Completely positive multipliers of quantum groups

Matthew Daws

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Abstract

We show that any completely positive multiplier of the convolution algebra of the dual of an operator algebraic quantum group $\mathbb{G}$ (either a locally compact quantum group, or a quantum group coming from a modular or manageable multiplicative unitary) is induced in a canonical fashion by a coisometric corepresentation of $\mathbb{G}$. In the locally compact quantum group case the corepresentation we construct is always unitary, and it follows that there is an order bijection between the completely positive multipliers of $L^1(\mathbb{G})$ and the positive functionals on the universal quantum group $C_0^u(\mathbb{G})$. We provide a direct link between the Junge, Neufang, Ruan representation result and the representing element of a multiplier, and use this to show that their representation map is always weak*-weak*-continuous. We also show that for any $\mathbb{G}$, for a completely positive multiplier, our constructed corepresentation can be chosen to be unitary if and only if the left multiplier comes from a double multiplier.
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1 Introduction

Consider a locally compact group G, and the Fourier algebra $A(G)$, [7]. As the predual of the group von Neumann algebra $VN(G)$, we can equip $A(G)$ with its standard operator space structure. Then consider $M_{cb}A(G)$, the algebra of completely bounded multipliers of $A(G)$, that is, the collection of completely bounded maps $T : A(G) \to A(G)$ which satisfy $T(ab) = T(a)b$ for all $a, b \in A(G)$. A result of Gilbert (see the simple presentation of [12]) shows that T can be identified with the map given by pointwise multiplication by a continuous function f, which must have the special form that there is a Hilbert space H and continuous maps $\alpha, \beta : G \to H$ with $f(t^{-1}s) = (\alpha(s)|\beta(t))$ for $s, t \in G$. An easy way to construct such maps α, β is to start with a unitary representation π of G on H, to fix vectors $\alpha_0, \beta_0 \in H$, and to define $\alpha(s) = \pi(s)\alpha_0$ and $\beta(s) = \pi(s)\beta_0$ for all $s \in G$. However, we can integrate π to get a $*$-representation of $L^1(G)$ and hence a $*$-representation of $C^*(G)$, and it hence follows that f is identified with the functional $\omega_{\alpha_0,\beta_0} \circ \pi$ on $C^*(G)$, that is, with a member of the Fourier-Stieltjes algebra $B(G)$. As $M_{cb}A(G) = B(G)$ if and only if G is amenable (this is shown in unpublished work of Losert and Ruan; compare [19] and [2]) we see that not every member of $M_{cb}A(G)$ can arise in this way.

However, suppose that our multiplier T is completely positive. It follows from Jolissaint's proof in [12] that we map choose $\alpha = \beta$ in this case, and so the function f is readily seen to be positive definite. Hence in this case, f is a member of $B(G)$ (and so we also see that $M_{cb}A(G)$ is the span of the completely positive multipliers if and only if G is amenable). Indeed, we can assume that α

is non-degenerate in the sense that H is the closed span of the image of α. Then we can construct a unitary representation π of G on H by setting $\pi(s)\alpha(t) = \alpha(st)$ for all $s, t \in G$, and α can be recovered as $\alpha(s) = \pi(s)\alpha(e)$, if $e \in G$ is the unit.

In this paper, we extend this result on completely positive multipliers to an arbitrary locally compact quantum group $\mathbb{G}$ (see Section 2 below for notation). As $L^1(\mathbb{G})$ is in general not commutative, we work with left multipliers (although analogous results hold on the right, by working with the opposite quantum group, or by directly using the unitary antipode). A unitary corepresentation $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H))$ gives rise to a completely bounded left multiplier of the dual $L^1(\hat{\mathbb{G}})$ by defining

$$L : L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}}); \quad \hat{x} \mapsto (\iota \otimes \omega_{\alpha,\alpha})(U(\hat{x} \otimes 1)U^*),$$

where $\alpha \in H$ is some vector. Then L is normal, and its preadjoint L_* is a left multiplier in the sense that $L_*(\hat{\omega}_1\hat{\omega}_2) = L_*(\hat{\omega}_1)\hat{\omega}_2$ for $\hat{\omega}_1, \hat{\omega}_2 \in L^1(\hat{\mathbb{G}})$. Alternatively, we can identify $L^1(\hat{\mathbb{G}})$ as an ideal in the dual of the universal quantum group $C_0^u(\hat{\mathbb{G}})$, and then L_* is given by left multiplication by the positive functional $\omega_{\alpha,\alpha} \circ \pi$, where $\pi : C_0^u(\hat{\mathbb{G}}) \to \mathcal{B}(H)$ is the unique $*$-representation corresponding to U. See Section 4 for further details.

Our main result, Theorem 5.10, is that any completely *positive* left multiplier of $L^1(\hat{\mathbb{G}})$ arises in this way from a unitary corepresentation, or equivalently from a positive functional on $C_0^u(\hat{\mathbb{G}})$. Indeed, Theorem 5.11 establishes an order bijection between the completely positive multipliers and $C_0^u(\hat{\mathbb{G}})^*_+$.

Our main technical tool is that "invariants are constant"; that is, if $x \in L^\infty(\mathbb{G})$ with $\Delta(x) = y \otimes 1$ (or $1 \otimes y$) then $x = y \in \mathbb{C}1$. We learnt the power of this simple condition from [20] which works in the more general setting of quantum groups given by manageable or modular multiplicative unitaries. Actually, most of our results hold in this more general setting with no further work– but we are only able to verify that the resulting corepresentation is coisometric, not unitary.

The outline of the paper is as follows. Section 2 is a very quick introduction (mainly to fix notation) to locally compact quantum groups, or quantum groups given by manageable multiplicative unitaries. Using the "invariants are constant" idea, together with some basic modular theory for weights, we show that for a locally compact quantum group $\mathbb{G}$, the algebra $L^1(\mathbb{G})$, treated as a completely contractive Banach algebra, is *self-induced*. This is a weakening of having a bounded approximate identity (see [5, Section 5] and [8, 9]) and it is interesting that this holds for all $\mathbb{G}$, without any coamenability assumption.

In Section 3 we review some of the constructions of Junge, Neufang and Ruan in [13]– we use these to construct corepresentations, and so we take the opportunity to show that the results (or at least, the ones we need) hold for general quantum groups. In Section 4 we show how corepresentations give rise to multipliers.

In Section 5 we start our programme of showing that every completely positive multiplier L arises from a corepresentation. The ideas of [13] allow us to extend L to a map Φ on all of $\mathcal{B}(L^2(\mathbb{G}))$. We start with a careful analysis of what the Stinespring representation for Φ gives us. From this, we can construct a coisometric corepresentation U which gives L in the sense of Section 4; the underlying idea, slightly hidden by the technical details, is a direct generalisation of the argument in the Fourier algebra case: see the comment after Proposition 5.3. When $\mathbb{G}$ is a locally compact quantum group, we can show that our corepresentation is unitary. We present a short proof which ultimately uses the modular theory of $L^\infty(\mathbb{G})$, and a second proof which is more corepresentation theoretic– it seems possible that this second proof could be extended to $\mathbb{G}$ coming from manageable multiplicative unitaries, but we don't see this at present.

In Section 6.1, we work again in the completely bounded case. Any completely bounded left multiplier L has a "representing element" $a_0 \in M(C_0(\mathbb{G}))$ so that, under the embedding $\hat{\lambda} : L^1(\hat{\mathbb{G}}) \to C_0(\mathbb{G})$, left multiplication by a_0 induces L. We show that a_0 determines the map Φ by

$$\big(\Phi(\theta_{\xi,\eta})\alpha\big|\beta\big) = \langle \Delta(a_0), \omega_{\alpha,\eta} \otimes \omega_{\xi,\beta}^{\sharp*}\rangle \qquad (\xi, \eta \in D(P^{1/2}), \alpha, \beta \in D(P^{-1/2})).$$

Here P is the positive operator which induces the scaling groups (τ_t) and $(\hat{\tau}_t)$. We remark that this presents a possible definition for a "positive definite function" in $M(C_0(\mathbb{G}))$, namely those a_0 such that Φ becomes completely positive– however, it is both unclear when this occurs, and whether Φ would then automatically be associated with a multiplier (and hence, by our work, a unitary corepresentation) or whether this would need to be made a further hypothesis on a_0.

In the remainder of Section 6.1, we show that for any $\mathbb{G}$, the corepresentation U associated with L is unitary if and only if L is the left part of a completely bounded double multiplier (which is automatically completely positive, if L is). Then in the final short section we apply this link between Φ and a_0 to remove certain technical "approximation property" type assumptions on the results of [11, Section 4], to show that the representation map in [13] is always weak*-weak*-continuous.

A final word on notation. Our Hilbert space inner products shall be linear in the first variable, and we write $(\cdot|\cdot)$ for an inner product (or more generally, a sesquilinear form). We write $\langle \cdot, \cdot \rangle$ for the bilinear pairing between a Banach space and its dual. For a Hilbert space H, we write $\mathcal{B}(H)$ for the algebra of all bounded operators on H, write $\mathcal{B}(H)_*$ for its predual (the trace class operators) and write $\mathcal{B}_0(H)$ for the ideal of compact operators. For $\xi, \eta \in H$, we denote by $\omega_{\xi,\eta}$ the normal functional in $\mathcal{B}(H)_*$, and by $\theta_{\xi,\eta}$ the rank one operator in $\mathcal{B}_0(H)$, which are defined by

$$\langle T, \omega_{\xi,\eta}\rangle = (T\xi|\eta), \qquad \theta_{\xi,\eta}(\gamma) = (\gamma|\eta)\xi \qquad (T \in \mathcal{B}(H), \gamma \in H).$$

Given a normal map T on a von Neumann algebra M, we write T_* for the pre-adjoint of T acting on the predual M_*. We write $\otimes$ to mean a completed tensor product, either of Hilbert spaces, or the minimal C*-algebraic tensor product. We write $\overline{\otimes}$ for the von Neumann algebraic tensor product, and $\odot$ for the purely algebraic tensor product. We write Σ for the tensor swap map of Hilbert spaces, say $\Sigma : H \otimes H \to H \otimes H; \xi \otimes \eta \mapsto \eta \otimes \xi$. To avoid confusion, we write σ for the tensor swap map of C* (or von Neumann) algebras.

We use the basic theory of Operator Spaces without comment; see, for example, [6] for further details.

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2 Operator algebraic quantum groups

In this paper, we shall be concerned with quantum groups in the operator algebraic setting– to be precise, either locally compact quantum groups, in the Vaes, Kustermans sense [16, 17, 18, 25], or C*-algebraic quantum groups built from manageable or modular multiplicative unitaries, in the Sołtan, Woronowicz sense [21, 22, 27] (the latter generalising the former). In fact, for many of our results, we shall need remarkably little– our main tool being that "invariants are constant" (see below– our inspiration here is [20, Section 2]).

A *locally compact quantum group* in the von Neumann algebraic setting is a Hopf-von Neumann algebra (M,Δ) equipped with left and right invariant weights. As usual, we use Δ to turn M_* into a Banach algebra, and we write the product by juxtaposition. We shall "work on the left"; so using the left invariant weight, we build the GNS space H, and a multiplicative unitary W acting on $L^2(\mathbb{G})\otimes L^2(\mathbb{G})$ (of course, the existence of a right weight is needed to show that W is unitary). There is a (in general unbounded) antipode S which admits a "polar decomposition" $S=R\tau_{-i/2}$, where R is the unitary antipode, and (τ_t) is the scaling group. There is a nonsingular positive operator P which implements (τ_t) as $\tau_t(x)=P^{it}xP^{-it}$. Then W is *manageable* with respect to P.

A manageable multiplicative unitary W acting on $H\otimes H$ has, by definition, a nonsingular positive operator P, and an operator $\tilde{W}$ acting on $H\otimes\overline{H}$ such that

$$\big(W(\xi\otimes\alpha)\big|\eta\otimes\beta\big)=\big(\tilde{W}(P^{-1/2}\xi\otimes\overline{\beta})\big|P^{1/2}\eta\otimes\overline{\alpha}\big),$$

for all $\alpha,\beta\in H$ and $\xi\in D(P^{-1/2}),\eta\in D(P^{1/2})$. A word on notation: we work with left multiplicative unitaries, whereas Sołtan and Woronowicz, in the conventions of [18], work with right multiplicative unitaries, and so we have translated everything to the left.

Given such a W, the space $\{(\iota\otimes\omega)W:\omega\in\mathcal{B}(H)_*\}$ is an algebra, and its closure is a C*-algebra, say A. There is a coassociative map $\Delta:A\to M(A\otimes A)$ given by $\Delta(a)=W^*(1\otimes a)W$. If we formed W from (M,Δ) with invariant weights, then A is σ-weakly dense in M, and the two definitions of Δ agree. Similarly, $\{(\omega\otimes\iota)W:\omega\in\mathcal{B}(H)_*\}$ is norm dense in a C*-algebra $\hat{A}$, and defining $\hat{\Delta}(\hat{a})=\hat{W}^*(1\otimes\hat{a})\hat{W}$, we get a non-degenerate $*$-homomorphism $\hat{\Delta}:\hat{A}\to M(\hat{A}\otimes\hat{A})$, where here $\hat{W}=\Sigma W^*\Sigma$. If we started with (M,Δ) having invariant weights, then we can construct invariant weights on $(\hat{A}'',\hat{\Delta})$. The unitary W is in the multiplier algebra $M(A\otimes\hat{A})\subseteq\mathcal{B}(H\otimes H)$.

