Thin Hessenberg Pairs and Double Vandermonde Matrices

Ali Godjali

July 26, 2011

Abstract

A square matrix is called *Hessenberg* whenever each entry below the subdiagonal is zero and each entry on the subdiagonal is nonzero. Let V denote a nonzero finite-dimensional vector space over a field $\mathbb{K}$. We consider an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ which satisfy both (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is Hessenberg and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is Hessenberg.

We call such a pair a *thin Hessenberg pair* (or *TH pair*). By the *diameter* of the pair we mean the dimension of V minus one. There is an "oriented" version of a TH pair called a TH system. In this paper we investigate a connection between TH systems and double Vandermonde matrices. We have two main results. For the first result we give a bijection between any two of the following three sets:

- The set of isomorphism classes of TH systems over $\mathbb{K}$ of diameter d.
- The set of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of parameter arrays over $\mathbb{K}$ of diameter d.

For the second result we give a bijection between any two of the following five sets:

- The set of affine isomorphism classes of TH systems over $\mathbb{K}$ of diameter d.
- The set of isomorphism classes of RTH systems over $\mathbb{K}$ of diameter d.
- The set of affine classes of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of normalized west-south Vandermonde matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of reduced parameter arrays over $\mathbb{K}$ of diameter d.

Keywords: Leonard pair, Hessenberg pair, Vandermonde matrix.
2010 Mathematics Subject Classification: 15A04.

1 Introduction

This paper is about a linear algebraic object called a thin Hessenberg pair [1]. To recall its definition, we will use the following term. A square matrix is called *Hessenberg* whenever each entry below the subdiagonal is zero and each entry on the subdiagonal is nonzero. Throughout the paper, $\mathbb{K}$ will denote a field.

Definition 1.1. [1, Definition 1.1] Let V denote a nonzero finite-dimensional vector space over $\mathbb{K}$. By a *thin Hessenberg pair* (or *TH pair*) on V, we mean an ordered pair of linear transformations $A : V \to V$ and $A^* : V \to V$ which satisfy both (i), (ii) below.

(i) There exists a basis for V with respect to which the matrix representing A is Hessenberg and the matrix representing A^* is diagonal.

(ii) There exists a basis for V with respect to which the matrix representing A is diagonal and the matrix representing A^* is Hessenberg.

We call V the *underlying* vector space and say that A, A^* is *over* $\mathbb{K}$. By the *diameter* of A, A^* we mean the dimension of V minus one.

Note 1.2. It is a common notational convention to use A^* to represent the conjugate-transpose of A. We are not using this convention. In a TH pair A, A^* the linear transformations A and A^* are arbitrary subject to (i), (ii) above.

A TH pair is a generalization of a Leonard pair [4]. Roughly speaking, a Leonard pair is a pair of linear transformations as in Definition 1.1, with the Hessenberg requirement replaced by an irreducible tridiagonal requirement. Leonard pairs have been extensively studied; for more information see [5] and the references therein.

In [1] we introduced the concept of a TH pair and began a systematic study of these objects. We now summarize the content of [1]. In [1, Definition 2.2] we introduced an "oriented" version of a TH pair called a TH system. A TH system is described as follows. Let A, A^* denote a TH pair on V of diameter d. By definition A is diagonalizable. It turns out that each eigenspace of A has dimension one [1, Lemma 2.1]. Therefore a basis from Definition 1.1(ii) induces an ordering $\{V_i\}_{i=0}^d$ of the eigenspaces of A. For $0 \leq i \leq d$, let E_i denote the primitive idempotent of A that corresponds to V_i. We call $\{E_i\}_{i=0}^d$ a standard ordering of the primitive idempotents of A. A standard ordering of the primitive idempotents of A^* is defined similarly. A TH system is a TH pair A, A^* together with a standard ordering of the primitive idempotents of A and a standard ordering of the primitive idempotents of A^*. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. In [1] we investigated six bases for V with respect to which the matrices representing A and A^* are attractive. We displayed these matrices along with the transition matrices relating the bases. We classified the TH systems up to isomorphism.

