Variations of gwistor space

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Abstract

We study variations of the G_2 structure on the unit tangent sphere bundle, introduced in [4, 5, 6] and now called gwistor space. We analize the equations of calibration and cocalibration, as well as those of W_3 pure type or nearly-parallel type.

Key Words: calibration, Einstein manifold, G_2-structure, gwistor space.

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1 Introduction

In [4, 5, 6] it was shown how a natural G_2 structure is associated to the unit tangent sphere bundle $\pi : SM \to M$ of any given oriented Riemannian 4-manifold M. The techniques are twistorial so we have chosen to give the name of gwistors to the theory.

One starts by a construction of the octonions over the 3-sphere fibre bundle. The Levi-Civita connection of the base induces a canonical splitting of the tangent bundle of TM. Both vertical and horizontal subbundles V, H become isometric to π^*TM with the pull-back metric. On the space $SM = \{u \in TM : \ \|u\| = 1\}$ each point u becomes the identity

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element, the generator of the real line in $\mathbb{O}$. Then we use the volume form coupled with $u = U_u \in V$, to induce a cross-product on $u^\perp \subset V$. This gives a quaternionic structure on V and then, applying the well-known Cayley-Dickson process, we obtain the $\mathbb{O}$-structure on $V \oplus H$. The pull-back of TM also inherits a metric connection $\nabla^* = \pi^*\nabla$ and thence parallel identifications of horizontals and verticals, passing through π^*TM, cf. loc. cit. and [14]. The manifold SM is endowed with the induced metric from the canonical or Sasaki metric on TM. Clearly TSM coincides with $V_1 \oplus H$ where $V_1 = \{v \in V : \langle u, v\rangle = 0\}$ at each point u. Since u is pointing outwards, our space SM inherits a G_2-structure, for which it receives the name of gwistor space. Recall $G_2 = \mathrm{Aut}\,\mathbb{O}$. Of course the structure is the extension of an $SO(3)$ structure. The connection induces a projection $\nabla^*_{\cdot}U : TSM \to V$ with kernel H, where the section U is the tautological vertical vector field.

It is known, by a Theorem of Y. Tashiro, that SM has an almost contact structure in any dimension of the base. As rigid geometrical objects these are, the contact structure is bound to be K-contact if and only if M is locally a radius 1 sphere. Then it is also Sasakian, cf. [7]. The model space is the trivial fibration $SO(5)/SO(3)$.

If we leave aside the Cayley-Dickson process and concentrate on the five invariant 3-forms which are naturally defined on SM, then we may try to find other interesting G_2 structures. This article is devoted to them, the variations of gwistor space, which should also be called g-natural G_2-structures on the unit tangent sphere bundle, in analogy with the terms used by [1, 2] and many references therein. On the other hand, the terms deformation or perturbation are also used in similar context by other authors, so we made an option.

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1.1 The basic 3-forms

We start by abbreviating the notation and write $SM = \mathcal{G}$. There is, as we have seen, an isometry map connecting H with V, which we denote by θ. We extend it by 0 to V. Therefore the tangent vector field $\theta^t U$ generates a real line bundle, contained in $T\mathcal{G}$. We now pass to the language of differential forms. We may write a splitting:

$$T^*\mathcal{G} = \mathbb{R}\mu \oplus H_1^* \oplus V_1^* \tag{1}$$

where $\mu = (\theta^t U)^\flat$ and $H_1 = \theta^t V_1$. This 1-form is the aforementioned contact structure, satisfying:

$$\mu_u(v) = \langle u, \mathrm{d}\pi(v)\rangle, \qquad \forall u \in \mathcal{G},\ v \in T\mathcal{G}. \tag{2}$$

The usual pull-back (horizontal) of the volume form of M is also denoted by vol. The vertical pull-back of $\mathrm{vol} \in \Omega^4(M)$ coupled with U is denoted by α; then we define analogously a 3-form $\alpha_3 = (\theta^t U)\lrcorner\mathrm{vol}$. Of course (we omit the wedge product symbol throughout the text),

$$\mu\alpha_3 = \mathrm{vol}, \qquad\qquad \mathrm{vol}\alpha = \mathrm{Vol}_{\mathcal{G}}. \tag{3}$$

As shown in [4], it is possible to find an 'adapted' direct orthonormal frame $e_0, e_1, \ldots, e_6$ such that

$$\mu = e^0, \qquad\qquad \alpha_3 = e^{123}, \qquad\qquad \alpha = e^{456}. \tag{4}$$

It is also known that $\mathrm{d}\mu = e^{41} + e^{52} + e^{63}$, which restricts to a symplectic 2-form on the vector bundle $H_1 \oplus V_1$.

