result, we have the following:

Corollary 2.1 *Let $a_i \geq 0$, $p_i \in \mathrm{R}$, $i = 1, 2, \ldots n$, $\sum\limits_{i=1}^{n} 1/p_i = 1$,*

(1) for $p_i > 1$, we have

$$\prod_{i=1}^{n} a_i^{\alpha} \leq \sum_{i=1}^{n} a_i^{p_i \alpha}/p_i, \tag{2.9}$$

where the equality holds if $a_j^{p_j} = a_k^{p_k}$ for $\forall j, k$.

(2) for $0 < p_1 < 1$, $p_i < 0$, $i = 2, \ldots n$,we have

$$\prod_{i=1}^{n} a_i^{\alpha} \geq \sum_{i=1}^{n} a_i^{p_i \alpha}/p_i, \tag{2.10}$$

where the equality holds if $a_j^{p_j} = a_k^{p_k}$ for $\forall j, k$.

3 Useful results based on generalized Young inequality

In the section we discuss some generalizations of holder inequality and Minkowski inequality. In order to proof our results, we first review the hölder inequality and Minkowski inequality [6]:

Theorem 3.1 *.(see [6]) Let $|x_i|$, $|y_i| \geq 0$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $i = 1, 2, \ldots n$, then*

$$\sum_{i=1}^{n} |x_i^{\alpha}| \, |y_i^{\alpha}| \leq \left(\sum_{i=1}^{n} |x_i^{\alpha}|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i^{\alpha}|^q \right)^{1/q}. \tag{3.1}$$

Equalities holding if and only if $\lambda_1 |x_i| = \lambda_2 |y_i|$, where λ_1 and λ_2 are constants.

Theorem 3.2 *.(see[6]) Let $|x_i|$, $|y_i| \geq 0$, $p > 1$, $i = 1, 2, \ldots n$, then*

$$\left(\sum_{i=1}^{n} |x_i^{\alpha} + y_i^{\alpha}|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} |x_i^{\alpha}|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |y_i^{\alpha}|^q \right)^{1/q}. \tag{3.2}$$

Equalities holding if and only if $\lambda_1 |x_i| = \lambda_2 |y_i|$, where λ_1 and λ_2 are constants.

Theorem 3.3 *. Let $|x_i|$, $|y_i| \geq 0$, $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $i = 1, 2, \ldots n$, then*

$$\sum_{i=1}^{n} |x_i^{\alpha}| \, |y_i^{\alpha}| \geq \left(\sum_{i=1}^{n} |x_i^{\alpha}|^p \right)^{1/p} \left(\sum_{i=1}^{n} |y_i^{\alpha}|^q \right)^{1/q}. \tag{3.3}$$

Equalities holding if and only if $\lambda_1 |x_i| = \lambda_2 |y_i|$, where λ_1 and λ_2 are constants.

Proof. Set $c = 1/p$, then we have $q = -pd$, $d = c/(c-1)$. By (3.1), we obtain

$$\begin{aligned} \sum_{i=1}^{n} |x_i^{\alpha}|^p &= \sum_{i=1}^{n} |x_i^{\alpha} y_i^{\alpha}|^p \, |y_i^{\alpha}|^{-p} \\ &\leq \left(\sum_{i=1}^{n} |x_i^{\alpha} y_i^{\alpha}|^{pc} \right)^{1/c} \left(\sum_{i=1}^{n} |y_i^{\alpha}|^{-pd} \right)^{1/d} \\ &= \left(\sum_{i=1}^{n} |x_i^{\alpha} y_i^{\alpha}| \right)^{1/p} \left(\sum_{i=1}^{n} |y_i^{\alpha}|^q \right)^{1-p}. \end{aligned} \tag{3.4}$$

In (3.4), multiplying both sides by $\left(\sum\limits_{i=1}^{n}|y_i^\alpha|^q\right)^{p-1}$ yields

$$\sum_{i=1}^{n}|x_i^\alpha|^p\left(\sum_{i=1}^{n}|y_i^\alpha|^q\right)^{p-1}\leq\left(\sum_{i=1}^{n}|x_i^\alpha y_i^\alpha|\right)^{1/p}. \tag{3.5}$$

Using (3.5) implies that

$$\sum_{i=1}^{n}|x_i^\alpha|\,|y_i^\alpha|\geq\left(\sum_{i=1}^{n}|x_i^\alpha|^p\right)^{1/p}\left(\sum_{i=1}^{n}|y_i^\alpha|^q\right)^{1/q}.$$