In the present paper, we continue our study of TH pairs and TH systems. Our focus is on a connection between TH systems and double Vandermonde matrices. We establish two main results. These results have the following form. In the first result we display three sets and show any two are in bijection. In the second result we display five sets and show any two are in bijection. We now describe the first result. To do this we display the three sets and then discuss the meaning. The three sets are:

- The set of isomorphism classes of TH systems over $\mathbb{K}$ of diameter d.
- The set of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of parameter arrays over $\mathbb{K}$ of diameter d.

We now describe the above three sets in more detail. The first set is clear, so consider the second set. For an indeterminate λ let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{K}$. Let $\{f_i\}_{i=0}^d$ denote a sequence of polynomials in $\mathbb{K}[\lambda]$. We say that $\{f_i\}_{i=0}^d$ is graded whenever $f_0 = 1$ and f_i has degree i for $0 \leq i \leq d$. By a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$ we mean a sequence $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ such that: (i) X is a matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$; (ii) $\{\theta_i\}_{i=0}^d$ is a sequence of mutually distinct scalars in $\mathbb{K}$; (iii) $\{\theta_i^*\}_{i=0}^d$ is a sequence of mutually distinct scalars in $\mathbb{K}$; (iv) there exists a graded sequence of polynomials $\{f_i\}_{i=0}^d$ in $\mathbb{K}[\lambda]$ such that $X_{ij} = f_j(\theta_i)$ for $0 \leq i, j \leq d$; (v) there exists a graded sequence of polynomials $\{f_i^*\}_{i=0}^d$ in $\mathbb{K}[\lambda]$ such that $X_{ij} = f_{d-i}^*(\theta_j^*)$ for $0 \leq i, j \leq d$. We now describe the third set. By a parameter array over $\mathbb{K}$ of diameter d we mean a sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ of scalars taken from $\mathbb{K}$ such that: (i) $\{\theta_i\}_{i=0}^d$ are mutually distinct; (ii) $\{\theta_i^*\}_{i=0}^d$ are mutually distinct; (iii) $\{\phi_i\}_{i=1}^d$ are all nonzero. We have now described the three sets. We now describe the bijections between these sets. We start by describing the bijection from the first set to the second set. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Associated with Φ is a certain matrix $\mathcal{P} \in \mathrm{Mat}_{d+1}(\mathbb{K})$. This is the transition matrix from a basis in Definition 1.1(ii) to a basis in Definition 1.1(i), where the bases are normalized so that each entry in the leftmost column and the bottom row of $\mathcal{P}$ is 1. For $0 \leq i \leq d$ let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) that corresponds to E_i (resp. E_i^*). Our bijection sends the isomorphism class of Φ to $(\mathcal{P}, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$. We now describe the bijection from the third set to the first set. Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ denote a parameter array over $\mathbb{K}$ of diameter d. Let A denote the lower bidiagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ with entries $A_{ii} = \theta_{d-i}$ for $0 \leq i \leq d$ and $A_{i,i-1} = \phi_i$ for $1 \leq i \leq d$. Let A^* denote the upper bidiagonal matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ with entries $A_{ii}^* = \theta_i^*$ for $0 \leq i \leq d$ and $A_{i-1,i}^* = 1$ for $1 \leq i \leq d$. Observe that $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is an ordering of the eigenvalues of A (resp. A^*). For $0 \leq i \leq d$ let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) that corresponds to θ_i (resp. θ_i^*). We show that $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a TH system. Our bijection sends $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ to the isomorphism class of Φ.

We now describe our second result, which is a variation on the first result. We mentioned above that the second result involves five sets. The five sets are:

- The set of affine isomorphism classes of TH systems over $\mathbb{K}$ of diameter d.
- The set of isomorphism classes of RTH systems over $\mathbb{K}$ of diameter d.
- The set of affine classes of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of normalized west-south Vandermonde matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of reduced parameter arrays over $\mathbb{K}$ of diameter d.