The endomorphism θ allows one to find two other 3-forms (see [4] for the invariant definition):

$$\alpha_1 = e^{156} + e^{264} + e^{345} \tag{5}$$

and

$$\alpha_2 = e^{126} + e^{234} + e^{315}. \tag{6}$$

One can prove the five 3-forms $\alpha, \ldots, \alpha_3, \mu\mathrm{d}\mu$ correspond to a basis for the space of invariants in $\Lambda^3(\mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3)$ under $SO(3)$, the underlying structure group of $\mathcal{G}$, ie. there are five irreducible 1-dimensional submodules[1].

The natural G_2 structure on $\mathcal{G}$ to which we have referred *is* given[2] by the 3-form

$$\sigma_0 = \alpha_2 - \alpha + \mu\mathrm{d}\mu. \tag{7}$$

Its integrability was studied first in the case of the torsion free metric connection on M and then in the case of metric connections with torsion (which clearly allow the same construction as the Levi-Civita). We know that the structure is co-calibrated, ie. $\mathrm{d}*\phi = 0$, if and only if the base M is an Einstein manifold.

1.2 Stability of G_2 structures

Let us recall the definition of stable forms from the theory of G_2-manifolds, [8, 9].

Let σ denote a linear G_2 structure on a 7-dimensional oriented vector space V. A consequence of the study of the Lie group $G_2 = \mathrm{Aut}\,\sigma \subset SO(7)$ is that it is connected and 14 dimensional; henceforth, that the orbit of σ under $GL(7, \mathbb{R})$ is an open set inside the module $\Lambda^3 V^*$. This orbit is denoted Λ^3_+ and known as the space of stable G_2-structures on V. We somehow detect the boundaries of such stability by the non-degeneracy of the induced Euclidean product. Indeed, the inner product is given by the map $(v, w) \mapsto v\lrcorner\sigma \wedge w\lrcorner\sigma \wedge \sigma$, required to be a positive multiple of the chosen orientation on the diagonal of V. The given

[1] The author acknowledges I. Agricola and Th. Friedrich for this computation.

[2] Actually the structure was given first by the opposite, $-\sigma_0$, but we take the opportunity here to change. The reason is that it gives the right *canonical* representation theory without changing the canonical orientation of $\mathcal{G}$; namely the G_2-modules $\Lambda^2_7, \Lambda^2_{14}$, which appear from opposite highest weights in [4].

σ satisfies this condition by assumption. Letting σ vary, we have a $GL(7,\mathbb{R})$-equivariant map

$$V\otimes V\otimes\Lambda^3V^*\longrightarrow\Lambda^7V^*.$$

Then of course Λ^3_+ is the reunion of two open orbits under the subgroup $GL^+(7,\mathbb{R})$, identified 1-1 by a $-$ sign as it is easy to see. Moreover, the orientation in V induced by the first map itself is preserved in each of these orbits.

Now we return to gwistor space $\mathcal{G}\to M$ and admit a variation of the 'canonical' structure σ_0. We let $f_0,\ldots,f_4$ be scalar functions on $\mathcal{G}$ and define

$$\sigma=f_0\alpha+f_1\alpha_1+f_2\alpha_2+f_3\alpha_3+f_4\mu\mathrm{d}\mu. \tag{8}$$

Clearly, at least for sufficiently close values to the preferred, we obtain new G_2-structures. For the fixed orientation $\mathrm{Vol}_{\mathcal{G}}=e^{0\cdots6}$, induced by the Sasaki structure on TM and the vector field U, we have that on any two vectors v,w:

$$v\lrcorner\sigma\wedge w\lrcorner\sigma\wedge\sigma=6\langle v,w\rangle_\sigma\mathrm{Vol}_\sigma=6\langle v,w\rangle_{\sigma_0}m\mathrm{Vol}_{\mathcal{G}}. \tag{9}$$

This identity defines the scalar function $m>0$, already assumed to be positive—as we may without loss of regularity or significant generality.

Detailed computations of the metric matrix on the adapted frame yield

$$[\langle e_i,e_j\rangle_\sigma]=t\begin{bmatrix} f_4^2 & & & & & & \\ & x & & & z & & \\ & & x & & & z & \\ & & & x & & & z \\ & z & & & y & & \\ & & z & & & y & \\ & & & z & & & y \end{bmatrix} \tag{10}$$

where

$$t=\frac{f_4}{m},\qquad x=f_2^2-f_1f_3,\qquad y=f_1^2-f_0f_2,\qquad z=f_1f_2-f_0f_3. \tag{11}$$

Notice σ_0 corresponds to the identity 1_7. Computing determinants, the metric is positive-definite if $f_4>0$, $x>0$ and $xy-z^2>0$. This proves the following result.