When W is a manageable multiplicative unitary, we can still form S,R and (τ_t) with the usual properties. The antipode S has elements of the form $(\iota\otimes\omega)W$ as a core, and $S((\iota\otimes\omega)W)=(\iota\otimes\omega)(W^*)$. Then also $S=R\tau_{-i/2}$ and (τ_t) is again implemented by P (the same map which appears in the definition of "manageable"). Thus we recover most of the objects associated with a locally compact quantum group. The exception is that we no longer have access to the Tomita-Takesaki theory of weights, and so, for example, have little control over the commutant M', and so forth.

There is a more general notation of a *modular multiplicative unitary*, see [22], which is more natural in certain examples. However, at the cost of changing our space H, we can recover (A,Δ) from a different, but related, manageable multiplicative unitary. Indeed, in [21], it is shown that if (A,Δ) is given by some modular multiplicative unitary, then $(\hat{A},\hat{\Delta})$, the σ-weak topologies on A and $\hat{A}$, the image of W in $M(A\otimes\hat{A})$, and all the maps $S,R,(\tau_t),\hat{S},\hat{R}$ and $(\hat{\tau}_t)$, are independent of the particular choice of modular multiplicative unitary giving (A,Δ). For this reason, we shall henceforth work only with manageable multiplicative unitaries (but all our results hold in the modular case as well).

We write $\mathbb{G}$ for an abstract object to be thought of as a quantum group. We write $C_0(\mathbb{G}),L^\infty(\mathbb{G})$ and $L^1(\mathbb{G})$ for A,M and M_* (and similar for the dual objects); as mentioned in the previous paragraph, these are well-defined. We also write $L^2(\mathbb{G})$ for H, but be aware that if $\mathbb{G}$ is given by a modular or manageable multiplicative unitary, then there is some arbitrary choice involved in $L^2(\mathbb{G})$. If $\mathbb{G}$ has invariant weights, then these weights unique up to a constant, and so $L^2(\mathbb{G})$ is unique.

This concludes our brief summary; we shall develop further theory as and when we need it. We finish this section with one of our major tools– that "invariants are constant". Notice that, by using the unitary antipode, we could replace y_{13} by y_{23} in the following; but we shall have no need of this variant.

Theorem 2.1. *For any $\mathbb{G}$ and a von Neumann algebra N, if $x, y \in L^\infty(\mathbb{G})\overline{\otimes}N$ satisfy $(\Delta\otimes\iota)x = y_{13}$, then $x = y \in \mathbb{C}\overline{\otimes}N$.*

Proof. We shall prove this when $N = \mathbb{C}$, the general case comes from considering $(\iota\otimes\omega)x$ and $(\iota\otimes\omega)y$, as $\omega \in N_*$ varies. For locally compact quantum groups, this was shown in [1, Lemma 4.6], compare also [17, Result 5.13]. For general $\mathbb{G}$, [20, Theorem 2.6] shows that if $a, b \in \mathcal{B}(L^2(\mathbb{G}))$ with $W^*(1\otimes a)W = b\otimes 1$ (working with left multiplicative unitaries) then $a = b \in \mathbb{C}1$, and this immediately implies the result. □

2.1 $L^1(\mathbb{G})$ is self-induced

A *completely contractive Banach algebra* is a Banach algebra A which is also an operator space, and such that the multiplication map $m : A\widehat{\otimes}A \to A$ is a complete contraction (here using the operator space projective tensor product), see [6]. Let X be the closure of $\{ab\otimes c - a\otimes bc : a, b, c \in A\}$ in $A\widehat{\otimes}A$, and let $A\widehat{\otimes}_A A$ be the quotient $A\widehat{\otimes}A/X$. Clearly X is contained in the kernel of m, and m induces a complete contraction $\tilde{m} : A\widehat{\otimes}_A A \to A$. When this is an isomorphism, we say that A is *self-induced*. We studied this concept in [5, Section 5]; our inspiration was papers of Gronbaek, [8, 9].

If A has a bounded approximate identity, it is not hard to see that A is self-induced; so this shows that $L^1(G)$ is self-induced for any locally compact group G. Using work of Tomita and Takesaki, we showed in [5, Theorem 6.5] that $A(G)$ is self-induced for any G. In fact, using that "invariants are constant", we can give a simple proof that $L^1(\mathbb{G})$ is self-induced for any $\mathbb{G}$.

In the following, we need to use invariant weights to get access to modular theory. We just quickly recall that given the modular conjugation J, we have that $JL^\infty(\mathbb{G})J = L^\infty(\mathbb{G})'$, the commutant. There is also a useful link with the quantum group structure of the dual, as the unitary antipode is given by $\hat{R}(\hat{x}) = J\hat{x}^*J$ for $\hat{x} \in L^\infty(\hat{\mathbb{G}})$.

Theorem 2.2. *For a locally compact quantum group $\mathbb{G}$, the algebra $L^1(\mathbb{G})$ is self-induced, as a completely contractive Banach algebra (and in fact $\tilde{m}$ becomes a completely isometric isomorphism).*

Proof. We can identify the dual of $A\widehat{\otimes}_A A$ with

$$X^\perp = \{\tau \in (A\widehat{\otimes}A)^* : \langle \tau, ab\otimes c\rangle = \langle \tau, a\otimes bc\rangle \ (a, b, c \in A)\}.$$

Thus $\tilde{m}$ is a (completely isometric) isomorphism if and only if $m^* : A^* \to X^\perp$ is a (completely isometric) isomorphism.

For us, $A = L^1(\mathbb{G})$ and so $(A\widehat{\otimes}A)^* = L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$. As m is the pre-adjoint of Δ, we need to prove that $\Delta : L^\infty(\mathbb{G}) \to X^\perp$ is an isomorphism (notice that it is automatically completely isometric onto its range). Similarly, we can identify $X^\perp$ as

$$X^\perp = \{x \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G}) : (\Delta\otimes\iota)x = (\iota\otimes\Delta)x\}.$$

So, let $x \in X^\perp$, and set $y = WxW^* \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(L^2(\mathbb{G}))$. Then, using that $(\Delta\otimes\iota)W = W_{13}W_{23}$, we have that

$$(\Delta\otimes\iota)y = W_{13}W_{23}((\Delta\otimes\iota)x)W_{23}^*W_{13}^* = W_{13}W_{23}((\iota\otimes\Delta)x)W_{23}^*W_{13}^*.$$

Now, $(\iota\otimes\Delta)x = W_{23}^*x_{13}W_{23}$, and so $(\Delta\otimes\iota)y = W_{13}x_{13}W_{13}$. By Theorem 2.1, there is $z \in \mathcal{B}(L^2(\mathbb{G}))$ with $y = 1\otimes z$, that is, $x = W^*(1\otimes z)W$.

To finish the proof, we need to show that $z \in L^\infty(\mathbb{G})$, which seems to require using modular theory. Let $W' = (J \otimes J)W(J \otimes J)$ (this is the fundamental unitary of the commutant quantum group, see [18, Section 4]). Then for $\hat{x} \in L^\infty(\hat{\mathbb{G}})$,

$$\begin{aligned} W'(\hat{x} \otimes 1)W'^* &= \Sigma(J \otimes J)\hat{W}^*(1 \otimes J\hat{x}J)\hat{W}(J \otimes J)\Sigma = \Sigma(J \otimes J)\hat{\Delta}(R(\hat{x}))^*(J \otimes J)\Sigma \\ &= \sigma(R \otimes R)\hat{\Delta}(R(\hat{x})) = \hat{\Delta}(\hat{x}). \end{aligned}$$

It hence follows that

$$W'_{23}W_{12} = (\iota \otimes \hat{\Delta})(W)W'_{23} = W_{13}W_{12}W'_{23}.$$

As $W \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\hat{\mathbb{G}})$, it follows that $W' \in L^\infty(\mathbb{G})'\overline{\otimes}L^\infty(\hat{\mathbb{G}})$. So

$$\begin{aligned} (W'_{23})^* x_{12} &= (W'_{23})^* W_{12}^*(1 \otimes z \otimes 1)W_{12} = W_{12}^*(W'_{23})^* W_{13}(1 \otimes z \otimes 1)W_{12} \\ &= x_{12}(W'_{23})^* = W_{12}^*(1 \otimes z \otimes 1)(W'_{23})^* W_{13}W_{12}. \end{aligned}$$

As W is unitary, we can cancel to get

$$(1 \otimes z \otimes 1)(W'_{23})^* W_{13} = (W'_{23})^* W_{13}(1 \otimes z \otimes 1) = (W'_{23})^*(1 \otimes z \otimes 1)W_{13}.$$

We conclude that $(z \otimes 1)W' = W'(z \otimes 1)$. As $\{(\iota \otimes \hat{\omega})W' : \hat{\omega} \in L^1(\hat{\mathbb{G}})\}$ is σ-weakly dense in $JL^\infty(\mathbb{G})J = L^\infty(\mathbb{G})'$, it follows that $z \in L^\infty(\mathbb{G})'' = L^\infty(\mathbb{G})$ as required. □

3 Multipliers of quantum groups

In this section, we review some of the ideas used by Junge, Neufang and Ruan in [13]. We shall actually need some constructions coming from the proofs in [13] (and not just the statements of the results). Rather than just give sketch proofs, we instead give quick, full proofs, and take the opportunity to show that some of their results also hold for quantum groups coming from manageable multiplicative unitaries. Further details and related ideas can be found in [4, 5, 11, 13].

Definition 3.1. A completely bounded left multiplier of $L^1(\mathbb{G})$ is a completely bounded map $L_* : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ with $L_*(\omega_1\omega_2) = L_*(\omega_1)\omega_2$ for $\omega_1, \omega_2 \in L^1(\mathbb{G})$.

Such maps are also often called "centralisers" in the literature (and in particular, in [13]). A simple calculation shows that a completely bounded map $L_* : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ is a left multiplier if and only if its adjoint $L = (L_*)^*$ satisfies $(L \otimes \iota)\Delta = \Delta L$.

Let us make a few remarks about normal completely bounded maps. As explained, for example, in the proof of [10, Theorem 2.5], as L is normal, we can find a normal $*$-representation $\pi : L^\infty(\mathbb{G}) \to \mathcal{B}(H)$ for some Hilbert space H, and bounded maps $P, Q : L^2(\mathbb{G}) \to H$, with $L(x) = P^*\pi(x)Q$ for each $x \in L^\infty(\mathbb{G})$. By the structure theory for normal $*$-representations (see [24, Theorem 5.5, Chapter IV]) by adjusting P and Q, and may suppose that $H = L^2(\mathbb{G}) \otimes H'$ for some Hilbert space H', and that $\pi(x) = x \otimes 1$. For example, it then follows that

$$(L \otimes \iota)(\hat{W}) = (P^* \otimes 1)\hat{W}_{13}(Q \otimes 1).$$

As $\hat{W} \in M(\mathcal{B}_0(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}))$, it follows easily from this that also $(L \otimes \iota)(\hat{W}) \in M(\mathcal{B}_0(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}))$, a fact we shall use in the following proof.

The following is a short unification of (the left version of) [13, Corollary 4.4] (compare [13, Theorem 4.10]) and [4, Theorem 4.2]; we make use of the "invariants are constant" technique.

Proposition 3.2. *Let L_* be a completely bounded left multiplier of $L^1(\hat{\mathbb{G}})$. There is $a \in M(C_0(\mathbb{G}))$ with $a\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(L_*(\hat{\omega}))$ for $\hat{\omega} \in L^1(\mathbb{G})$, or equivalently, with $(1 \otimes a)\hat{W} = (L \otimes \iota)(\hat{W})$.*

Proof. That $a\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(L_*(\hat{\omega}))$ for each $\hat{\omega} \in L^1(\mathbb{G})$, if and only if $(1 \otimes a)\hat{W} = (L \otimes \iota)(\hat{W})$ follows easily from the definition that $\hat{\lambda}(\hat{\omega}) = (\hat{\omega} \otimes \iota)(\hat{W})$. Consider now

$$\begin{aligned}(\hat{\Delta} \otimes \iota)\big((L \otimes \iota)(\hat{W})\hat{W}^*\big) &= \big((L \otimes \iota \otimes \iota)(\hat{\Delta} \otimes \iota)\hat{W}\big)(\hat{\Delta} \otimes \iota)(\hat{W}^*) \\ &= \big((L \otimes \iota \otimes \iota)(\hat{W}_{13}\hat{W}_{23})\big)\hat{W}_{23}^*\hat{W}_{13}^* \\ &= (L \otimes \iota)(\hat{W})_{13}\hat{W}_{13}^*,\end{aligned}$$

where we have used that $\hat{\Delta}L = (L \otimes \iota)\hat{\Delta}$, that $\hat{\Delta}$ is a $*$-homomorphism, and that $(\hat{\Delta} \otimes \iota)(\hat{W}) = \hat{W}_{13}\hat{W}_{23}$. By Theorem 2.1, it follows that there is $a \in L^\infty(\mathbb{G})$ with $(L\otimes\iota)(\hat{W})\hat{W}^* = 1\otimes a$. However, as $\hat{W} \in M(\mathcal{B}_0(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}))$, we see that

$$1 \otimes a = (L \otimes \iota)(\hat{W})\hat{W}^* \in M(\mathcal{B}_0(L^2(\mathbb{G})) \otimes C_0(\mathbb{G})),$$

from which it follows immediately that $a \in M(C_0(\mathbb{G}))$. □

In the language of [4], the previous lemma says that L_* is "represented"; in the language of [13], the element a is the "multiplier" associated to the "centraliser" L_*.