We now describe the above five sets in more detail. Throughout the description let $\alpha, \beta, \alpha^*, \beta^*$ denote scalars in $\mathbb{K}$ with α, α^* nonzero. We now describe the first set. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system over $\mathbb{K}$. Observe that the sequence $(\alpha A + \beta I; \{E_i\}_{i=0}^d; \alpha^* A^* + \beta^* I; \{E_i^*\}_{i=0}^d)$ is a TH system over $\mathbb{K}$, said to be an affine transformation of Φ. We now describe the second set. By an RTH system over $\mathbb{K}$ we mean the sequence $(\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ induced by a TH system $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ over $\mathbb{K}$. We now describe the third set. Let $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ denote a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. One checks that $(X, \{\alpha\theta_i + \beta\}_{i=0}^d, \{\alpha^*\theta_i^* + \beta^*\}_{i=0}^d)$ is a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. These two systems are said to be affine related. We now describe the fourth set. By a normalized west-south Vandermonde matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ we mean the matrix X induced by a normalized west-south Vandermonde system $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ in $\mathrm{Mat}_{d+1}(\mathbb{K})$. We now describe the fifth set. Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ denote a parameter array over $\mathbb{K}$. Observe that $(\{\alpha\theta_i + \beta\}_{i=0}^d, \{\alpha^*\theta_i^* + \beta^*\}_{i=0}^d, \{\alpha\alpha^*\phi_i\}_{i=1}^d)$ is a parameter array over $\mathbb{K}$. These two parameter arrays are said to be affine related. This affine relation is an equivalence relation; the equivalence classes are called reduced parameter arrays. We have now described the five sets. We omit the description of the bijections between these sets as they are not hard to guess.

This paper is organized as follows. In Sections 2, 3 we review some basic concepts regarding TH pairs and TH systems. In Section 4 we summarize the classification of TH systems given in [1]. In Section 5 we discuss affine transformations of a TH system. In Sections 6, 7 we discuss how a given TH system yields three more TH systems called the relatives. In Sections 8, 9 we discuss some scalars that are helpful in describing a given TH system. In Section 10 we use these scalars to describe the relatives of a given TH system. In Sections 11, 12 we discuss the transition matrix $\mathcal{P}$ and a related matrix P. In Section 13 we define the notion of a Vandermonde system. In Sections 14–16 we discuss the connection between Vandermonde systems, graded sequences of polynomials, and Hessenberg matrices. In Section 17 we discuss the double Vandermonde structure of the transition matrices $\mathcal{P}$ and P. Sections 18, 19 contain the main results of the paper.

2 Thin Hessenberg systems

In our study of a TH pair, it is often helpful to consider a closely related object called a TH system. Before defining this notion, we make some definitions and observations. For the rest of the paper, fix an integer $d \geq 0$. Let $\mathrm{Mat}_{d+1}(\mathbb{K})$ denote the $\mathbb{K}$-algebra consisting of the $(d+1) \times (d+1)$ matrices that have all entries in $\mathbb{K}$. We index the rows and columns by $0, 1, \ldots, d$. Let $\mathbb{K}^{d+1}$ denote the $\mathbb{K}$-vector space consisting of the $(d+1) \times 1$ matrices that have all entries in $\mathbb{K}$. We index the columns by $0, 1, \ldots, d$. Observe that $\mathrm{Mat}_{d+1}(\mathbb{K})$ acts on $\mathbb{K}^{d+1}$ by left multiplication. For the rest of the paper, fix a vector space V over $\mathbb{K}$ with dimension $d+1$. Let $\mathrm{End}(V)$ denote the $\mathbb{K}$-algebra consisting of the linear transformations from V to V. Suppose that $\{v_i\}_{i=0}^d$ is a basis for V. For $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ and $Y \in \mathrm{End}(V)$, we say that X *represents* Y *with respect to* $\{v_i\}_{i=0}^d$ whenever $Yv_j = \sum_{i=0}^d X_{ij} v_i$ for $0 \leq j \leq d$. For $A \in \mathrm{End}(V)$ and $W \subseteq V$, we call W an *eigenspace* of A whenever $W \neq 0$ and there exists $\theta \in \mathbb{K}$ such that $W = \{v \in V \mid Av = \theta v\}$. In this case θ is called the *eigenvalue* of A corresponding to W. We say that A is *diagonalizable* whenever V is spanned by the

eigenspaces of A. We say that A is *multiplicity-free* whenever A is diagonalizable and each eigenspace of A has dimension one.