Theorem 1.1. *If a set of scalar functions $f_0,\ldots,f_4$ induces a G_2 structure on $\mathcal{G}$, then it satisfies $f_4>0$, $f_2^2-f_1f_3>0$ and*

$$3f_0f_1f_2f_3-f_0f_2^3-f_0^2f_3^2-f_3f_1^3>0. \tag{12}$$

Remarks. 1. The homogeneous fourth degree polynomial is irreducible and has no critical values in the domain. 2. The metrics obtained are all natural metrics in the sense of [1, 2] and other references therein.

Now, by Gram-Schmidt process on the new metric, we obtain the direct orthonormal frame, $i = 1, 2, 3$,

$$\tilde{e}_0 = \frac{1}{f_4\sqrt{t}}e_0, \qquad \tilde{e}_i = \frac{1}{\sqrt{tx}}e_i, \qquad \tilde{e}_{i+3} = \sqrt{\frac{x}{th}}(e_{i+3} - \frac{z}{x}e_i), \tag{13}$$

where $h = xy - z^2$, the polynomial in (12). A dual co-frame is then

$$\tilde{e}^0 = f_4\sqrt{t}e^0, \qquad \tilde{e}^i = \sqrt{tx}e^i + z\sqrt{\frac{t}{x}}e^{i+3}, \qquad \tilde{e}^{i+3} = \sqrt{\frac{th}{x}}e^{i+3}. \tag{14}$$

We obtain also the useful formulas

$$e^0 = \frac{1}{f_4\sqrt{t}}\tilde{e}^0, \qquad e^i = \frac{1}{\sqrt{txh}}(\sqrt{h}\tilde{e}^i - z\tilde{e}^{i+3}), \qquad e^{i+3} = \sqrt{\frac{x}{th}}\tilde{e}^{i+3}. \tag{15}$$

Indeed the frame (13) is direct, ie. $\tilde{e}^{0123456}$ is a positive multiple of the chosen orientation. Immediately we find

$$m = f_4 h^{\frac{1}{3}}. \tag{16}$$

1.3 Exterior derivatives for σ preserving the Sasaki metric

Let σ be a variation of σ_0.

Proposition 1.1. *The metric induced by σ coincides with the Sasaki metric on $\mathcal{G}$ if and only if*

$$f_0^2 + f_1^2 = 1, \qquad f_2 = -f_0, \qquad f_3 = -f_1, \qquad f_4 = 1. \tag{17}$$

The orbit under $SO(7)$ of 3-forms which can be written in the form (8) is a circle S^1.

Proof. By hypothesis, we have $tf_4^2 = tx = ty = 1$ and $z = 0$. Hence $f_4^3 = f_4x = f_4y = m$ and $h = xy = f_4^4$. By (16) we get all these equal to 1, except z. Now solving the system (11) we deduce the equivalence in the first part of the result. The second follows from the orbit of $\sigma_0 = \alpha_2 - \alpha + \mu\mathrm{d}\mu$ intersected with our set of 3-forms, observed through typical methods. Indeed already $U(3) \subset SO(7)$ acts as a real group on the vector space $E = H_1 \oplus V_1$, which has a natural complex structure, and fixing e_0. We notice

$$(e^1 + \sqrt{-1}e^4)(e^2 + \sqrt{-1}e^5)(e^3 + \sqrt{-1}e^6) = \alpha_3 - \alpha_1 + \sqrt{-1}(\alpha_2 - \alpha) =: \eta \in \Lambda^3 E^{(1,0)^*}$$

As $SU(3) \subset G_2$ we only have to consider maps $g = \mathrm{e}^{is}1_E$ for $s \in \mathbb{R}$ (restricted to E). Immediately we see such g fixes the 3-form $\mu\mathrm{d}\mu = e^{041+052+063}$. Finally $g \cdot \eta = g^3\eta$. Letting g be such that $g^3 = f_0 + \sqrt{-1}f_1 \in S^1$ we find that the real map g solves

$$g \cdot \sigma_0 = -f_0\alpha - f_1\alpha_1 + f_0\alpha_2 + f_1\alpha_3 + \mu\mathrm{d}\mu.$$

The invariant statement follows (relevant due to $SO(7)/G_2$ being 7-dimensional). ∎

For the following computations we apply formulas which have been deduced in [4, 5, 6]. We start by the particular case found above, when the Sasaki metric is preserved. The manifold M is assumed connected.