Proposition 3.3. *Let L_* be a completely bounded left multiplier of $L^1(\hat{\mathbb{G}})$. There is a completely bounded, normal map $\Phi : \mathcal{B}(L^2(\mathbb{G})) \to \mathcal{B}(L^2(\mathbb{G}))$ which extends L, and which is a $L^\infty(\mathbb{G})'$-bimodule map. Indeed, Φ satisfies*

$$1 \otimes \Phi(x) = \hat{W}\big((L \otimes \iota)(\hat{W}^*(1 \otimes x)\hat{W})\big)\hat{W}^* \qquad (x \in \mathcal{B}(L^2(\mathbb{G}))).$$

Proof. We closely follow [13, Proposition 4.3], while translating "to the left" and using that "invariants are constant". Define

$$T : \mathcal{B}(L^2(\mathbb{G})) \to L^\infty(\hat{\mathbb{G}})\overline{\otimes}\mathcal{B}(L^2(\mathbb{G})), \quad T(x) = \hat{W}\big((L \otimes \iota)(\hat{W}^*(1 \otimes x)\hat{W})\big)\hat{W}^*.$$

We now perform a similar calculation to that in the previous lemma:

$$\begin{aligned}(\hat{\Delta} \otimes \iota)T(x) &= \hat{W}_{13}\hat{W}_{23}(L \otimes \iota \otimes \iota)\big((\hat{\Delta} \otimes \iota)(\hat{W}^*(1 \otimes x)\hat{W})\big)\hat{W}_{23}^*\hat{W}_{13}^* \\ &= \hat{W}_{13}\hat{W}_{23}(L \otimes \iota \otimes \iota)\big(\hat{W}_{23}^*\hat{W}_{13}^*(1 \otimes 1 \otimes x)\hat{W}_{13}\hat{W}_{23}\big)\hat{W}_{23}^*\hat{W}_{13}^* \\ &= \hat{W}_{13}(L \otimes \iota \otimes \iota)\big(\hat{W}_{13}^*(1 \otimes 1 \otimes x)\hat{W}_{13}\big)\hat{W}_{13}^* \\ &= T(x)_{13}.\end{aligned}$$

So by Theorem 2.1, there is $\Phi(x) \in \mathcal{B}(L^2(\mathbb{G}))$ with $T(x) = 1 \otimes \Phi(x)$. It is easy to see that Φ is completely bounded and normal.

For $x \in L^\infty(\hat{G})$

$$1 \otimes \Phi(x) = T(x) = \hat{W}((L \otimes \iota)\hat{\Delta}(x))\hat{W}^* = \hat{W}\hat{\Delta}(L(x))\hat{W}^* = 1 \otimes L(x),$$

and so Φ extends L. For $y, z \in L^\infty(\mathbb{G})'$, as $\hat{W} \in L^\infty(\hat{\mathbb{G}})\overline{\otimes}L^\infty(\mathbb{G})$, it is easy to see that

$$T(yxz) = (1 \otimes y)T(x)(1 \otimes z) \qquad (x \in \mathcal{B}(L^2(\mathbb{G}))),$$

and so $\Phi(yxz) = y\Phi(x)z$ as required. □

In the language of [13], we have thus constructed a map from the set of completely bounded left multipliers of $L^1(\hat{\mathbb{G}}))$ to $\mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$. It seems that, to continue with the arguments of [13], we start to need to use arguments that involve the relative position of $L^\infty(\mathbb{G})$ and its commutant in $\mathcal{B}(L^2(\mathbb{G}))$. In particular, to show that every $\Phi \in \mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$ comes from a left multiplier would require us to know that $L^\infty(\mathbb{G}) \cap L^\infty(\hat{\mathbb{G}}) = \mathbb{C}$ (at least if one is following the proof of [13, Proposition 3.2]), and we have no proof of this in the Manageable Multiplicative Unitary setting.

4 Multipliers coming from invertible corepresentations

A *corepresentation* of $\mathbb{G}$ shall be, for us, an element $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ with $(\Delta\otimes\iota)(U) = U_{13}U_{23}$ (so, we don't assume that U is unitary). We state the following in a little generality, but note that it obviously applies to unitary corepresentations. Similar ideas are explored in [4, Section 6].

Proposition 4.1. *Let U be a corepresentation of $\mathbb{G}$, and suppose there is $V \in \mathcal{B}(L^2(\mathbb{G}) \otimes H)$ with $VU^* = 1$ (that is, U has a right inverse). For each $\alpha, \beta \in H$, there is a completely bounded left multiplier of $L^1(\hat{\mathbb{G}})$ represented by $a = (\iota \otimes \omega_{\alpha,\beta})(U^*)$. If U^* is an isometry (so we may take $V = U$) and $\alpha = \beta$, then the multiplier is completely positive.*

Proof. We have that $(\Delta\otimes\iota)(U^*) = U^*_{23}U^*_{13}$, or equivalently, $W^*_{12}U^*_{23} = U^*_{23}U^*_{13}W^*_{12}$. Thus also $V_{23}W^*_{12}U^*_{23} = U^*_{13}W^*_{12}$, and using that $\hat{W} = \Sigma W^*\Sigma$, it follows that $V_{13}\hat{W}_{12}U^*_{13} = U^*_{23}\hat{W}_{12}$. Thus define $L : L^\infty(\hat{\mathbb{G}}) \to \mathcal{B}(L^2(\mathbb{G}))$ by

$$L(\hat{x}) = (\iota \otimes \omega_{\alpha,\beta})(V(\hat{x} \otimes 1)U^*) \qquad (\hat{x} \in L^\infty(\hat{\mathbb{G}})).$$

Clearly L is a normal, completely bounded map. Then immediately we see that $(L \otimes \iota)\hat{W} = (1 \otimes a)\hat{W}$, and it is now easy to see (compare [4, Proposition 2.3]) that L maps into $L^\infty(\hat{\mathbb{G}})$, and that L is the adjoint of left multiplier on $L^1(\hat{\mathbb{G}})$, represented by a.

When U^* is an isometry, $V = U$ and $\alpha = \beta$, clearly L is completely positive. ☐

For U, V as in the proposition, we could weaken the condition on U to asking that $U \in \mathcal{B}(L^2(\mathbb{G}) \otimes H)$ with $W^*_{12}U_{23}W_{12} = U_{13}U_{23}$. Then, arguing as in [27, Page 142], we see that $U_{13} = W^*_{12}U_{23}W_{12}V^*_{23} \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(L^2(\mathbb{G})) \otimes \mathcal{B}_0(H))$, and so $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H))$, in particular, U is a corepresentation in our sense.

Let us just remark that if also V is a corepresentation, then consider forming Φ as in Section 3, using the L given as in the proposition. So, for $x \in \mathcal{B}(L^2(\mathbb{G}))$,

$$\begin{aligned} 1 \otimes \Phi(x) &= (\iota \otimes \omega_{\alpha,\beta} \otimes \iota)\big(\hat{W}_{13}V_{12}\hat{W}^*_{13}(1 \otimes 1 \otimes x)\hat{W}_{13}U^*_{12}\hat{W}^*_{13}\big) \\ &= (\iota \otimes \iota \otimes \omega_{\alpha,\beta})\big(\hat{W}_{12}V_{13}\hat{W}^*_{12}(1 \otimes x \otimes 1)\hat{W}_{12}U^*_{13}\hat{W}^*_{12}\big). \end{aligned}$$

Now, $\hat{W}(a \otimes 1)\hat{W}^* = \Sigma W^*(1 \otimes a)W\Sigma = \Sigma\Delta(a)\Sigma$ for $a \in L^\infty(\mathbb{G})$, and so

$$1 \otimes \Phi(x) = (\iota \otimes \iota \otimes \omega_{\alpha,\beta})\big(V_{23}V_{13}(1 \otimes x \otimes 1)U^*_{13}U^*_{23}\big) = 1 \otimes (\iota \otimes \omega_{\alpha,\beta})\big(V(x \otimes 1)U^*\big).$$

Hence Φ has the same "defining formula" as L.

4.1 Links with universal quantum groups

Universal quantum groups are constructed in [15] and [22, Section 5]. We write $C_0^u(\hat{\mathbb{G}})$ for the universal dual of $C_0(\mathbb{G})$. For us, the important properties are that:

- There is a coassociative non-degenerate $*$-homomorphism $\hat{\Delta}_u : C_0^u(\hat{\mathbb{G}}) \to M(C_0^u(\hat{\mathbb{G}}) \otimes C_0^u(\hat{\mathbb{G}}))$;
- There is a surjective $*$-homomorphism $\hat{\pi}_u : C_0^u(\hat{\mathbb{G}}) \to C_0(\hat{G})$ with $\hat{\Delta}\hat{\pi}_u = (\hat{\pi}_u \otimes \hat{\pi}_u)\hat{\Delta}_u$;
- There is a unitary corepresentation $\mathcal{W} \in M(C_0(\mathbb{G}) \otimes C_0^u(\hat{\mathbb{G}}))$ of $C_0(\mathbb{G})$ such that $(\iota \otimes \hat{\pi}_u)\mathcal{W} = W$ and $(\iota \otimes \hat{\Delta}_u)\mathcal{W} = \mathcal{W}_{13}\mathcal{W}_{12}$.
- The space $\{(\omega \otimes \iota)\mathcal{W} : \omega \in L^1(\mathbb{G})\}$ is dense in $C_0^u(\hat{\mathbb{G}})$.
- There is a bijection between unitary corepresentations U of $C_0(\mathbb{G})$ and non-degenerate $*$-homomorphisms $\pi : C_0^u(\hat{\mathbb{G}}) \to \mathcal{B}(H)$ given by the relation that $U = (\iota \otimes \pi)\mathcal{W}$.

Note that our $\mathcal{W}$ is denoted by $\hat{\mathcal{V}}$ in the notation of [15]; and is the "left analogue" of $\mathbb{W}$ in the notation of [22].

The map $\hat{\pi}_u^* : L^1(\hat{\mathbb{G}}) \to C_0^u(\hat{\mathbb{G}})^*$ is an isometry and an algebra homomorphism. We know (see for example [5, Proposition 8.3]) that this identifies $L^1(\hat{\mathbb{G}})$ with an ideal in $C_0^u(\hat{\mathbb{G}})^*$, and hence that members of $C_0^u(\hat{\mathbb{G}})^*$ induce multipliers on $L^1(\hat{\mathbb{G}})$. Let us make links with Proposition 4.1.

Proposition 4.2. *Let U be a unitary corepresentation of $\mathbb{G}$ on H, and let $\alpha, \beta \in H$. Let π be the $*$-representation of $C_0^u(\hat{\mathbb{G}})$ on H associated with U. Then the multiplier represented by $(\iota \otimes \omega_{\alpha,\beta})(U^*)$ is given by left multiplication by $\mu = \omega_{\alpha,\beta} \circ \pi \in C_0^u(\hat{\mathbb{G}})^*$.*

Proof. Let $L : L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}})$ be the adjoint of the completely bounded left multiplier represented by $a = (\iota \otimes \omega_{\alpha,\beta})(U^*)$, as constructed in Proposition 4.1. Then $(L \otimes \iota)(\hat{W}) = (1 \otimes a)\hat{W}$, or equivalently, $(\iota \otimes L)(W^*) = (a \otimes 1)W^*$.

Define $L^\dagger(\hat{x}) = L(\hat{x}^*)^*$ for $\hat{x} \in L^\infty(\hat{\mathbb{G}})$, so $L^\dagger$ is a normal, completely bounded map on $L^\infty(\hat{\mathbb{G}})$. For any von Neumann algebra M and $X \in M \overline{\otimes} L^\infty(\hat{\mathbb{G}})$, we see that $(\iota \otimes L^\dagger)(X^*) = (\iota \otimes L)(X)^*$. In particular, it follows that $(L^\dagger \otimes \iota)\hat{\Delta} = \hat{\Delta}L^\dagger$, and so $L^\dagger$ is the adjoint of a completely bounded left multiplier of $L^1(\hat{\mathbb{G}})$, represented by b say. The proof of Proposition 4.1 shows that $b = (\iota \otimes \omega_{\beta,\alpha})(U^*)$.