Lemma 2.1. [1, Lemma 2.1] *Let A, A^* denote a TH pair on V. Then each of A, A^* is multiplicity-free.*

We recall a few more concepts from linear algebra. Let A denote a multiplicity-free element of $\mathrm{End}(V)$. Let $\{V_i\}_{i=0}^d$ denote an ordering of the eigenspaces of A and let $\{\theta_i\}_{i=0}^d$ denote the corresponding ordering of the eigenvalues of A. For $0 \leq i \leq d$, define $E_i \in \mathrm{End}(V)$ such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ $(0 \leq j \leq d)$. Here I denotes the identity of $\mathrm{End}(V)$. We call E_i the *primitive idempotent* of A corresponding to V_i (or θ_i). Observe that (i) $I = \sum_{i=0}^d E_i$; (ii) $E_iE_j = \delta_{i,j}E_i$ $(0 \leq i, j \leq d)$; (iii) $V_i = E_iV$ $(0 \leq i \leq d)$; (iv) $A = \sum_{i=0}^d \theta_i E_i$. Moreover

$$E_i = \prod_{\substack{0 \leq j \leq d \\ j \neq i}} \frac{A - \theta_j I}{\theta_i - \theta_j} \qquad (0 \leq i \leq d). \tag{1}$$

Note that each of $\{A^i\}_{i=0}^d$, $\{E_i\}_{i=0}^d$ is a basis for the $\mathbb{K}$-subalgebra of $\mathrm{End}(V)$ generated by A. Moreover $\prod_{i=0}^d (A - \theta_i I) = 0$.

We now define a TH system.

Definition 2.2. By a *thin Hessenberg system* (or *TH system*) on V we mean a sequence

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

which satisfies (i)–(v) below.

(i) Each of A, A^* is a multiplicity-free element of $\mathrm{End}(V)$.

(ii) $\{E_i\}_{i=0}^d$ is an ordering of the primitive idempotents of A.

(iii) $\{E_i^*\}_{i=0}^d$ is an ordering of the primitive idempotents of A^*.

(iv) $E_iA^*E_j = \begin{cases} 0, & \text{if } i - j > 1 \\ \neq 0, & \text{if } i - j = 1 \end{cases} \qquad (0 \leq i, j \leq d).$

(v) $E_i^*AE_j^* = \begin{cases} 0, & \text{if } i - j > 1 \\ \neq 0, & \text{if } i - j = 1 \end{cases} \qquad (0 \leq i, j \leq d).$

We refer to d as the *diameter* of Φ. We call V the *underlying* vector space and say that Φ is *over* $\mathbb{K}$.

We comment on how TH pairs and TH systems are related. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. For $0 \leq i \leq d$, let v_i (resp. v_i^*) denote a nonzero vector in E_iV (resp. E_i^*V). Then the sequence $\{v_i\}_{i=0}^d$ (resp. $\{v_i^*\}_{i=0}^d$) is a basis for V which satisfies Definition 1.1(ii) (resp. Definition 1.1(i)). Therefore the pair A, A^* is a TH pair on V. Conversely, let A, A^* denote a TH pair on V. Then each of A, A^* is multiplicity-free by Lemma 2.1. Let $\{v_i\}_{i=0}^d$ (resp. $\{v_i^*\}_{i=0}^d$) denote a basis for V which satisfies Definition 1.1(ii) (resp. Definition 1.1(i)). For $0 \leq i \leq d$, the vector v_i (resp. v_i^*) is an eigenvector for A (resp. A^*); let E_i (resp. E_i^*) denote the corresponding primitive idempotent. Then $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a TH system on V.

Definition 2.3. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Observe that A, A^* is a TH pair on V. We say that this pair is *associated* with Φ.

Remark 2.4. With reference to Definition 2.3, conceivably a given TH pair is associated with many TH systems.

We now recall several definitions and results on TH systems.