Theorem 3.4 . *Let* $|x_i|$,$|y_i|\geq 0$, $0<p<1$,$i=1,2,\ldots n$, *then*

$$\left(\sum_{i=1}^{n}|x_i^\alpha+y_i^\alpha|^p\right)^{1/p}\geq\left(\sum_{i=1}^{n}|x_i^\alpha|^p\right)^{1/p}+\left(\sum_{i=1}^{n}|y_i^\alpha|^p\right)^{1/p}. \tag{3.6}$$

Equalities holding if and only if $\lambda_1\,|x_i|=\lambda_2\,|y_i|$, *where* λ_1 *and* λ_2 *are constants.*

Proof. $A_n=\sum\limits_{i=1}^{n}|x_i^\alpha|^p B_n=\sum\limits_{i=1}^{n}|y_i^\alpha|^p$, $C_n=\left(\sum\limits_{i=1}^{n}|x_i^\alpha|^p\right)^{1/p}+\left(\sum\limits_{i=1}^{n}|y_i^\alpha|^q\right)^{1/q}$, by Hölder inequality, in view of $0<p<1$, we have

$$\begin{aligned}C_n&=\sum_{i=1}^{n}(|x_i^\alpha|^p\,A_n^{1/p-1}+|y_i^\alpha|^p\,B_n^{1/p-1})\\&\leq\sum_{i=1}^{n}|x_i^\alpha+y_i^\alpha|^p\,(A_n^{1/p}+B_n^{1/p})^{1-p}=C_n^{1-p}\sum_{i=1}^{n}|x_i^\alpha+y_i^\alpha|^p.\end{aligned} \tag{3.7}$$

By inequality (3.7), we arrive at reverse Minkowski's inequality and the theorem is completely proved.

Corollary 3.1 *Let* $|x_{ij}|\geq 0$, $p_j\in\mathrm{R}$, $i=1,2,\ldots n$, $j=1,2,\ldots m$ $\sum\limits_{j=1}^{m}1/p_j=1$.

(1) for $p_j>1$, *we have*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}|x_{ij}^\alpha|\leq\prod_{j=1}^{m}\left(\sum_{i=1}^{n}|x_{ij}^\alpha|^{p_j}\right)^{1/p_j}. \tag{3.8}$$

Equalities holding if and only if $\lambda_j\,|x_{ij}|=\lambda_k\,|x_{ik}|$ *for* $\forall j,k$, *where* λ_j *and* λ_k *are constants.*

(2) for $0<p_1<1$, $p_j<0$, $j=2,\ldots m$,*we have*

$$\sum_{i=1}^{n}\prod_{j=1}^{m}|x_{ij}^\alpha|\geq\prod_{j=1}^{m}\left(\sum_{i=1}^{n}|x_{ij}^\alpha|^{p_j}\right)^{1/p_j}. \tag{3.9}$$

Equalities holding if and only if $\lambda_j\,|x_{ij}|=\lambda_k\,|x_{ik}|$ for $\forall j,k$, where λ_j and λ_k are constants.

Corollary 3.2 *Let* $|x_{ij}| \geq 0$, $i = 1, 2, \ldots n$,$j = 1, 2, \ldots m$,

(1)for $p > 1$, *we have*

$$\left(\sum_{i=1}^{n}\left|\sum_{j=1}^{m} x_{ij}^{\alpha}\right|^{p}\right)^{1/p} \leq \sum_{i=1}^{n}\left(\sum_{j=1}^{m}|x_{ij}^{\alpha}|^{p}\right)^{1/p}. \tag{3.10}$$

Equalities holding if and only if $\lambda_j |x_{ij}| = \lambda_k |x_{ik}|$ *for* $\forall j, k$, *where* λ_j *and* λ_k *are constants.*

(2)for $0 < p < 1$ *,we have*

$$\left(\sum_{i=1}^{n}\left|\sum_{j=1}^{m} x_{ij}^{\alpha}\right|^{p}\right)^{1/p} \geq \sum_{i=1}^{n}\left(\sum_{j=1}^{m}|x_{ij}^{\alpha}|^{p}\right)^{1/p}. \tag{3.11}$$

Equalities holding if and only if $\lambda_j |x_{ij}| = \lambda_k |x_{ik}|$ *for* $\forall j, k$, *where* λ_j *and* λ_k *are constants.*