Theorem 1.2. *Let* $\sigma=-f_0\alpha-f_1\alpha_1+f_0\alpha_2+f_1\alpha_3+\mu\mathrm{d}\mu$ *with* $(f_0,f_1):\mathcal{G}\rightarrow S^1$ *smooth.*
1. Always $\mathrm{d}\sigma\neq 0$.
2. If $(f_0,f_1)\neq(\pm1,0)$*, then* $\mathrm{d}*\sigma=0$ *if and only if the functions* f_0,f_1 *are constant and the Riemannian base* M *has constant sectional curvature.*
3. If $(f_0,f_1)=(\pm1,0)$*, then* $\mathrm{d}*\sigma=0$ *if and only if* M *is Einstein.*

The proof follows by recalling the list of derivatives of the fundamental 3-forms in (31), which were deduced in [4, Proposition 2.3]. Result (1) is the particular case of Theorem 1.3. For (2) we may easily compute $\mathrm{d}*\sigma$. If it is to vanish, then we deduce a curvature equation $R_{0123}=0$, which implies constant sectional curvature on the base, and that $f_0\mathrm{d}f_0=-f_1\mathrm{d}f_1$ is a multiple of μ, which implies (f_0,f_1) constant. Finally, if the base metric has constant sectional curvature k, then (see below) $\mathcal{R}^U\alpha=-k\mu\alpha_1$, and we find this is the solution required in case $f_1\neq 0$.

The Theorem shows that the original gwistor space structure we found, the preferred σ_0, has greater interest than the other on the circle (of course besides its antipodal, a duality which we shall not explore here).

We shall now see a result concerning the type of $\mathrm{d}\sigma$ with respect to the G_2-decomposition of $\Lambda^4T^*\mathcal{G}$, following the description by [10] reproduced in several good references.

Proposition 1.2. *The gwistor space* $(\mathcal{G},\sigma)$ *of a constant sectional curvature* k *manifold with* σ *given as before, with* f_0,f_1 *constant, is of pure type* W_3 *if and only if* $k=-2$.

Proof. Our always invoked Riemann tensor gives $R_{ijpq}=k(\delta_i^q\delta_j^p-\delta_i^p\delta_j^q)$ for constant sectional curvature metrics. By definitions in (32,33) below, we have $\mathcal{R}^U\alpha=-k\mu\alpha_1$, $\mathcal{R}^U\alpha_1=-2k\mu\alpha_2$. Now, we know $\mathrm{d}*\sigma=0$ and thence $\mathrm{d}\sigma=\lambda*\sigma+*\tau_3$, with τ_3 the so called W_3 part characterized by $\tau_3\sigma=\tau_3*\sigma=0$. The condition $\lambda=0\in\mathbb{R}$ resumes to $(\mathrm{d}\sigma)\sigma=0$ by a simple duality argument. Computing from the formulas and repeatedly using $f_0^2+f_1^2=1$, we find $k=-2$. ■

The following formula is used in the proof:

$$\mathrm{d}\sigma=\mu\big(-3f_1\alpha+f_0(k+2)\alpha_1+f_1(2k+1)\alpha_2-3f_0k\alpha_3\big)+(\mathrm{d}\mu)^2. \qquad (18)$$

The Proposition recovers, in particular, the result in [4, Corollary 3.1] for the preferred $\sigma_0=\alpha_2-\alpha+\mu\mathrm{d}\mu$ on hyperbolic space of curvature -2. However, the result now is independent of the pair $(f_0,f_1)\in S^1$, just as the result $\|\mathrm{d}\sigma\|^2=48$.

1.4 Exterior derivatives for σ in the general case

Suppose $(f_0,\ldots,f_4):\mathcal{G}\rightarrow\mathbb{R}^5$ is a vectorial function satisfying the conditions in Theorem 1.1. We study the possibly G_2-structure on $\mathcal{G}\rightarrow M$

$$\sigma=f_0\alpha+f_1\alpha_1+f_2\alpha_2+f_3\alpha_3+f_4\mu\mathrm{d}\mu. \qquad (19)$$