Definition 2.5. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. For $0 \leq i \leq d$, let θ_i (resp. θ_i^*) denote the eigenvalue of A (resp. A^*) corresponding to E_i (resp. E_i^*). We refer to $\{\theta_i\}_{i=0}^d$ as the *eigenvalue sequence* of Φ. We refer to $\{\theta_i^*\}_{i=0}^d$ as the *dual eigenvalue sequence* of Φ. We observe that $\{\theta_i\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{K}$. Similarly $\{\theta_i^*\}_{i=0}^d$ are mutually distinct and contained in $\mathbb{K}$.

Definition 2.6. Let A, A^* denote a TH pair. By an *eigenvalue sequence* (resp. *dual eigenvalue sequence*) of A, A^*, we mean the eigenvalue sequence (resp. dual eigenvalue sequence) of an associated TH system. We emphasize that a given TH pair could have many eigenvalue and dual eigenvalue sequences.

Let $\mathbb{K}[\lambda]$ denote the $\mathbb{K}$-algebra consisting of the polynomials in λ that have all coefficients in $\mathbb{K}$.

Notation 2.7. Let $\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d$ denote two sequences of scalars taken from $\mathbb{K}$. For $0 \leq i \leq d+1$, let τ_i, τ_i^*, η_i, η_i^* denote the following polynomials in $\mathbb{K}[\lambda]$.

$$\tau_i = \prod_{h=0}^{i-1}(\lambda - \theta_h), \qquad \eta_i = \prod_{h=0}^{i-1}(\lambda - \theta_{d-h}),$$
$$\tau_i^* = \prod_{h=0}^{i-1}(\lambda - \theta_h^*), \qquad \eta_i^* = \prod_{h=0}^{i-1}(\lambda - \theta_{d-h}^*).$$

We observe that each of τ_i, η_i, τ_i^*, η_i^* is monic with degree i.

By (1), for $0 \leq i \leq d$

$$E_i = \frac{\tau_i(A)\eta_{d-i}(A)}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)}, \qquad E_i^* = \frac{\tau_i^*(A^*)\eta_{d-i}^*(A^*)}{\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)}. \tag{2}$$

By a *decomposition* of V we mean a sequence $\{U_i\}_{i=0}^d$ consisting of one-dimensional subspaces of V such that

$$V = U_0 + U_1 + \cdots + U_d \qquad \text{(direct sum)}.$$

For notational convenience, set $U_{-1} = 0$ and $U_{d+1} = 0$.

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Then $\{E_i^*V\}_{i=0}^d$ is a decomposition of V, said to be Φ-*standard*. Let $0 \neq \xi_0 \in E_0V$. The sequence $\{E_i^*\xi_0\}_{i=0}^d$ is a basis for V [1, Lemma 8.1], said to be Φ-*standard*. We recall another decomposition of V associated with Φ. For $0 \leq i \leq d$, let

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_0V + E_1V + \cdots + E_{d-i}V). \tag{3}$$

The sequence $\{U_i\}_{i=0}^d$ is a decomposition of V [1, Section 4], said to be Φ-*split*. Moreover for $0 \leq i \leq d$, both

$$(A - \theta_{d-i}I)U_i = U_{i+1}, \tag{4}$$
$$(A^* - \theta_i^*I)U_i = U_{i-1}. \tag{5}$$

Setting $i = d$ in (3) we find $U_d = E_0V$. Combining this with (5) we find

$$U_i = \eta_{d-i}^*(A^*)E_0V \qquad (0 \leq i \leq d). \tag{6}$$

Recall $0 \neq \xi_0 \in E_0V$. From (6) we find that for $0 \leq i \leq d$, the vector $\eta_{d-i}^*(A^*)\xi_0$ is a basis for U_i. By this and since $\{U_i\}_{i=0}^d$ is a decomposition of V, the sequence

$$\eta_{d-i}^*(A^*)\xi_0 \qquad (0 \leq i \leq d) \tag{7}$$

is a basis for V, said to be Φ-*split*. Let $1 \leq i \leq d$. By (5) we have $(A^* - \theta_i^*I)U_i = U_{i-1}$, and by (4) we have $(A - \theta_{d-i+1}I)U_{i-1} = U_i$. Therefore U_i is invariant under $(A - \theta_{d-i+1}I)(A^* - \theta_i^*I)$ and the corresponding eigenvalue is a nonzero element of $\mathbb{K}$. We denote this eigenvalue by ϕ_i. We call the sequence $\{\phi_i\}_{i=1}^d$ the *split sequence* of Φ. For notational convenience, set $\phi_0 = 0$ and $\phi_{d+1} = 0$.