Given $\hat{\omega} \in L^1(\hat{\mathbb{G}})$, we wish to show that $\mu\hat{\pi}_u^*(\hat{\omega}) = \hat{\pi}_u^*(L_*(\hat{\omega}))$. Let $\omega \in L^1(\mathbb{G})$ and set $x = (\omega \otimes \iota)\mathcal{W} \in C_0^u(\hat{\mathbb{G}})$. Then

$$\begin{aligned}\langle \mu\hat{\pi}_u^*(\hat{\omega}), x\rangle &= \langle \mu \otimes \hat{\pi}_u^*(\hat{\omega}), \hat{\Delta}_u((\omega \otimes \iota)\mathcal{W})\rangle = \langle \omega \otimes \mu \otimes \hat{\pi}_u^*(\hat{\omega}), \mathcal{W}_{13}\mathcal{W}_{12}\rangle \\ &= \langle \omega \otimes \omega_{\alpha,\beta} \otimes \hat{\omega}, (\iota \otimes \hat{\pi}_u)(\mathcal{W})_{13}(\iota \otimes \pi)(\mathcal{W})_{12}\rangle = \langle W_{13}U_{12}, \omega \otimes \omega_{\alpha,\beta} \otimes \hat{\omega}\rangle,\end{aligned}$$

and also

$$\langle \hat{\pi}_u^*(L_*(\hat{\omega})), x\rangle = \langle (\iota \otimes \hat{\pi}_u)\mathcal{W}, \omega \otimes L_*(\hat{\omega})\rangle = \langle W, \omega \otimes L_*(\hat{\omega})\rangle = \langle (\iota \otimes L)(W), \omega \otimes \hat{\omega}\rangle.$$

Now, $(\iota \otimes L)(W) = (\iota \otimes L^\dagger)(W^*)^* = ((b \otimes 1)W^*)^* = W(b^* \otimes 1)$, and so, using that $b^* = (\iota \otimes \omega_{\alpha,\beta})(U)$, we have that

$$\langle \hat{\pi}_u^*(L_*(\hat{\omega})), x\rangle = \langle W(b^* \otimes 1), \omega \otimes \hat{\omega}\rangle = \langle W_{13}U_{12}, \omega \otimes \omega_{\alpha,\beta} \otimes \hat{\omega}\rangle.$$

As such x are dense in $C_0^u(\hat{\mathbb{G}})$, the proof is complete. ☐

In particular, taking $U = \mathcal{W}$, we see that every positive functional on $C_0^u(\hat{\mathbb{G}})$ induces a completely positive left multiplier of $L^1(\hat{\mathbb{G}})$. We shall prove that the converse is also true, for any locally compact quantum group $\mathbb{G}$.

5 Completely positive multipliers

In this section, we study completely *positive* multipliers of $L^1(\hat{\mathbb{G}})$. Motivated by Proposition 3.3, we will first study completely positive normal maps on $\mathcal{B}(L^2(\mathbb{G}))$. As $\mathcal{B}_0(L^2(\mathbb{G}))$ is σ-weakly-dense in $\mathcal{B}(L^2(\mathbb{G}))$, it suffices to consider completely positive maps $\mathcal{B}_0(L^2(\mathbb{G})) \to \mathcal{B}(L^2(\mathbb{G}))$. The following adaptation of the Stinespring construction is surely folklore, but we give the details, as they are central to our argument. For $\xi, \eta \in L^2(\mathbb{G})$, let $\theta_{\xi,\eta} \in \mathcal{B}_0(L^2(\mathbb{G}))$ be the rank-one operator $\alpha \mapsto (\alpha|\eta)\xi$.

Let $\Phi : \mathcal{B}_0(L^2(\mathbb{G})) \to \mathcal{B}(L^2(\mathbb{G}))$ be a completely positive map, and let Φ_0 be the normal extension of Φ to $\mathcal{B}(L^2(\mathbb{G}))$. Let H be the completion of the algebraic tensor product $\overline{L^2(\mathbb{G})} \odot L^2(\mathbb{G})$ for the pre-inner-product

$$\big(\xi \otimes \alpha \big| \eta \otimes \beta\big)_H = \big(\Phi(\theta_{\eta,\xi})\alpha|\beta\big).$$

That this is a *positive* sesquilinear form follows from the fact that Φ is completely positive (compare with [24, Theorem 3.6]). We shall abuse notation, and continue to write $\xi \otimes \alpha$ for the equivalence class it defines in H. Let (e_i) be an orthonormal basis of $L^2(\mathbb{G})$, and define

$$V : L^2(\mathbb{G}) \to L^2(\mathbb{G}) \otimes H; \quad \alpha \mapsto \sum_i e_i \otimes (e_i \otimes \alpha).$$

This makes sense, as

$$\sum_i \|e_i \otimes \alpha\|_H^2 = \sum_i (\Phi(\theta_{e_i,e_i})\alpha|\alpha) = (\Phi_0(1)\alpha|\alpha).$$

Then, for $\alpha, \beta, \xi, \eta \in L^2(\mathbb{G})$,

$$\begin{aligned}(V^*(\theta_{\xi,\eta} \otimes 1)V\alpha|\beta) &= \sum_{i,j} (\theta_{\xi,\eta}(e_i) \otimes (e_i \otimes \alpha)|e_j \otimes (e_j \otimes \beta))_H \\ &= \sum_i (e_i|\eta)(\xi|e_j)(\Phi(\theta_{e_j,e_i})\alpha|\beta) = (\Phi(\theta_{\xi,\eta})\alpha|\beta).\end{aligned}$$

Thus we have a Stinespring dilation of Φ. Now let (f_i) be an orthonormal basis of H, and define a family (a_i) in $\mathcal{B}(L^2(\mathbb{G}))$ by setting

$$V(\alpha) = \sum_i a_i(\alpha) \otimes f_i \in L^2(\mathbb{G}) \otimes H \qquad (\alpha \in L^2(\mathbb{G})).$$

It follows that $\Phi_0(x) = \sum_i a_i^* x a_i$ for each $x \in \mathcal{B}(L^2(\mathbb{G}))$. Furthermore, for $\xi, \eta, \alpha, \beta \in L^2(\mathbb{G})$,

$$(\xi \otimes \alpha|\eta \otimes \beta)_H = \sum_i (a_i^* \theta_{\eta,\xi} a_i \alpha|\beta) = \sum_i (a_i\alpha|\xi)(\eta|a_i\beta),$$

and so $\xi \otimes \alpha = \sum_i (a_i\alpha|\xi) f_i$ in H.

Proposition 5.1. *Suppose further that M is a von Neumann algebra on $L^2(\mathbb{G})$, and that Φ is an M-bimodule map. Then $(x \otimes 1)V = Vx$ for each $x \in M$, and $a_i \in M'$ for each i.*

Proof. For $x \in M$ and $\xi, \eta, \alpha, \beta \in L^2(\mathbb{G})$,

$$(x^*(\xi) \otimes \alpha|\eta \otimes \beta)_H = (\Phi(\theta_{\eta,\xi}x)\alpha|\beta) = (\Phi(\theta_{\eta,\xi})x\alpha|\beta) = (\xi \otimes x(\alpha)|\eta \otimes \beta)_H.$$

Thus $x^*(\xi)\otimes\alpha = \xi\otimes x(\alpha)$ in H. It follows that

$$V(x(\alpha)) = \sum_i e_i\otimes(e_i\otimes x(\alpha)) = \sum_i e_i\otimes(x^*(e_i)\otimes\alpha) = \sum_{i,j} e_i\otimes((x^*(e_i)|e_j)e_j\otimes\alpha)$$
$$= \sum_{i,j}(x(e_j)|e_i)e_i\otimes(e_j\otimes\alpha) = \sum_j x(e_j)\otimes(e_j\otimes\alpha) = (x\otimes 1)V(\alpha),$$

remembering that H is the completion of $\overline{L^2(\mathbb{G})}\otimes L^2(\mathbb{G})$. It now follows that $xa_i = a_i x$ for each i, and so as $x\in M$ was arbitrary, $a_i\in M'$ for each i. ☐

The previous result (in the more general completely bounded setting) is well-known, see for example [23, Theorem 3.1] and unpublished work of Haagerup. However, the actual construction will be central to our arguments.

5.1 Constructing a corepresentation

Now let L be a completely positive left multiplier on $L^1(\hat{\mathbb{G}})$. Form $\Phi:\mathcal{B}(L^2(\mathbb{G}))\to\mathcal{B}(L^2(\mathbb{G}))$ using Proposition 3.3, and apply the construction of the previous section to find H and $V: L^2(\mathbb{G})\to L^2(\mathbb{G})\otimes H$. Fixing an orthonormal basis (f_i) for H, we find a_i such that $\Phi(x)=\sum_i a_i^* x a_i$ for each $x\in\mathcal{B}(L^2(\mathbb{G}))$. By Proposition 5.1, we see that $(x\otimes 1)V = Vx$ for each $x\in L^\infty(\mathbb{G})'$, equivalently, that $a_i\in L^\infty(\mathbb{G})$ for each i.

Proposition 5.2. *There is a unique isometry U^* on $L^2(\mathbb{G})\otimes H$ which satisfies*

$$U^*\big(\xi\otimes\sum_i (a_i\alpha|\eta)f_i\big) = \sum_i(\omega_{\alpha,\eta}\otimes\iota)\Delta(a_i)\xi\otimes f_i,$$

for all $\xi,\eta,\alpha\in L^2(\mathbb{G})$.

Proof. As $\xi\otimes\alpha = \sum_i(a_i\alpha|\xi)f_i$ in H, uniqueness of U^* follows by density. We know that $L(x) = \Phi_0(x) = \sum_i a_i^* x a_i$ for $x\in L^\infty(\hat{\mathbb{G}})$. We now use Proposition 3.3, which tells us that

$$1\otimes\Phi(x) = \sum_i \hat{W}(a_i^*\otimes 1)\hat{W}^*(1\otimes x)\hat{W}(a_i\otimes 1)\hat{W}^* \qquad (x\in\mathcal{B}_0(L^2(\mathbb{G}))).$$

For $\xi_1,\eta_1,\alpha_1,\xi_2,\eta_2,\alpha_2\in L^2(\mathbb{G})$, we have that

$$\big(\xi_1\otimes(\eta_1\otimes\alpha_1)\big|\xi_2\otimes(\eta_2\otimes\alpha_2)\big)_{L^2(\mathbb{G})\otimes H} = \big((1\otimes\Phi(\theta_{\eta_2,\eta_1}))\xi_1\otimes\alpha_1\big|\xi_2\otimes\alpha_2\big)$$
$$= \sum_i\big(\hat{W}(a_i^*\otimes 1)\hat{W}^*(1\otimes\theta_{\eta_2,\eta_1})\hat{W}(a_i\otimes 1)\hat{W}^*\xi_1\otimes\alpha_1\big|\xi_2\otimes\alpha_2\big)$$
$$= \sum_i\big((1\otimes\theta_{\eta_2,\eta_1})\Sigma\Delta(a_i)\Sigma\xi_1\otimes\alpha_1\big|\Sigma\Delta(a_i)\Sigma\xi_2\otimes\alpha_2\big)$$
$$= \sum_i\big((\theta_{\eta_2,\eta_1}\otimes 1)\Delta(a_i)\alpha_1\otimes\xi_1\big|\Delta(a_i)\alpha_2\otimes\xi_2\big)$$
$$= \sum_i\big((\omega_{\alpha_1,\eta_1}\otimes\iota)\Delta(a_i)\xi_1\big|(\omega_{\alpha_2,\eta_2}\otimes\iota)\Delta(a_i)\xi_2\big),$$

using that $\hat{W}(a\otimes 1)\hat{W}^* = \Sigma\Delta(a)\Sigma$ for $a\in L^\infty(\mathbb{G})$. It follows immediately that U^* exists and is an isometry. ☐

Proposition 5.3. *The operator U is a member of $L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$, and is a corepresentation, that is, $(\Delta\otimes\iota)U = U_{13}U_{23}$.*

Proof. Let $x \in L^\infty(\mathbb{G})'$, so for $\xi,\alpha,\beta \in L^2(\mathbb{G})$,

$$U^*(x\xi\otimes(\beta\otimes\alpha)) = \sum_i (\omega_{\alpha,\beta}\otimes\iota)\Delta(a_i)x\xi\otimes f_i = (x\otimes 1)U^*(\xi\otimes(\beta\otimes\alpha)).$$

Thus $U^* \in (L^\infty(\mathbb{G})'\overline{\otimes}\mathbb{C})' = L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$, and of course the same is true of U.