Theorem 3.5 *Let* $|x_i|$, $|y_i| \geq 0$, $0 < r < 1 < p$, $i = 1, 2, \ldots n$, *then*

$$\left(\frac{\sum\limits_{i=1}^{n}|x_i^{\alpha}+y_i^{\alpha}|^{p}}{\sum\limits_{i=1}^{n}|x_i^{\alpha}+y_i^{\alpha}|^{r}}\right)^{1/(p-r)} \leq \left(\frac{\sum\limits_{i=1}^{n}|x_i^{\alpha}|^{p}}{\sum\limits_{i=1}^{n}|x_i^{\alpha}|^{r}}\right)^{1/(p-r)} + \left(\frac{\sum\limits_{i=1}^{n}|y_i^{\alpha}|^{p}}{\sum\limits_{i=1}^{n}|y_i^{\alpha}|^{r}}\right)^{1/(p-r)}. \tag{3.12}$$

Equalities holding if and only if $\lambda_1 |x_i| = \lambda_2 |y_i|$,*where* λ_1 *and* λ_2 *are constants.*

Proof. By Theorem 3.1 and Theorem 3.2 We have

$$\begin{aligned}
&\left(\sum_{i=1}^{n}|x_i^{\alpha}+y_i^{\alpha}|^{p}\right)^{1/(p-r)} \leq \left(\left(\sum_{i=1}^{n}|x_i^{\alpha}|^{p}\right)^{1/p} + \left(\sum_{i=1}^{n}|y_i^{\alpha}|^{p}\right)^{1/p}\right)^{p/(p-r)}\\
&= \left(\left(\frac{\sum\limits_{i=1}^{n}|x_i^{\alpha}|^{p}}{\sum\limits_{i=1}^{n}|x_i^{\alpha}|^{r}}\right)^{1/p}\left(\sum_{i=1}^{n}|x_i^{\alpha}|^{r}\right)^{1/p} + \left(\frac{\sum\limits_{i=1}^{n}|y_i^{\alpha}|^{p}}{\sum\limits_{i=1}^{n}|y_i^{\alpha}|^{r}}\right)^{1/p}\left(\sum_{i=1}^{n}|y_i^{\alpha}|^{r}\right)^{1/p}\right)^{p/(p-r)} \qquad (3.13)\\
&\leq \left(\left(\frac{\sum\limits_{i=1}^{n}|x_i^{\alpha}|^{p}}{\sum\limits_{i=1}^{n}|x_i^{\alpha}|^{r}}\right)^{1/(p-r)} + \left(\frac{\sum\limits_{i=1}^{n}|y_i^{\alpha}|^{p}}{\sum\limits_{i=1}^{n}|y_i^{\alpha}|^{r}}\right)^{1/(p-r)}\right)\left(\left(\sum_{i=1}^{n}|x_i^{\alpha}|^{r}\right)^{1/r} + \left(\sum_{i=1}^{n}|y_i^{\alpha}|^{r}\right)^{1/r}\right)^{r/(p-r)}.
\end{aligned}$$

Using reverse Minkowski inequality implies that

$$\left(\left(\sum_{i=1}^{n}|x_i^{\alpha}|^{r}\right)^{1/r} + \left(\sum_{i=1}^{n}|y_i^{\alpha}|^{r}\right)^{1/r}\right)^{r} \leq \sum_{i=1}^{n}|x_i^{\alpha}+y_i^{\alpha}|^{r}. \tag{3.14}$$

By (3.13) and (3.14), we get (3.12). Hence, the theorem is completely proved.

Corollary 3.3 *Let* $|x_{ij}| \geq 0,\ 0 < r < 1 < p,\ i = 1, 2, \dots n, j = 1, 2, \dots m$ *then*

$$\left(\frac{\sum\limits_{i=1}^{n} \left| \sum\limits_{j=1}^{m} x_{ij}^{\alpha} \right|^{p}}{\sum\limits_{i=1}^{n} \left| \sum\limits_{j=1}^{m} x_{ij}^{\alpha} \right|^{r}} \right)^{1/(p-r)} \leq \sum_{i=1}^{n} \left(\frac{\sum\limits_{j=1}^{m} \left| x_{ij}^{\alpha} \right|^{p}}{\sum\limits_{j=1}^{m} \left| x_{ij}^{\alpha} \right|^{r}} \right)^{1/(p-r)} . \tag{3.15}$$

Equalities holding if and only if $\lambda_j |x_{ij}| = \lambda_k |x_{ik}|$ *for* $\forall j, k$, *where* λ_j *and* λ_k *are constants.*

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