Proposition 2.8. [1, Proposition 4.4] *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V with eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$, and split sequence $\{\phi_i\}_{i=1}^d$. Then the matrices representing A and A^* with respect to a Φ-split basis for V are*

$$\begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ \phi_1 & \theta_{d-1} & & & & \\ & \phi_2 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & \phi_d & \theta_0 \end{pmatrix}, \qquad \begin{pmatrix} \theta_0^* & 1 & & & & \mathbf{0} \\ & \theta_1^* & 1 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix} \tag{8}$$

respectively.

Next we describe the matrices representing the primitive idempotents of A, A^* with respect to a Φ-split basis for V.

Proposition 2.9. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V with eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$, and split sequence $\{\phi_i\}_{i=1}^d$. For $0 \leq r \leq d$, consider the matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$ that represent E_r and E_r^* with respect to a Φ-split basis. For $0 \leq i, j \leq d$, their (i,j)-entry is described as follows. For E_r this entry is*

$$\frac{\phi_1\phi_2\cdots\phi_i}{\phi_1\phi_2\cdots\phi_j} \frac{\tau_{d-i}(\theta_r)\eta_j(\theta_r)}{\tau_r(\theta_r)\eta_{d-r}(\theta_r)}, \tag{9}$$

and for E_r^* this entry is

$$\frac{\tau_i^*(\theta_r^*)\eta_{d-j}^*(\theta_r^*)}{\tau_r^*(\theta_r^*)\eta_{d-r}^*(\theta_r^*)}. \tag{10}$$

Proof: Fix a Φ-split basis for V. For notational convenience, identify each element of $\mathrm{End}(V)$ with the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ that represents it with respect to this basis. We first show that the (i,j)-entry of E_r^* is given by (10). Computing the (i,j)-entry of $A^*E_r^* = \theta_r^* E_r^*$ using matrix multiplication, and taking into account the form of A^* in (8), we find

$$(E_r^*)_{i+1,j} = (\theta_r^* - \theta_i^*)(E_r^*)_{ij}$$

if $i \leq d-1$. Replacing i by $i-1$ in the above line, we find

$$(E_r^*)_{ij} = (\theta_r^* - \theta_{i-1}^*)(E_r^*)_{i-1,j} \tag{11}$$

if $i \geq 1$. Using the recursion (11), we routinely find

$$\begin{aligned} (E_r^*)_{ij} &= (\theta_r^* - \theta_{i-1}^*)(\theta_r^* - \theta_{i-2}^*)\cdots(\theta_r^* - \theta_0^*)(E_r^*)_{0j} \\ &= \tau_i^*(\theta_r^*)(E_r^*)_{0j}. \end{aligned} \tag{12}$$

Computing the $(0,j)$-entry of $E_r^*A^* = \theta_r^* E_r^*$ using matrix multiplication, and taking into account the form of A^*, we find

$$(E_r^*)_{0,j-1} = (\theta_r^* - \theta_j^*)(E_r^*)_{0j}$$

if $j \geq 1$. Replacing j by $j+1$ in the above line we find

$$(E_r^*)_{0j} = (\theta_r^* - \theta_{j+1}^*)(E_r^*)_{0,j+1} \tag{13}$$

if $j \leq d-1$. Using the recursion (13), we routinely find

$$\begin{aligned} (E_r^*)_{0j} &= (\theta_r^* - \theta_{j+1}^*)(\theta_r^* - \theta_{j+2}^*)\cdots(\theta_r^* - \theta_d^*)(E_r^*)_{0d} \\ &= \eta_{d-j}^*(\theta_r^*)(E_r^*)_{0d}. \end{aligned} \tag{14}$$