We shall prove that $(\Delta\otimes\iota)(U^*) = U_{23}^*U_{13}^*$. It is easy to see that this is equivalent to $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H); \omega \mapsto (\omega\otimes\iota)(U^*)$ being an anti-homomorphism of the Banach algebra $L^1(\mathbb{G})$. However, notice that for $\omega_1,\omega_2 \in L^1(\mathbb{G})$ and $\xi,\eta \in L^2(\mathbb{G})$,

$$\begin{aligned}\Big(\pi(\omega_{\xi,\eta})\sum_i\langle a_i,\omega_1\rangle f_i\Big|\sum_j\langle a_j,\omega_2\rangle f_j\Big) &= \Big(U^*\Big(\xi\otimes\sum_i\langle a_i,\omega_1\rangle f_i\Big)\Big|\eta\otimes\sum_j\langle a_j,\omega_2\rangle f_j\Big)\\ &= \sum_i\big((\omega_1\otimes\iota)\Delta(a_i)\xi\big|\eta\big)\overline{\langle a_i,\omega_2\rangle}\\ &= \Big(\sum_i\langle a_i,\omega_1\omega_{\xi,\eta}\rangle f_i\Big|\sum_j\langle a_j,\omega_2\rangle f_j\Big).\end{aligned}$$

Thus

$$\pi(\omega)\Big(\sum_i\langle a_i,\omega'\rangle f_i\Big) = \sum_i\langle a_i,\omega'\omega\rangle f_i \qquad (\omega,\omega' \in L^1(\mathbb{G})),$$

and it is now immediate that π is an anti-homomorphism. □

We remark that we can view H as being a completion of $L^1(\mathbb{G})$, where we identify $\omega \in L^1(\mathbb{G})$ with $\sum_i\langle a_i,\omega\rangle f_i \in H$. Then π in the above proof (that is, the anti-homomorphism from $L^1(\mathbb{G})$ to $\mathcal{B}(H)$ induced by U^*) is simply the map $\pi(\omega) : \omega' \mapsto \omega'\omega$. We can view this as an analogue of the construction of a representation of G in the commutative setting, as sketched in the introduction.

5.2 Characterising when U is unitary

We only see how to prove that U is unitary in the locally compact quantum group setting. Notice that U is unitary precisely when U^* is surjective. Recall the operator $V : L^2(\mathbb{G}) \to L^2(\mathbb{G})\otimes H$ which we used to construct the family (a_i) at the start of Section 5.

Lemma 5.4. *The closed image of U^* is equal to the closed linear span of $\{(\hat{a}\otimes 1)V(\xi) : \xi \in L^2(\mathbb{G}), \hat{a} \in C_0(\hat{\mathbb{G}})\}$. In particular, the image of U^* contains the image of V, and so $U^*UV = V$.*

Proof. Let $\xi_1,\xi_2,\eta \in L^2(\mathbb{G})$, and let $\sum_i \xi_i\otimes f_i \in L^2(\mathbb{G})\otimes H$, and observe that

$$\begin{aligned}\Big(U^*\big(\xi_1\otimes\sum_i(a_i\xi_2|\eta)f_i\big)\Big|\sum_j\xi_j\otimes f_j\Big) &= \sum_i\big((\omega_{\xi_2,\eta}\otimes\iota)\Delta(a_i)\xi_1\big|\xi_i\big)\\ &= \sum_i\big((1\otimes a_i)W(\xi_2\otimes\xi_1)\big|W(\eta\otimes\xi_i)\big).\end{aligned}$$

As W is unitary, we see that the image of U^* is the closed linear span of vectors of the form

$$\sum_i(\omega\otimes\iota)(W)^*a_i(\xi)\otimes f_i \qquad (\omega \in L^1(\mathbb{G}), \xi \in L^2(\mathbb{G})).$$

Now, $\{(\omega\otimes\iota)(W)^* : \omega \in L^1(\mathbb{G})\}$ is dense in $C_0(\hat{\mathbb{G}})$, and so the result follows, as $V(\xi) = \sum_i a_i(\xi)\otimes f_i$. As $C_0(\hat{\mathbb{G}})$ contains a bounded approximate identity, clearly the image of U^* contains the image of V. As U^*U is the orthogonal projection onto the image of U^* (as U^* is an isometry) it follows immediately that $U^*UV = V$. □

Remark 5.5. While we started with a completely positive left multiplier L, we immediately used Proposition 3.3 to extend L to a completely positive map Φ on all of $\mathcal{B}(L^2(\mathbb{G}))$. Remember that the representation $\Phi(x) = V^*(x\otimes 1)V$ is unique (up to unitary isomorphism), as this dilation is *minimal*. This is equivalent to the non-degeneracy condition that $\{(x\otimes 1)V\xi : x \in \mathcal{B}_0(L^2(\mathbb{G})), \xi \in L^2(\mathbb{G})\}$ is linearly dense in $L^2(\mathbb{G})\otimes H$.

As Φ extends L, we hence have a normal Stinespring representation of L, as $L(\hat{x}) = V^*(\hat{x}\otimes 1)V$. However, this might not be minimal; indeed, the previous lemma shows immediately (as $C_0(\hat{\mathbb{G}})$ is σ-weakly dense in $L^\infty(\hat{\mathbb{G}})$) that this representation will be minimal if and only if U is unitary.

However, we don't see a way to use this observation to prove that U is unitary, because constructing a Stinespring representation directly from L would give a general $*$-homomorphism $\pi : C_0(\hat{\mathbb{G}}) \to \mathcal{B}(K)$ for some K, and not a representation of the special form $\hat{x} \mapsto \hat{x}\otimes 1$ on $L^2(\mathbb{G})\otimes H$ for some H.

When $\mathbb{G}$ is a locally compact quantum group, we have a number of ways (all of which ultimately rely upon invariant weights) to show that U is unitary. Firstly, we give a proof which is similar in nature to arguments in [13]. This uses that $L^\infty(\hat{\mathbb{G}})L^\infty(\mathbb{G})' = \{\hat{x}x' : \hat{x} \in L^\infty(\hat{\mathbb{G}}), x' \in L^\infty(\mathbb{G})'\}$ is σ-weakly dense in $\mathcal{B}(L^2(\mathbb{G}))$. This is well-known to experts, but self-contained proofs can be hard to find, so we give a quick sketch. It is easy to adapt the nice presentation in the proof of [26, Proposition 5.13] to show that the norm closure of $X = \{\hat{a}JaJ : \hat{a} \in C_0(\hat{\mathbb{G}}), a \in C_0(\mathbb{G})\}$ is a C*-algebra acting non-degenerately on $L^2(\mathbb{G})$. Here J is the modular conjugation given by the left Haar weight on $L^\infty(\mathbb{G})$, but also J implements the unitary antipode on the dual by $\hat{R}(\hat{x}) = J\hat{x}^*J$ for $\hat{x} \in L^\infty(\hat{\mathbb{G}})$. The proof of [26, Proposition 5.13] uses the Pentagonal equation for W, together with the fact that $(R\otimes\hat{R})(W) = W$. Then $X' = C_0(\hat{\mathbb{G}})' \cap (JC_0(\mathbb{G})J)' = L^\infty(\hat{\mathbb{G}})' \cap L^\infty(\mathbb{G})$, and we claim that this is equal to $\mathbb{C}1$. Indeed, if $x \in L^\infty(\hat{\mathbb{G}})' \cap L^\infty(\mathbb{G})$ then $\Delta(x) = W^*(1\otimes x)W = W^*W(1\otimes x) = 1\otimes x$, as $W \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\hat{\mathbb{G}})$, and so by Theorem 2.1, $x \in \mathbb{C}1$ as required. Thus $X'' = \mathcal{B}(L^2(\mathbb{G}))$, and obviously $X \subseteq L^\infty(\hat{\mathbb{G}})L^\infty(\mathbb{G})'$, which completes the argument.

Proposition 5.6. *If $\mathbb{G}$ is a locally compact quantum group, then U is unitary.*

Proof. Suppose that $\sum_i \xi_i\otimes f_i \in L^2(\mathbb{G})\otimes H$ is orthogonal to the image of U^*. By Lemma 5.4, this means that

$$0 = \sum_i \big(\hat{x}a_i\xi\big|\xi_i\big) = \sum_i \big(\hat{x}x'a_i\xi\big|\xi_i\big) \qquad (\xi \in L^2(\mathbb{G}), \hat{x} \in L^\infty(\hat{\mathbb{G}}), x' \in L^\infty(\mathbb{G})'),$$

using that $C_0(\hat{\mathbb{G}})$ is strongly dense in $L^\infty(\mathbb{G})$, and that $a_i \in L^\infty(\mathbb{G})$ for each i. As elements of the form $\hat{x}x'$ are σ-weakly dense in $\mathcal{B}(L^2(\mathbb{G}))$, this shows in particular that

$$0 = \sum_i \big(xa_i\xi\big|\xi_i\big) = \Big((x\otimes 1)V(\xi)\Big|\sum_i \xi_i\otimes f_i\Big) \qquad (x \in \mathcal{B}_0(L^2(\mathbb{G})), \xi \in L^2(\mathbb{G})).$$

However, we know that $\{(x\otimes 1)V(\xi) : x \in \mathcal{B}_0(L^2(\mathbb{G})), \xi \in L^2(\mathbb{G})\}$ is linearly dense in $L^2(\mathbb{G})\otimes H$. Hence $\sum_i \xi_i\otimes f_i = 0$, and so U^* has dense range, as required. □

We can also argue more abstractly, using the results of [3] or [15]. Indeed, for the moment, suppose that U is any coisometric corepresentation— that is, $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ for some H, that $(\Delta \otimes \iota)(U) = U_{13}U_{23}$, and that $UU^* = 1$. Firstly, we make links with the antipode– this result is well-known when U is unitary (see [27, Theorem 1.6] and compare with [26, Proposition 5.6]).

Proposition 5.7. *For any $\mathbb{G}$, let $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ be a coisometric corepresentation. For all $\omega \in \mathcal{B}(H)_*$, we have that $(\iota \otimes \omega)(U) \in D(S)$ and $S((\iota \otimes \omega)(U)) = (\iota \otimes \omega)(U^*)$.*

Proof. For a locally compact quantum group, we could slice against an orthonormal basis, and appeal to [17, Corollary 5.34]; for the details of this approach, see [4, Section 5.2].

For general $\mathbb{G}$, we argue as follows. By Proposition 4.1, $L(\hat{x}) = (\iota \otimes \omega_{\alpha,\beta})(U(\hat{x} \otimes 1)U^*)$ defines a completely bounded left multiplier of $L^1(\hat{\mathbb{G}})$, which is represented by $a = (\iota \otimes \omega_{\alpha,\beta})(U^*)$. That is,

$$(\iota \otimes L)(W^*) = (a \otimes 1)W^*.$$

As in the proof of Theorem 4.2, if we define $L^\dagger(\hat{x}) = L(\hat{x}^*)^*$, then $L^\dagger$ is a completely bounded left multiplier of $L^1(\hat{\mathbb{G}})$, represented by $b = (\iota \otimes \omega_{\beta,\alpha})(U^*)$. Then

$$(b \otimes 1)W^* = (\iota \otimes L^\dagger)(W^*) = (\iota \otimes L)(W)^*.$$

Let $\hat{\omega} \in L^1(\hat{\mathbb{G}})$; recall (see [27, Theorem 1.5]) that $(\iota \otimes \hat{\omega})(W) \in D(S)$ with $S((\iota \otimes \hat{\omega})(W)) = (\iota \otimes \hat{\omega})(W^*)$. Hence

$$(\iota \otimes \hat{\omega})(W)b^* = (\iota \otimes L_*(\hat{\omega}))(W) \in D(S),$$

and we see that

$$S\big((\iota \otimes \hat{\omega})(W)b^*\big) = (\iota \otimes L_*(\hat{\omega}))(W^*) = a(\iota \otimes \hat{\omega})(W^*) = aS\big((\iota \otimes \hat{\omega})(W)\big).$$

As $\{(\iota\otimes\hat{\omega})(W) : \hat{\omega} \in L^1(\hat{\mathbb{G}})\}$ is a core for S, this is enough to show that $b^* \in D(S)$ with $S(b^*) = a$, that is, $(\iota \otimes \omega_{\alpha,\beta})(U) \in D(S)$ with $S((\iota \otimes \omega_{\alpha,\beta})(U)) = (\iota \otimes \omega_{\alpha,\beta})(U^*)$, as required.