From the formulas in (15) we deduce

$$\mu=\frac{1}{f_4t^{\frac12}}\tilde{\mu},\qquad \mathrm{d}\mu=\frac{1}{th^{\frac12}}\widetilde{\mathrm{d}\mu},\qquad \alpha=\frac{x^{\frac32}}{(th)^{\frac32}}\tilde{\alpha}, \tag{20}$$

$$\alpha_1=\frac{x^{\frac12}}{t^{\frac32}h}\big(\tilde{\alpha}_1-\frac{z}{h^{\frac12}}\tilde{\alpha}\big),\qquad \alpha_2=\frac{1}{x^{\frac12}(th)^{\frac32}}(h\tilde{\alpha}_2-2h^{\frac12}z\tilde{\alpha}_1+3z^2\tilde{\alpha}), \tag{21}$$

$$\alpha_3=\frac{1}{(txh)^{\frac32}}\big(h^{\frac32}\tilde{\alpha}_3-hz\tilde{\alpha}_2+h^{\frac12}z^2\tilde{\alpha}_1-z^3\tilde{\alpha}\big). \tag{22}$$

The forms with a tilde are defined algebraically using the orthonormal basis for σ, formally introduced on the respective $\mu,\mathrm{d}\mu,\alpha,\ldots,\alpha_3$ (it is the $SO(3)$ structure of the tangent sphere bundle revealing itself). In particular, we may use the so called *first structure equations* from [4] but with a tilde. We also need the inversed formulas of the above:

$$\widetilde{\mu\mathrm{d}\mu}=f_4t^{\frac32}h^{\frac12}\mu\mathrm{d}\mu,\qquad \tilde{\alpha}=\frac{(th)^{\frac32}}{x^{\frac32}}\alpha, \tag{23}$$

$$\tilde{\alpha}_1=\frac{ht^{\frac32}}{x^{\frac32}}\big(x\alpha_1+3z\alpha\big),\qquad \tilde{\alpha}_2=\frac{h^{\frac12}t^{\frac32}}{x^{\frac32}}\big(x^2\alpha_2+2xz\alpha_1+3z^2\alpha\big), \tag{24}$$

$$\tilde{\alpha}_3=\frac{t^{\frac32}}{x^{\frac32}}\big(x^3\alpha_3+x^2z\alpha_2+xz^2\alpha_1+z^3\alpha\big). \tag{25}$$

Using the 'first structure equations' in [4, Proposition 2.1], but for the Hodge operator of the metric and orientation induced by σ, and writing back in terms of the usual frame, we obtain:

$$*_\sigma\,(\mu\mathrm{d}\mu)=\frac{t^{\frac12}h^{\frac12}}{2f_4}(\mathrm{d}\mu)^2, \tag{26}$$

$$*_\sigma\,\alpha=\frac{f_4t^{\frac12}}{h^{\frac32}}\mu\big(x^3\alpha_3+x^2z\alpha_2+xz^2\alpha_1+z^3\alpha\big), \tag{27}$$

$$*_\sigma\,\alpha_1=-\frac{f_4t^{\frac12}}{xh^{\frac32}}\mu\big(3x^3z\alpha_3+x^2(h+3z^2)\alpha_2+x(2hz+3z^3)\alpha_1+(3hz^2+3z^4)\alpha\big), \tag{28}$$

$$*_\sigma\alpha_2=\frac{f_4t^{\frac12}}{x^2h^{\frac32}}\mu\big(3x^3z^2\alpha_3+x^2(2hz+3z^3)\alpha_2+x(h^2+4hz^2+3z^4)\alpha_1+(3h^2z+6hz^3+3z^5)\alpha\big), \tag{29}$$

$$\begin{aligned}*_\sigma\alpha_3=-\frac{f_4t^{\frac12}}{x^3h^{\frac32}}\mu\big(&x^3z^3\alpha_3+x^2(hz^2+z^4)\alpha_2+\\ &+x(h^2z+2hz^3+z^5)\alpha_1+(h^3+3h^2z^2+3hz^4+z^6)\alpha\big).\end{aligned} \tag{30}$$