Combining (12), (14), we find

$$(E_r^*)_{ij} = \tau_i^*(\theta_r^*)\eta_{d-j}^*(\theta_r^*)c, \tag{15}$$

where we abbreviate $c = (E_r^*)_{0d}$. We now find c. Since A^* is upper triangular, and since E_r^* is a polynomial in A^*, we see E_r^* is upper triangular. Recall $E_r^{*2} = E_r^*$, so the diagonal entry of $(E_r^*)_{rr}$ equals 0 or 1. We show $(E_r^*)_{rr} = 1$. Setting $i = r$, $j = r$ in (15),

$$(E_r^*)_{rr} = \tau_r^*(\theta_r^*)\eta_{d-r}^*(\theta_r^*)c. \tag{16}$$

Observe $\tau_r^*(\theta_r^*) \neq 0$ and $\eta_{d-r}^*(\theta_r^*) \neq 0$ by Notation 2.7, and since $\{\theta_i^*\}_{i=0}^d$ are distinct. Observe $c \neq 0$; otherwise $E_r^* = 0$ in view of (15). Apparently the right side of (16) is not 0, so $(E_r^*)_{rr} \neq 0$, and we conclude $(E_r^*)_{rr} = 1$. Setting $(E_r^*)_{rr} = 1$ in (16), solving for c, and evaluating (15) using the result, we find the (i,j)-entry of E_r^* is given by (10).
We now show that the (i,j)-entry of E_r is given by (9). Let $G \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the

diagonal matrix with (i,i)-entry $\phi_1\phi_2\cdots\phi_i$ for $0 \le i \le d$ and set $A' := GA^tG^{-1}$, where A is the matrix on the left of (8). The matrix A' is equal to

$$\begin{pmatrix} \theta_d & 1 & & & \mathbf{0} \\ & \theta_{d-1} & 1 & & \\ & & \theta_{d-2} & \cdot & \\ & & & \cdot & \cdot \\ & & & & \cdot \quad 1 \\ \mathbf{0} & & & & \theta_0 \end{pmatrix} \tag{17}$$

Let E'_r denote the primitive idempotent of A' associated with the eigenvalue θ_r. We find E'_r in two ways. On one hand, applying (10) to A', we find E'_r has (i,j)-entry

$$\frac{\tau_{d-j}(\theta_r)\eta_i(\theta_r)}{\tau_r(\theta_r)\eta_{d-r}(\theta_r)} \tag{18}$$

for $0 \le i,j \le d$. On the other hand, by elementary linear algebra

$$E'_r = GE_r^tG^{-1},$$

so E'_r has (i,j)-entry

$$G_{ii}(E_r)_{ji}G_{jj}^{-1} = \frac{\phi_1\phi_2\cdots\phi_i}{\phi_1\phi_2\cdots\phi_j}(E_r)_{ji} \tag{19}$$

for $0 \le i,j \le d$. Equating (18) and the right side of (19), and solving for $(E_r)_{ji}$, we routinely obtain the result. □

Example 2.10. Referring to Proposition 2.9, assume $d = 2$. With respect to a Φ-split basis, the matrices representing E_0, E_1, E_2 are

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\phi_1\phi_2}{(\theta_0-\theta_2)(\theta_0-\theta_1)} & \frac{\phi_2}{\theta_0-\theta_1} & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ \frac{\phi_1}{\theta_1-\theta_2} & 1 & 0 \\ \frac{\phi_1\phi_2}{(\theta_1-\theta_0)(\theta_1-\theta_2)} & \frac{\phi_2}{\theta_1-\theta_0} & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ \frac{\phi_1}{\theta_2-\theta_1} & 0 & 0 \\ \frac{\phi_1\phi_2}{(\theta_2-\theta_0)(\theta_2-\theta_1)} & 0 & 0 \end{pmatrix},$$

respectively. Moreover the matrices representing E_0^*, E_1^*, E_2^* are

$$\begin{pmatrix} 1 & \frac{1}{\theta_0^*-\theta_1^*} & \frac{1}{(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{\theta_1^*-\theta_0^*} & \frac{1}{(\theta_1^*-\theta_0^*)(\theta_1^*-\theta_2^*)} \\ 0 & 1 & \frac{1}{\theta_1^*-\theta_2^*} \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & \frac{1}{(\theta_2^*-\theta_1^*)(\theta_2^*-\theta_0^*)} \\ 0 & 0 & \frac{1}{\theta_2^*-\theta_1^*} \\ 0 & 0 & 1 \end{pmatrix},$$

respectively.