To justify this, we work with analytic extensions of the scaling group (τ_t). Firstly we extend this to $M(C_0(\mathbb{G}))$, say giving a group of automorphisms $(\overline{\tau}_t)$; this is all very carefully explained in [14]. In particular, [14, Proposition 2.42] shows that if $y, z \in M(C_0(\mathbb{G}))$ are such that for each $x \in D(\tau_{-i/2})$, we have that $xy \in D(\tau_{-i/2})$ with $\tau_{-i/2}(xy) = \tau_{-i/2}(x)z$, then $y \in D(\overline{\tau}_{-i/2})$ and $\overline{\tau}_{-i/2}(y) = z$. As R and S commute, we have just shown that this condition holds with $y = b^*$ and $z = R(a)$, at least when $x = (\iota \otimes \hat{\omega})(W)$ (recall that $U \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H))$ and so $a, b \in M(C_0(\mathbb{G}))$). The case of general $x \in D(\tau_{-i/2}) = D(S)$ follows by continuity, as $\{(\iota \otimes \hat{\omega})(W) : \omega \in L^1(\hat{\mathbb{G}})\}$ is a core for S, and S is a closed operator. □

Theorem 5.8. *Let $\mathbb{G}$ be a locally compact quantum group, and let $U \in L^\infty(\mathbb{G})\overline{\otimes}\mathcal{B}(H)$ be a coisometric corepresentation. Then U is unitary.*

Proof. Recall (see [15, Section 3] for example) that we may define a dense subalgebra $L^1_\sharp(\mathbb{G})$ of $L^1(\mathbb{G})$ by setting $\omega \in L^1_\sharp(\mathbb{G})$ if and only if there is $\omega^\sharp \in L^1(\mathbb{G})$ with $\langle x, \omega^\sharp \rangle = \overline{\langle S(x)^*, \omega \rangle}$ for all $x \in D(S)$. Let $\pi : L^1(\mathbb{G}) \to \mathcal{B}(H)$ be the completely bounded map $\omega \mapsto (\omega \otimes \iota)(U)$. As U is a corepresentation, π is a homomorphism. As in [3, Section 4] define $\pi^* : L^1_\sharp(\mathbb{G}) \to \mathcal{B}(H)$ by $\pi^*(\omega) = \pi(\omega^\sharp)^*$. Then, for $\alpha, \beta \in H$, by the previous proposition,

$$\overline{(\pi^*(\omega)\alpha|\beta)} = (\pi(\omega^\sharp)\beta|\alpha) = \langle (\iota \otimes \omega_{\beta,\alpha})(U), \omega^\sharp \rangle = \overline{\langle (\iota \otimes \omega_{\beta,\alpha})(U^*)^*, \omega \rangle} = \overline{(\pi(\omega)\alpha|\beta)}.$$

Thus $\pi = \pi^*$, that is, π restricts to a $*$-homomorphism on $L^1_\sharp(\mathbb{G})$.

It follows immediately from [3, Theorem 4.7], that $U^*U = UU^*$, and so U is unitary. Alternatively, assuming that π is non-degenerate, we can invoke [15, Corollary 4.3] to find a unitary $V \in M(C_0(\mathbb{G}) \otimes \mathcal{B}_0(H))$ with $\pi(\omega) = (\omega \otimes \iota)(V)$ for all $\omega \in L^1_\sharp(\mathbb{G})$. As $L^1_\sharp(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, it follows that $V = U$, and so U is unitary. We finish by observing that π is indeed non-degenerate, as U is coisometric. The map $\tilde{\pi} : L^1(\mathbb{G}) \to \mathcal{B}(H); \omega \mapsto \pi(\omega^*)^*$ is non-degenerate if and only if $\pi^* = \pi$ is non-degenerate, and furthermore, $\tilde{\pi}(\omega) = (\omega \otimes \iota)(U^*)$. Now, if $\beta \in H$ is such that $(\pi^*(\omega)\alpha|\beta) = 0$ for all ω and α, then $(U^*(\xi \otimes \alpha)|\eta \otimes \beta) = 0$ for all ξ, η and α. As U^* surjects, this shows that $\beta = 0$, which shows that π^* is indeed non-degenerate. □

The proof of [3, Theorem 4.7] uses [3, Theorem 3.4] which states that if $(a_i), (b_i) \subseteq L^\infty(\mathbb{G})$ with $\sum_i a_i^* a_i < \infty$, with $b_i^* \in D(S)$ for each i with $\sum_i S(b_i^*)^* S(b_i^*) < \infty$, and such that $\Delta(a) = \sum_i a_i \otimes b_i$ for some a, then there is a completely bounded left multiplier L represented by a. The proof makes extensive use of invariant weights; it would be interesting to know if similar results are true for quantum groups coming from Manageable Multiplicative Unitaries.

5.3 Recovering the multiplier

Finally, we show that for any $\mathbb{G}$, we can recover L from the corepresentation U using Proposition 4.1.

Proposition 5.9. *There is $\alpha_0 \in H$ such that $U^*(\xi \otimes \alpha_0) = \sum_i a_i(\xi) \otimes f_i$ for all $\xi \in L^2(\mathbb{G})$.*

Proof. Recall again the map V which satisfies $V(\xi) = \sum_i a_i(\xi) \otimes f_i$ for $\xi \in L^2(\mathbb{G})$. Let the left multiplier L be represented by $a_0 \in M(C_0(\mathbb{G}))$, so that $(1 \otimes a_0)\hat{W} = (L \otimes \iota)(\hat{W}) = \sum_i (a_i^* \otimes 1)\hat{W}(a_i \otimes 1)$. Equivalently, $\sum_i (1 \otimes a_i^*)\Delta(a_i) = a_0 \otimes 1$. For $\omega \in L^1(\mathbb{G})$ and $\xi, \eta \in L^2(\mathbb{G})$, observe that

$$\Big(U^*\big(\xi \otimes \sum_i \langle a_i, \omega\rangle f_i\big)\Big|V(\eta)\Big) = \sum_i \big((\omega \otimes \iota)\Delta(a_i)\xi\big|a_i(\eta)\big)$$
$$= \Big((\omega \otimes \iota)\big((1 \otimes a_i^*)\Delta(a_i)\big)\xi\Big|\eta\Big) = \langle a_0, \omega\rangle(\xi|\eta).$$

So the Riesz representation theorem for Hilbert spaces provides $\alpha_0 \in H$ such that

$$\Big(\sum_i \langle a_i, \omega\rangle f_i\Big|\alpha_0\Big) = \langle a_0, \omega\rangle.$$

By continuity,

$$\big(U^*(\xi \otimes \alpha)\big|V(\eta)\big) = (\xi \otimes \alpha|\eta \otimes \alpha_0) \qquad (\xi, \eta \in L^2(\mathbb{G}), \alpha \in H),$$

that is, $UV(\eta) = \eta \otimes \alpha_0$ for all $\eta \in H$. By Lemma 5.4, as $U^*UV = V$, it follows that $V(\eta) = U^*UV(\eta) = U^*(\eta \otimes \alpha_0)$ as required. □

We now take slices of U against this vector α_0, and find that this constructs the multiplier L, in the sense of Proposition 4.1.

Theorem 5.10. *Let L_* be a completely positive left multiplier of $L^1(\hat{\mathbb{G}})$. There is a coisometric corepresentation (unitary if $\mathbb{G}$ is a locally compact quantum group) U of $\mathbb{G}$ on H such that L is induced by U, using $\alpha_0 \in H$.*

Proof. Form U as above and form α_0 as in the previous proposition. It is immediate that $\alpha_i = (\iota\otimes\alpha_{\alpha_0,f_i})(U^*)$ for all i. So the multiplier constructed by Proposition 4.1 for α_0 is $L(\hat{x}) = \sum_i a_i^*\hat{x}a_i$, that is, the original L which we started with. □

For locally compact quantum groups $\mathbb{G}$, we could equivalently state this in terms of the universal dual $C_0^u(\hat{\mathbb{G}})$. Indeed, there is a representation π of $C_0^u(\hat{\mathbb{G}})$ on H associated to U, and then L is given by left multiplication by the positive functional $\omega_{\alpha_0,\alpha_0}\circ\pi \in C_0^u(\hat{\mathbb{G}})^*$.

Theorem 5.11. *Let $\mathbb{G}$ be a locally compact quantum group. There is an isometric, order preserving bijection between the completely positive multipliers of $L^1(\hat{\mathbb{G}})$ and $C_0^u(\hat{\mathbb{G}})^*_+$.*

Proof. That we have a bijection is immediate from the above work. If $\mu \in C_0^u(\hat{\mathbb{G}})^*_+$ is a state, then suppose that $C_0^u(\hat{\mathbb{G}}) \subseteq \mathcal{B}(H)$ is the universal representation, so $\mu = \omega_{\alpha,\alpha}$ for some $\alpha \in H$. Then $\mathcal{W}$ can be identified with a member of $\mathcal{B}(L^2(\mathbb{G})\otimes H)$, and Proposition 4.2 and Proposition 4.1 show that left multiplication by μ induces the completely positive multiplier L, where in particular,

$$L(1) = (\iota\otimes\omega_{\alpha,\alpha})(\mathcal{W}\mathcal{W}^*) = 1\langle\mu, 1\rangle = 1.$$

So $\|L\| = 1$, and hence our bijection is an isometry.

Finally, if $\mu \leq \lambda$ in $C_0^u(\hat{\mathbb{G}})^*_+$ then form the associated completely positive multipliers L_μ and L_λ. Let L be the multiplier formed from $\lambda-\mu$, so by uniqueness, $L = L_\lambda - L_\mu$. As L is completely positive, $L_\lambda \geq L_\mu$. The converse is simply a case of reversing the argument. Thus our bijection is order preserving. □

6 From left multipliers to double multipliers

Suppose for the moment that $\mathbb{G}$ is a locally compact quantum group. As $C_0^u(\hat{\mathbb{G}})$ has a unitary antipode $\hat{R}_u$ which extends $\hat{R}$ in the sense that $\hat{\pi}_u\hat{R}_u = \hat{R}\hat{\pi}_u$, it is easy to see that actually $\hat{\pi}_u^*(L^1(\hat{\mathbb{G}}))$ is a two-sided ideal in $C_0^u(\hat{\mathbb{G}})^*$. Combining this with Theorem 5.10 shows immediately that any completely positive left multiplier L is the "left half" of a completely positive double multiplier (L, R). We now explore what this means for unitary corepresentations.

Following [22, Section 3.3.3], we define the contragradient corepresentation as follows. Let $U \in \mathcal{B}(L^2(\mathbb{G})\otimes H)$ be a corepresentation of $\mathbb{G}$. Fix an involution J_H on H– so J_H is an antilinear isometry $H \to H$ with J_H^2 the identity. For example, if H has an orthonormal basis (f_i), then we can define J_H by $J_H(\sum_i \lambda_i f_i) = \sum_i \overline{\lambda_i} f_i$. Then we define the contragradient corepresentation U^c by

$$U^c = (\hat{J}\otimes J_H)U^*(\hat{J}\otimes J_H).$$

Again, $\hat{J}$ is the modular conjugation given by the left Haar weight on $L^\infty(\hat{\mathbb{G}})$, but used here because $R(x) = \hat{J}x^*\hat{J}$ for $x \in L^\infty(\mathbb{G})$. Notice that then

$$(\iota\otimes\omega_{\alpha,\beta})(U^c) = R\big((\iota\otimes\omega_{J_H\beta,J_H\alpha})(U)\big),$$

and using this, it is easy to show that U^c is a corepresentation. Indeed, we could use this relation to define U^c, which is essentially the construction in [22] (which avoids having to use that $R(\cdot) = \hat{J}(\cdot)^*\hat{J}$). Clearly U^c is unitary if U is, and also $(U^c)^c = U$.

Let L be a completely positive left multiplier of $L^1(\hat{\mathbb{G}})$, and form the corepresentation U and the vector $\alpha_0 \in H$, as in the previous sections. Applying Proposition 4.1 to U^c, we find that if

$$L'(\hat{x}) = (\iota\otimes\omega_{J_H(\alpha_0),J_H(\alpha_0)}(U^c(\hat{x}\otimes 1)U^{c*}) \qquad (\hat{x} \in L^\infty(\hat{\mathbb{G}})),$$

then L' is the adjoint of a completely bounded left multiplier on $L^1(\hat{\mathbb{G}})$, represented by

$$b = (\iota \otimes \omega_{J_H(\alpha_0),J_H(\alpha_0)})(U^{c*}) = R\big((\iota \otimes \omega_{\alpha_0,\alpha_0})(U^*)\big).$$

An easy calculation (see [4, Section 5]) shows that $\hat{R}L'\hat{R}$ is a right multiplier, represented by $R(b) = (\iota \otimes \omega_{\alpha_0,\alpha_0})(U^*)$, that is, the same element which L is represented by. So $(L, \hat{R}L'\hat{R})$ is a double multiplier.

6.1 Double multiplier implies unitary corepresentation

We shall now reverse the argument of the previous section– we will show that for any $\mathbb{G}$, if (L,T) is a completely positive double multiplier, then the corepresentation U constructed for L is a unitary; the obvious proof strategy is to use the extra information which T provides.

Notice that while Proposition 3.2 shows that all completely bounded multipliers are "represented", this fact took a back seat in our arguments, until Proposition 5.9. Here we show how to use the representing element more directly.