Now we recall the formulas from [4, Proposition 2.3]:

$$\mathrm{d}\alpha=\mathcal{R}^U\alpha,\qquad \mathrm{d}\alpha_1=3\mu\alpha+\mathcal{R}^U\alpha_1,\qquad \mathrm{d}\alpha_2=2\mu\alpha_1-\underline{r}\mathrm{vol},\qquad \mathrm{d}\alpha_3=\mu\alpha_2 \tag{31}$$

where $\mathcal{R}^U\alpha, \mathcal{R}^U\alpha_1$ are linearly independent forms depending on the curvature of M, and $\underline{r} = r(u,u)$ is a function on $\mathcal{G}$ (of course R and r are the usual Riemannian and Ricci curvature tensors). Concretely, cf. [4, formulas 25 and 26],

$$\mathcal{R}^U\alpha = \sum_{0\leq i<j\leq 3} R_{ij01}e^{ij56} + R_{ij02}e^{ij64} + R_{ij03}e^{ij45}, \tag{32}$$

$$\mathcal{R}^U\alpha_1 = \sum_{0\leq i<j\leq 3} R_{ij01}(e^{ij26}+e^{ij53}) + R_{ij02}(e^{ij61}+e^{ij34}) + R_{ij03}(e^{ij15}+e^{ij42}). \tag{33}$$

In particular $\mu\mathcal{R}^U\alpha_1 = -\rho\mathrm{vol}$ where $\rho = \sum_{i=1}^3 r(e_i,e_0)e^{i+3}$.

Theorem 1.3. *For any functions* $f_0,\ldots,f_4$*, we have* $\mathrm{d}\sigma \neq 0$.

Proof. Indeed, since $(\mathrm{d}\mu)\alpha_i = 0, \forall i = 0,1,2,3,\ \alpha_0 = \alpha$, a moments thought gives

$$\mu(\mathrm{d}\mu)\mathrm{d}\sigma = (6f_4 + f_0(R_{2301}+R_{3102}+R_{1203}))\mathrm{Vol}_{\mathcal{G}} = 6f_4\mathrm{Vol}_{\mathcal{G}}$$

by Bianchi identity. However, we saw f_4 must be positive. ∎

From now on we assume the functions $f_0,\ldots,f_4$ are constant.

A metric almost contact structure is said to be K-contact if the characteristic vector field is Killing. In the case of the Sasaki metric, $(\mathcal{G},\mu,\theta^tU)$ is K-contact if and only if M is locally isometric to S^4 of radius 1, a result due to Y. Tashiro. In general, since our metrics turn out to be natural metrics, we have the question in the larger setting solved in [1].

Another feature of gwistor theory is that never a G_2-structure varying from the usual is preserved by the vector field θ^tU (known both as the geodesic spray or the geodesic flow vector field, cf. [13, 14]). Indeed, computations for constant f_i have shown that $\mathcal{L}_{\theta^tU}\sigma \neq 0$.

Returning to the Hodge duals, then we have by simple reasons

$$\begin{aligned}
\mathrm{d}(*_\sigma(\mu\mathrm{d}\mu)) =&\ 0,\\
\mathrm{d}(*_\sigma\alpha) =&\ -\frac{f_4t^{\frac12}}{h^{\frac32}}\mu\big(xz^2\mathcal{R}^U\alpha_1 + z^3\mathcal{R}^U\alpha\big),\\
\mathrm{d}(*_\sigma\alpha_1) =&\ \frac{f_4t^{\frac12}}{xh^{\frac32}}\mu\big(x(2hz+3z^3)\mathcal{R}^U\alpha_1 + (3hz^2+3z^4)\mathcal{R}^U\alpha\big),\\
\mathrm{d}(*_\sigma\alpha_2) =&\ -\frac{f_4t^{\frac12}}{x^2h^{\frac32}}\mu\big(x(h^2+4hz^2+3z^4)\mathcal{R}^U\alpha_1 + (3h^2z+6hz^3+3z^5)\mathcal{R}^U\alpha\big),\\
\mathrm{d}(*_\sigma\alpha_3) =&\ \frac{f_4t^{\frac12}}{x^3h^{\frac32}}\mu\big(x(h^2z+2hz^3+z^5)\mathcal{R}^U\alpha_1 + (h^3+3h^2z^2+3hz^4+z^6)\mathcal{R}^U\alpha\big).
\end{aligned} \tag{34}$$

Hence the vanishing of the two polynomials

$$-f_0x^3z^2 + f_1x^2(2hz+3z^3) - f_2x(h^2+4hz^2+3z^4) + f_3(h^2z+2hz^3+z^5), \tag{35}$$

$$f_0x^3z^3 - f_1x^2(3hz^2+3z^4) + f_2x(3h^2z+6hz^3+3z^5) - f_3(h^3+3h^2z^2+3hz^4+z^6) \tag{36}$$

is a sufficient condition for the vanishing of $\mathrm{d}(*_\sigma\sigma)$. Multiplying the first by z, adding to the second and factoring out a $h(>0)$ from the result, we obtain:

$$-f_1x^2z^2+2f_2xhz+2f_2z^3x-f_3h^2-2f_3hz^2-f_3z^4. \tag{37}$$

Finally recurring to some computer algebra software, we are able to find two independent expressions in the original parameters $f_0,\ldots,f_3$:

$$-f_0\left(f_1^2-f_0f_2\right)\left(-f_2^2+f_1f_3\right)^2 \qquad (=(35)), \tag{38}$$

$$\begin{aligned}(f_2^2-f_1f_3)^3\big(&-2f_0f_1^3f_2^3+3f_0^2f_1f_2^4-f_1^6f_3+6f_0f_1^4f_2f_3-6f_0^2f_1^2f_2^2f_3\\ &-2f_0^3f_2^3f_3-3f_0^2f_1^3f_3^2+6f_0^3f_1f_2f_3^2-f_0^4f_3^3\big) \qquad (=(36)).\end{aligned} \tag{39}$$

Notice they are homogeneous, as expected, and notice the factor $y=f_1^2-f_0f_2$ in the second polynomial and the common factor $x=f_2^2-f_1f_3$, which must both be positive. From equivalence we get the simple expression

$$(f_1^3-2f_0f_1f_2+f_0^2f_3)(f_2^2-f_1f_3)^3 \quad (=(37)). \tag{40}$$

Theorem 1.4. *A 3-form σ as above, with $f_0,\ldots,f_4$ constant, satisfies $\mathrm{d}*_\sigma\sigma=0$ if and only if any of the following occurs:*
(i) the polynomial (39) vanishes and M is Einstein.
(ii) M has constant sectional curvature.

Proof. Notice first that, if $f_0=0$, then neither f_1 or f_3 can vanish (otherwise we would get $y=0$ or $h=0$ from definition). So the two main polynomials cannot vanish simultaneously, as we see directly, or from the implied equation (40).

Now, if the polynomial (39) vanishes, then we may conclude $f_0\neq0$, ie. the first polynomial (38) does not vanish. So the cocalibration equation is equivalent to the vanishing of $\mu\mathcal{R}^U\alpha_1=-\rho vol$, which happens if and only if M is Einstein. On the contrary, if the polynomial does not vanish, then the equation is on metrics such that $\mu\mathcal{R}^U\alpha=0$; equivalently, $R_{1201}=R_{2301}=0$, etc. This is the same as M having constant sectional curvature. In particular, being Einstein. ■

For example, if $f_0=0$, then we are certainly bound to the second case.

Noteworthy is the case when $f_1f_2=f_0f_3$ (or $z=0$), which generalizes Proposition 1.2.

A question put to the author by colleagues was: if we could always find, invariant of the metric on M, a natural G_2 structure which would be co-closed. The answer is no, because the two polynomials do not vanish altogether.

We thus stress the relevance of G_2 cocalibration goes much beyond the known cases and examples.

1.5 Nearly-parallel G_2-structures

Nearly-parallel G_2-structures on 7-dimensional manifolds are defined by $\delta\sigma = 0$ and $\mathrm{d}\sigma = c *_\sigma \sigma$ for some constant c. Clearly, if $c \neq 0$, the condition is simply the latter equation.

We consider a variation of the G_2 structure on $\mathcal{G}$, as in (19). In order to find a nearly-parallel structure σ, we may assume already it is cocalibrated ($c \neq 0$). We notice the Hodge $*$ operator is homogeneous of degree 1/3 on 3-forms seen as a map $\sigma \rightsquigarrow *_\sigma$ (the simplest way to see this is by (26), but from the definition will also do). Hence if we find a solution to the above in our subspace of $\sigma \in \Lambda^3_+$, we find a line of solutions: $\mathrm{d}(s\sigma) = \frac{c}{s^{\frac{1}{3}}} *_{s\sigma} s\sigma$, $s \in \mathbb{R}^+$.

We restrict here to the case $z = f_1f_2 - f_0f_3 = 0$, the less 'prohibitive' condition.

Theorem 1.5. *Under the previous condition, the only metric on an oriented Riemannian 4-manifold M for which a $(\mathcal{G}, \sigma)$ is nearly-parallel is the constant sectional curvature 1 metric. Then there are two classes of solutions, represented by the following two G_2-structures:*

$$\sigma_\pm = \pm\frac{\sqrt{2}}{2}(\alpha_2 - \alpha + \alpha_3 - \alpha_1) + \sqrt{\frac{3}{2}}\mu\mathrm{d}\mu, \tag{41}$$

both satisfying $\mathrm{d}\sigma = \sqrt{6} *_\sigma \sigma$.