As before, we form the $*$-algebra $L^1_\sharp(\mathbb{G})$, which is dense in $L^1(\mathbb{G})$. Our reference here is [15], and we note that the elementary properties of $L^1_\sharp(\mathbb{G})$ can be developed *mutatis mutandis* for $\mathbb{G}$ coming from manageable multiplicative unitaries.

Recall that the scaling group (τ_t) is implemented as $\tau_t(x) = P^{it}xP^{-it}$, where P is a certain positive injective operator. As R and τ_t commute for all t, and $S = R\tau_{-i/2}$, it follows that R leaves $D(S)$ invariant, and $RS = SR$. It is then easy to see that R_* leaves $L^1_\sharp(\mathbb{G})$ invariant, and $R_*(\omega^\sharp) = R_*(\omega)^\sharp$ for $\omega \in L^1_\sharp(\mathbb{G})$. Given $\beta \in D(P^{-1/2})$ and $\xi \in D(P^{1/2})$, we have that for $x \in D(S) = D(\tau_{-i/2})$,

$$\langle x, \omega_{P^{-1/2}\beta,P^{1/2}\xi}\rangle = \big(P^{1/2}xP^{-1/2}\beta\big|\xi\big) = \langle \tau_{-i/2}(x), \omega_{\beta,\xi}\rangle = \langle S(R(x)), \omega^*_{\xi,\beta}\rangle = \langle x, R_*(\omega^\sharp_{\xi,\beta})\rangle,$$

and so $\omega_{\xi,\beta} \in L^1_\sharp(\mathbb{G})$ with $\omega^\sharp_{\xi,\beta} = R_*(\omega_{P^{-1/2}\beta,P^{1/2}\xi})$.

Proposition 6.1. *Let L be a completely bounded left multiplier of $L^1(\hat{\mathbb{G}})$, represented by $a_0 \in M(C_0(\mathbb{G}))$. For $\xi,\eta \in D(P^{1/2})$ and $\alpha,\beta \in D(P^{-1/2})$, we have that*

$$\big(\Phi(\theta_{\xi,\eta})\alpha\big|\beta\big) = \langle \Delta(a_0), \omega_{\alpha,\eta} \otimes \omega^{\sharp*}_{\xi,\beta}\rangle.$$

Proof. Let $\xi_0 \in L^2(\mathbb{G})$ be a unit vector, let (e_i) be an orthonormal basis for $L^2(\mathbb{G})$, and let $W(\alpha \otimes \xi_0) = \sum_i \alpha_i \otimes e_i$ and $W(\beta \otimes \xi_0) = \sum_i \beta'_i \otimes e_i$. For $\epsilon > 0$, we can find a family (β_i) in $D(P^{-1/2})$ with

$$\Big\|W(\beta \otimes \xi_0) - \sum_i \beta_i \otimes e_i\Big\| < \epsilon.$$

Using Proposition 3.3, and that $\hat{W} = \Sigma W^* \Sigma$, we see that

$$\begin{aligned}
\big(\Phi(\theta_{\xi,\eta})\alpha\big|\beta\big) &= \big((\iota \otimes L)(W(\theta_{\xi,\eta} \otimes 1)W^*)W(\alpha \otimes \xi_0)\big|W(\beta \otimes \xi_0)\big) \\
&= \sum_{i,j} \big((\omega_{\alpha_i,\beta'_j} \otimes \iota)(\iota \otimes L)(W(\theta_{\xi,\eta} \otimes 1)W^*)e_i\big|e_j\big) \\
&= \sum_{i,j} \big(L((\omega_{\xi,\beta'_j} \otimes \iota)(W)(\omega_{\alpha_i,\eta} \otimes \iota)(W^*))e_i\big|e_j\big).
\end{aligned}$$

A similar calculation establishes that if

$$x = \sum_{i,j} \big(L((\omega_{\xi,\beta_j} \otimes \iota)(W)(\omega_{\alpha_i,\eta} \otimes \iota)(W^*))e_i \big| e_j\big),$$

then

$$\big|\big(\Phi(\theta_{\xi,\eta})\alpha\big|\beta\big) - x\big| < \epsilon\|L\|_{cb}\|\alpha\|\|\xi\|\|\eta\|.$$

That is, we may replace (β'_j) by (β_j), at the cost of a small error term.

As $(\omega \otimes \iota)(W)^* = (\omega^\sharp \otimes \iota)(W)$ for $\omega \in L^1_\sharp(\mathbb{G})$, we see that $(\omega_{\xi,\beta_j} \otimes \iota)(W) = (\omega^\sharp_{\xi,\beta_j} \otimes \iota)(W)^* = (\omega^{\sharp *}_{\xi,\beta_j} \otimes \iota)(W^*)$. This makes sense, as $\beta_j \in D(P^{-1/2})$ and $\xi \in D(P^{1/2})$. Thus

$$(\omega_{\xi,\beta_j} \otimes \iota)(W)(\omega_{\alpha_i,\eta} \otimes \iota)(W^*) = (\omega^{\sharp *}_{\xi,\beta_j} \otimes \iota)(W^*)(\omega_{\alpha_i,\eta} \otimes \iota)(W^*) = (\omega_{\alpha_i,\eta}\omega^{\sharp *}_{\xi,\beta_j} \otimes \iota)(W^*).$$

Recall that $(\iota \otimes L)(W^*) = (a_0 \otimes 1)W^*$, and that $(\Delta \otimes \iota)(W^*) = W^*_{23}W^*_{13}$, and so

$$\begin{aligned} x &= \sum_{i,j} \big((\omega_{\alpha_i,\eta}\omega^{\sharp *}_{\xi,\beta_j} \otimes \iota)((\iota \otimes L)(W^*))e_i \big| e_j\big) \\ &= \sum_{i,j} \langle (\Delta(a_0) \otimes 1)W^*_{23}W^*_{13}, \omega_{\alpha_i,\eta} \otimes \omega^{\sharp *}_{\xi,\beta_j} \otimes \omega_{e_i,e_j}\rangle \\ &= \sum_{j} \langle (\Delta(a_0) \otimes 1)W^*_{23}, \omega_{\alpha,\eta} \otimes \omega^{\sharp *}_{\xi,\beta_j} \otimes \omega_{\xi_0,e_j}\rangle. \end{aligned}$$

Let $a \in D(S)^*$, so that

$$\begin{aligned} \sum_j \langle (a \otimes 1)W^*, \omega^{\sharp *}_{\xi,\beta_j} \otimes \omega_{\xi_0,e_j}\rangle &= \sum_j \langle aS((\iota \otimes \omega_{\xi_0,e_j})(W)), \omega^{\sharp *}_{\xi,\beta_j}\rangle \\ &= \sum_j \overline{\langle S((\iota \otimes \omega_{\xi_0,e_j})(W))^* a^*, \omega^\sharp_{\xi,\beta_j}\rangle} = \sum_j \langle (\iota \otimes \omega_{\xi_0,e_j})(W) S(a^*)^*, \omega_{\xi,\beta_j}\rangle \\ &= \sum_j \langle W(S(a^*)^* \otimes 1), \omega_{\xi,\beta_j} \otimes \omega_{\xi_0,e_j}\rangle = \Big(W(S(a^*)^* \otimes 1)(\xi \otimes \xi_0)\Big| \sum_j \beta_j \otimes e_j\Big). \end{aligned}$$

By comparison,

$$\begin{aligned} \Big(W(S(a^*)^* \otimes 1)(\xi \otimes \xi_0)\Big| \sum_j \beta'_j \otimes e_j\Big) &= \big((S(a^*)^* \otimes 1)(\xi \otimes \xi_0)\big| W^*W(\beta \otimes \xi_0)\big) \\ &= \langle S(a^*)^*, \omega_{\xi,\beta}\rangle = \overline{\langle a^*, \omega^\sharp_{\xi,\beta}\rangle} = \langle a, \omega^{\sharp *}_{\xi,\beta}\rangle. \end{aligned}$$

If it so happens that $a = (\omega_{\alpha,\eta} \otimes \iota)\Delta(a_0)$ is in $D(S)^*$, then we have

$$\big|x - \langle a, \omega^{\sharp *}_{\xi,\beta}\rangle\big| \leq \epsilon\|\xi\|\|S(a^*)\|.$$

However, observe that for this choice of a,

$$\langle a, \omega^{\sharp *}_{\xi,\beta}\rangle = \langle \Delta(a_0), \omega_{\alpha,\eta} \otimes \omega^{\sharp *}_{\xi,\beta}\rangle,$$

and so as $\epsilon > 0$, this gives the required result.

So it remains to show that $a = (\omega_{\alpha,\eta} \otimes \iota)\Delta(a_0) \in D(S)^*$. By [4, Theorem 5.9], we know that $a_0 \in D(S)^*$, and by hypothesis, $\omega_{\eta,\alpha} \in L^1_\sharp(\mathbb{G})$. Thus, for $\omega \in L^1_\sharp(\mathbb{G})$,

$$\langle a^*, \omega^\sharp\rangle = \langle (\omega_{\eta,\alpha} \otimes \iota)\Delta(a_0^*), \omega^\sharp\rangle = \langle a_0^*, \omega_{\eta,\alpha}\omega^\sharp\rangle = \overline{\langle S(a_0^*)^*, \omega\omega^\sharp_{\eta,\alpha}\rangle} = \overline{\langle (\iota \otimes \omega^\sharp_{\eta,\alpha})\Delta(S(a_0^*)^*), \omega\rangle}.$$

This is enough to show that $a^* \in D(S)$ with $S(a^*) = (\iota \otimes \omega^\sharp_{\eta,\alpha})\Delta(S(a_0^*)^*)$, see for example [3, Appendix A.2.]. □

Lemma 6.2. *Let L, L' be completely positive multipliers of $L^1(\hat{\mathbb{G}})$, represented by a_0 and $R(a_0)$ respectively, and use these to define Φ and Φ'. Let H and H' be the Hilbert spaces formed as in Section 5, using Φ and Φ'. Then, for $\alpha, \beta \in D(P^{-1/2})$ and $\xi, \eta \in D(P^{1/2})$,*

$$\big(\eta \otimes \alpha \big| \xi \otimes \beta\big)_H = \big(P^{-1/2}\beta \otimes P^{1/2}\xi \big| P^{-1/2}\alpha \otimes P^{1/2}\eta\big)_{H'}.$$

Proof. By the definition of Φ and Φ', and using the previous proposition,

$$\begin{aligned}\big(\eta \otimes \alpha \big| \xi \otimes \beta\big)_H &= \big(\Phi(\theta_{\xi,\eta})\alpha \big| \beta\big) = \langle \Delta(a_0), \omega_{\alpha,\eta} \otimes \omega^{\sharp *}_{\xi,\beta}\rangle = \langle \Delta(R(a_0)), R_*(\omega^{\sharp *}_{\xi,\beta}) \otimes R_*(\omega_{\alpha,\eta})\rangle \\ &= \langle \Delta(R(a_0)), \omega^*_{P^{-1/2}\beta, P^{1/2}\xi} \otimes \omega^{\sharp *}_{P^{-1/2}\alpha, P^{1/2}\eta}\rangle \\ &= \big(P^{-1/2}\beta \otimes P^{1/2}\xi \big| P^{-1/2}\alpha \otimes P^{1/2}\eta\big)_{H'},\end{aligned}$$

where we used that $\Delta R = \sigma(R \otimes R)\Delta$, and that $R_*(\omega_{\alpha,\eta}) = R_*(\omega_{\eta,\alpha})^* = (\omega^\sharp_{P^{-1/2}\alpha, P^{1/2}\eta})^*$. ☐

Lemma 6.3. *Let (L, T) be a completely bounded double multiplier of $L^1(\mathbb{G})$, with L completely positive. Then T is completely positive.*

Proof. We shall prove that $L' = \hat{R}T\hat{R}$ is completely positive; in fact, we show that Φ', formed from L', is completely positive. Positive elements of $M_n(\mathcal{B}_0(L^2(\mathbb{G})))$ are sums of matrices of the form $(a_i^* a_j)_{i,j=1}^n$ where $(a_i)_{i=1}^n \subseteq \mathcal{B}_0(L^2(\mathbb{G}))$. By density, we may suppose that

$$a_i = \sum_{k=1}^m \theta_{\eta_{i,k}, \xi_{i,k}},$$

for some $(\eta_{i,k}) \subseteq L^2(\mathbb{G})$ and $(\xi_{i,k}) \subseteq D(P^{1/2})$. Then, for $(\eta_i)_{i=1}^n \subseteq D(P^{-1/2})$, by (the proof of) the previous lemma,

$$\begin{aligned}\sum_{i,j} \big(\Phi'(a_i^* a_j)\eta_j \big| \eta_i\big) &= \sum_{i,j,k,l} \big(\eta_{j,l} \big| \eta_{i,k}\big)\big(\Phi'(\theta_{\xi_{i,k},\xi_{j,l}})\eta_j \big| \eta_i\big) \\ &= \sum_{i,j,k,l} \big(P^{-1/2}\eta_j \otimes P^{1/2}\xi_{j,l} \otimes \eta_{j,l} \big| P^{-1/2}\eta_i \otimes P^{1/2}\xi_{i,k} \otimes \eta_{i,k}\big)_{H \otimes L^2(\mathbb{G})} \geq 0.\end{aligned}$$

By the density of $D(P^{-1/2})$ in $L^2(\mathbb{G})$, this shows that $(\Phi'(a_i^* a_j)) \geq 0$ in $M_n(\mathcal{B}(L^2(\mathbb{G})))$, and so Φ', and hence also L', is completely positive. ☐

We now come to the main result of this section.

Theorem 6.4. *For any $\mathbb{G}$, a completely positive left multiplier L is given by a unitary corepresentation if and only if L is the left part of a completely bounded double multiplier.*

Proof. We need only prove that if L is the left part of a double multiplier, then the U we constructed before is actually unitary. Form Φ and H for L, and using the previous lemma, find L' completely positive, and form Φ' and H'. Then there is an anti-linear unitary $J_0 : H' \to H$ which satisfies

$$J_0\big(\xi \otimes \alpha\big) = P^{-1/2}\alpha \otimes P^{1/2}\xi \qquad (\xi \in D(P^{1/2}), \alpha \in D(P^{-1/2})).$$

Then the map $R_0 : \mathcal{B}(H) \to \mathcal{B}(H'); x \mapsto J_0^* x J_0$ is an anti-$*$-isomorphism.

Let (f_i) be an orthonormal basis for H, so $(J_0 f_i)$ is an orthonormal basis for H'. We then define families (a_i) and (b_i) by $V(\alpha) = \sum_i a_i(\alpha) \otimes f_i$ and $V'(\alpha) = \sum_i b_i(\alpha) \otimes J_0 f_i$. As before, this means that $\xi \otimes \alpha = \sum_i \langle a_i, \omega_{\alpha,\xi} \rangle f_i$ in H, and similarly for H'. By Lemma 6.2,

$$\begin{aligned}\sum_i \langle a_i^*, \omega_{\xi,\alpha} \rangle J_0 f_i = J_0\Big(\sum_i \langle a_i, \omega_{\alpha,\xi} \rangle f_i \Big) = J_0(\xi \otimes \alpha) = P^{-1/2}\alpha \otimes P^{1/2}\xi \\ = \sum_i \langle b_i, \omega_{P^{1/2}\xi, P^{-1/2}\alpha} \rangle J_0 f_i = \sum_i \langle b_i, R_*(\omega_{\xi,\alpha}^{\sharp *}) \rangle J_0 f_i.\end{aligned}$$

Form the coisometric corepresentations U and U' associated to L and L', respectively. Let $X = (R \otimes R_0)((U')^*) \in L^\infty(\mathbb{G}) \overline{\otimes} \mathcal{B}(H)$. Then $XX^* = (R \otimes R_0)(U'(U')^*) = 1$, so X is also coisometric. Let $\xi, \eta \in L^2(\mathbb{G})$ be such that $\omega_{\eta,\xi} \in L^1_\sharp(\mathbb{G})$, and set $R_*(\omega_{\xi,\eta}) = \sum_k \omega_{\xi_k,\eta_k}$ and $\omega_{\eta,\xi}^{\sharp *} = \sum_k \omega_{\xi'_k,\eta'_k}$, the sums converging absolutely. Let $\alpha = \sum_i \langle a_i, \omega_1 \rangle f_i, \beta = \sum_i \langle a_i, \omega_2 \rangle f_i$ in H, with $\omega_2^* \in L^1_\sharp(\mathbb{G})$. Then

$$\begin{aligned}\big(X(\xi \otimes \alpha) \big| \eta \otimes \beta \big) &= \sum_k \big((U')^*(\xi_k \otimes J_0\beta) \big| \eta_k \otimes J_0\alpha \big) \\ &= \sum_{k,i,j} \big((U')^*(\xi_k \otimes \langle b_i, R_*(\omega_2^{*\sharp *}) \rangle J_0 f_i) \big| \eta_k \otimes \langle a_i^*, \omega_1^* \rangle J_0 f_i) \big) \\ &= \sum_{k,i} \big((R_*(\omega_2^{*\sharp *}) \otimes \iota)\Delta(b_i)\xi_k \big| \eta_k \big) \langle a_i, \omega_1 \rangle \\ &= \sum_{k,i} \langle b_i, R_*(\omega_2^{*\sharp *}) \omega_{\xi_k,\eta_k} \rangle \langle a_i, \omega_1 \rangle \\ &= \sum_{k,i} \overline{\langle a_i, \omega_2 R_*(\omega_{\eta_k,\xi_k})^{\sharp *} \rangle} \langle a_i, \omega_1 \rangle = \sum_i \overline{\langle a_i, \omega_2 \omega_{\eta,\xi}^{\sharp *} \rangle} \langle a_i, \omega_1 \rangle \\ &= \sum_{i,k} \overline{\langle a_i, \omega_2 \omega_{\xi'_k,\eta'_k} \rangle} \langle a_i, \omega_1 \rangle = \sum_k \big(\eta'_k \otimes \alpha \big| U^*\big(\xi'_k \otimes \beta\big) \big).\end{aligned}$$

By density, it follows that for any $\alpha, \beta \in H$,

$$\begin{aligned}\langle (\iota \otimes \omega_{\alpha,\beta})(X), \omega_{\xi,\eta} \rangle = \sum_k \langle (\iota \otimes \omega_{\alpha,\beta})(U), \omega_{\eta'_k,\xi'_k} \rangle = \langle (\iota \otimes \omega_{\alpha,\beta})(U), \omega_{\xi,\eta}^{*\sharp} \rangle \\ = \langle S((\iota \otimes \omega_{\alpha,\beta})(U)), \omega_{\xi,\eta} \rangle = \langle (\iota \otimes \omega_{\alpha,\beta})(U^*), \omega_{\xi,\eta} \rangle,\end{aligned}$$

where here we used Proposition 5.7. Thus $X = U^*$, and so U is unitary, as required. ☐

In the light of this result, we could phrase the results of Section 5.2 as saying that for a locally compact quantum group $\mathbb{G}$, a completely positive left multiplier is always the left part of a double multiplier. It would be interesting to know if the same is true for completely bounded left multipliers.

7 Weak*-continuity of the Junge, Neufang, Ruan representation

As explained in Section 3 above, [13] shows that for a locally compact quantum group $\mathbb{G}$, there is a bijection between the completely bounded left multipliers of $L^1(\hat{\mathbb{G}})$, say $M_{cb}^l(L^1(\hat{\mathbb{G}}))$, and

$\mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$. In [11], it is shown that this map is weak*-weak* continuous, at least when $\hat{\mathbb{G}}$ has the *left co-AP* property, see [11, Corollary 4.10] (and [11, Theorem 4.7] for the version for right multipliers). In this final section of the paper, we show that this weak*-continuity result is true for all $\mathbb{G}$.

Firstly, we recall from [11] the proof that $M^l_{cb}(L^1(\hat{\mathbb{G}}))$ is a dual space. Proposition 3.2 shows that we have a map $\Lambda : M^l_{cb}(L^1(\hat{\mathbb{G}})) \to L^\infty(\mathbb{G})$ (actually, this maps into $M(C_0(\mathbb{G}))$, but this is unimportant here). Then [11, Proposition 3.4] shows that if we denote by X the image of Λ, equipped with the norm coming from $M^l_{cb}(L^1(\hat{\mathbb{G}}))$, then the closed unit ball of X is weak*-closed in $L^\infty(\mathbb{G})$. Indeed, giving $M^l_{cb}(L^1(\hat{\mathbb{G}}))$ its canonical operator space structure, the closed unit ball of $M_n(X)$ is weak*-closed in $M_n(L^\infty(\mathbb{G}))$. Using this, [11, Theorem 3.5] shows that if we let $Q^l_{cb}(L^1(\hat{\mathbb{G}}))$ be the closure in $M^l_{cb}(L^1(\hat{\mathbb{G}}))^*$ of the image of $L^1(\mathbb{G})$ under the adjoint of Λ, then $Q^l_{cb}(L^1(\hat{\mathbb{G}}))^*$ is completely isometrically isomorphic to $M^l_{cb}(L^1(\hat{\mathbb{G}}))$.

In [5, Section 8] we independently gave an analogous construction of a weak*-topology on the space of double multipliers. In fact, the first part of the proof of [5, Proposition 8.11] already works for merely left multipliers, and then one can apply the abstract result which is [5, Proposition 8.12] to construct $Q^l_{cb}(L^1(\hat{\mathbb{G}}))$. In [5] we found a very "Banach algebraic" way to construct preduals for double multiplier algebras (see [5, Theorem 7.7] for example), but it seems that at several crucial points, it really is necessary to work with double multipliers. It would be interesting to know how to adapt these ideas to one-sided multipliers.

For us, the important point is that if (L_α) is a bounded net in $M^l_{cb}(L^1(\hat{\mathbb{G}}))$, then (L_α) is weak*-null with respect to $Q^l_{cb}(L^1(\hat{\mathbb{G}}))$ if and only if $(\Lambda(L_\alpha))$ is weak*-null in $L^\infty(\mathbb{G})$.

We next consider the space $\mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$. Firstly, we consider the larger space $\mathcal{CB}^{\sigma}(\mathcal{B}(L^2(\mathbb{G})))$ which can be identified with $\mathcal{CB}(\mathcal{B}_0(L^2(\mathbb{G})), \mathcal{B}(L^2(\mathbb{G})))$. This in turn is the dual space of $\mathcal{B}_0(L^2(\mathbb{G}))\widehat{\otimes}\mathcal{B}(L^2(\mathbb{G}))_*$ the operator space projective tensor product of the compact operators $\mathcal{B}_0(L^2(\mathbb{G}))$ with the trace-class operators $\mathcal{B}(L^2(\mathbb{G}))_*$. By restriction, we have a weak*-topology on $\mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$. Again, for us the important point is that a bounded net (Φ_α) in $\mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$ is weak*-null if and only if $(\Phi_\alpha(\theta))$ is a weak*-null net in $\mathcal{B}(L^2(\mathbb{G}))$, for each $\theta \in \mathcal{B}_0(L^2(\mathbb{G}))$. All this is explained in [11, Section 4] and the references therein.

The following improves [11, Theorem 4.7] (which is stated for right multipliers) in that we need make no approximation property type assumptions.

Theorem 7.1. *For any* $\mathbb{G}$*, the map* $M^l_{cb}(L^1(\hat{\mathbb{G}})) \to \mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$ *is weak*-weak*-continuous. If* $\mathbb{G}$ *is a locally compact quantum group, this correspondence is a weak*-weak*-continuous homeomorphism.*

Proof. Denote by ϕ the map $M^l_{cb}(L^1(\hat{\mathbb{G}})) \to \mathcal{CB}^{\sigma,L^\infty(\hat{\mathbb{G}})}_{L^\infty(\mathbb{G})'}(\mathcal{B}(L^2(\mathbb{G})))$. To show that ϕ is weak*-continuous, it suffices to show that if (L_i) is a bounded, weak*-null net in $M^l_{cb}(L^1(\hat{\mathbb{G}}))$, then the corresponding bounded net, say (Φ_i), in $\mathcal{CB}^{\sigma}(\mathcal{B}(L^2(\mathbb{G})))$ is weak*-null. When $\mathbb{G}$ is a locally compact quantum group, we know from [13] that ϕ is a completely isometric isomorphism, and then if ϕ is weak*-continuous, it is automatically a weak*-weak*-continuous homeomorphism. This is perhaps not well-known (in the operator space setting) but see [5, Lemma 10.1] for example.

We fix a bounded weak*-null net (L_i) of left multipliers, with corresponding net (Φ_i). For each i let L_i be represented by $a_i \in L^\infty(\mathbb{G})$. That (L_i) is weak*-null means that (a_i) is weak*-null. As $(L_i \otimes \iota)(\hat{W})\hat{W}^* = (1 \otimes a_i)$ for each i, we see that (a_i) is a bounded net. By Proposition 6.1, we have that

$$(\Phi_i(\theta_{\xi,\eta})\alpha|\beta) = \langle a_i, \omega_{\alpha,\eta}\omega^{\sharp *}_{\xi,\beta}\rangle \qquad (\xi, \eta \in D(P^{1/2}), \alpha, \beta \in D(P^{-1/2})).$$

As $D(P^{1/2})$ and $D(P^{-1/2})$ are dense in $L^2(\mathbb{G})$, we immediately see that

$$\lim_i \langle \Phi_i(\theta), \omega \rangle = 0$$

for a dense collection of $\theta \in \mathcal{B}_0(L^2(\mathbb{G}))$ and $\omega \in \mathcal{B}(L^2(\mathbb{G}))_*$. As (Φ_i) is a bounded net, this is enough to show that (Φ_i) is weak*-null, as required. □

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Matthew Daws
School of Mathematics,
University of Leeds,
LEEDS LS2 9JT
United Kingdom
Email: `matt.daws@cantab.net`