Proof. Since we assume $z = 0$ and this is maintained on the line $\mathbb{R}^+\sigma$, there exists a positive multiple of σ such that (f_0, f_1) is in the unit circle. Then we easily deduce $x = y = 1$ and $f_2 = -f_0,\ f_3 = -f_1$. Hence $h = 1 = t$ and $m = f_4$, cf. (16).

From formulas (26...30) and the hypothesis of σ being nearly-parallel, we see the 4-form $\mathrm{d}\sigma$ is again $SO(3)$-invariant. Then we easily deduce the curvature restriction: it must be of the constant kind. The equation $\mathrm{d}\sigma = c *_\sigma \sigma$ is solved using those same formulas, with $z = 0$ proving a major advantage. Looking at components, we find a system (k is the sectional curvature)

$$\begin{cases} c = 2f_4 \\ f_0f_1 - kf_0^2 = 0 \\ 2f_0f_1k + f_0f_1 - 3f_1^2 = 0 \\ 3f_1 - 2f_0f_4^2 = 0 \\ 2f_0 + kf_0 - 2f_0f_4^2 = 0 \end{cases}.$$

This yields $f_0 = f_1$, which occurs twice in the circle; and $k = 1,\ f_4 = \sqrt{3/2},\ c = \sqrt{6}$. The given 3-forms satisfy the equation and are genuine G_2-structures. ■

Notice the metric on $\mathcal{G}$ is the same on both solutions. Now we recall the classification of nearly-parallel G_2 structures in [11]. The ones we got correspond to the Stiefel manifold $V_{5,2} = SO(5)/SO(3)$ in their Table 2, which is of course the unit tangent sphere bundle of S^4. The G_2 is constructed as a $U(1)$-bundle over the complex quadric $G_{5,2}$, the Grassmannian of 2-planes, with a Kähler-Einstein metric. The resulting nearly parallel G_2 is said to be Einstein-Sasakian for *some* homogeneous $SO(5)$-invariant metric. We have thus found just some more details of this case. It is also most interesting to see that our result gives a metric coinciding precisely with the Einstein metric on $V_{5,2}$ deduced in [2, Theorem 4]. It has Riemannian scalar curvature $\frac{63}{4}$, by a formula there.

References

[1] M. T. K. Abbassi and G. Calvaruso, *g-Natural Contact Metrics on Unit Tangent Sphere Bundles*, Monatsh. Math. 151 (2006), 89–109.

[2] M.T.K. Abbassi and O.Kowalski, *On Einstein Riemannian g-natural metrics on unit tangent sphere bundles*, Ann. Global. Anal. Geom. 38, 1 (2010), 11–20.

[3] I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Archi. Mathematicum (Brno), Tomus 42 (2006), Suppl., 5–84.

[4] R. Albuquerque, *On the G_2 bundle of a Riemannian 4-manifold*, J. Geom. Phys. 60 (2010), 924–939.

[5] R. Albuquerque and I. Salavessa, *The G_2 sphere of a 4-manifold*, Monatsh. Math. 158, Issue 4 (2009), 335–348.

[6] R. Albuquerque and I. Salavessa, *Erratum to: The G_2 sphere of a 4-manifold*, Monatsh. Math. 160, Issue 1 (2010), 109–110.

[7] D. Blair, *Riemannian geometry of contact and symplectic manifolds*, Progress in Mathematics, vol. 203, Birkhaüser Boston Inc., Boston, MA, 2002.

[8] R. L. Bryant, *Metrics with exceptional holonomy*, Annals of Mathematics (2), vol. 126 no. 3 (1987), 525–576.

[9] R. L. Bryant, *Some remarks on G_2 structures*, Proceedings of the 2004 Gokova Conference on Geometry and Topology (May, 2003).

[10] M. Fernández and A. Gray, *Riemannian manifolds with structure group G_2*, Ann. Mat. Pura Appl. 4 132 (1982), 19–45.

[11] Th. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On nearly parallel G_2-structures*, J. Geom. Phys. 23 (1997), 259–286.

[12] Th. Friedrich and S. Ivanov, *Killing spinor equations in dimension 7 and geometry of integrable G_2-manifolds*, J. Geom. Phys. 48 (2003), 1–11.

[13] H. Geiges, *An Introduction to Contact Topology*, Cambridge Studies in Advanced Mathematics 109, Cambridge University Press, 2008.

[14] T. Sakai, *Riemannian Geometry*, Transl. Math. Mono. Vol 149 AMS, 1996.