We now give some characterizations of the split sequence.

Lemma 2.11. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$, and split sequence $\{\phi_i\}_{i=1}^d$. Then*

$$E_0^*\eta_i(A)E_0^* = \frac{\phi_1\phi_2\cdots\phi_i}{(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_i^*)}E_0^* \qquad (0 \le i \le d). \tag{20}$$

Proof: Let Φ denote the TH system in question and assume V is the underlying vector space. Let $\{U_i\}_{i=0}^d$ denote the Φ-split decomposition of V. Setting $i=0$ in (3) we find $U_0 = E_0^* V$. By this and (4), (5) we obtain

$$(A^* - \theta_1^* I)(A^* - \theta_2^* I)\cdots(A^* - \theta_i^* I)\eta_i(A) = \phi_1\phi_2\cdots\phi_i I \tag{21}$$

on $E_0^* V$. To obtain (20), multiply both sides of (21) on the left by E_0^* and use $E_0^* A^* = \theta_0^* E_0^*$. □

Corollary 2.12. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$, and split sequence $\{\phi_i\}_{i=1}^d$. Then for $0 \le i \le d$,*

$$\phi_1\phi_2\cdots\phi_i = (\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*)\cdots(\theta_0^* - \theta_i^*)\mathrm{trace}(\eta_i(A)E_0^*). \tag{22}$$

Moreover $\eta_i(A)E_0^*$ has nonzero trace.

Proof: To obtain (22), in (20) take the trace of each side and simplify the result using the fact that $\mathrm{trace}(E_0^*) = 1$ and $\mathrm{trace}(E_0^*\eta_i(A)E_0^*) = \mathrm{trace}(\eta_i(A)E_0^*E_0^*) = \mathrm{trace}(\eta_i(A)E_0^*)$. This gives (22). The last assertion follows since $\phi_i \neq 0$ for $1 \le i \le d$. □

Corollary 2.13. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$, dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$, and split sequence $\{\phi_i\}_{i=1}^d$. Then*

$$\phi_i = (\theta_0^* - \theta_i^*)\mathrm{trace}(\eta_i(A)E_0^*)/\mathrm{trace}(\eta_{i-1}(A)E_0^*) \qquad (1 \le i \le d). \tag{23}$$

Proof: Routine by Corollary 2.12. □

In Section 7 we give some more characterizations of the split sequence.

3 Isomorphisms for TH pairs and TH systems

In this section we discuss the notion of isomorphism for TH pairs and TH systems.

Lemma 3.1. *For $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ the following* (i)–(iii) *are equivalent.*

(i) *X is diagonal.*

(ii) *$DX = XD$ for all diagonal $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$.*

(iii) *There exists a diagonal $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ that has mutually distinct diagonal entries and $DX = XD$.*

Proof: (i) ⇒ (ii) Clear.
(ii) ⇒ (iii) Clear.
(iii) ⇒ (i) For $0 \le i, j \le d$ with $i \neq j$, we show $X_{ij} = 0$. Comparing the (i,j)-entry of DX and XD, we find $D_{ii}X_{ij} = X_{ij}D_{jj}$. By assumption $D_{ii} \neq D_{jj}$, so $X_{ij} = 0$. □

Let A, A^* denote a TH pair on V. In general, $\mathrm{End}(V)$ may not be generated by A, A^*. Moreover there may exist a subspace W of V such that $AW \subseteq W, A^*W \subseteq W, W \neq 0, W \neq V$. However we do have the following result.

Lemma 3.2. *Let A, A^* denote a TH pair on V. Let Δ denote an element of* $\mathrm{End}(V)$ *such that $\Delta A = A\Delta$ and $\Delta A^* = A^*\Delta$. Then $\Delta \in \mathbb{K}I$.*

Proof: Pick a basis for V from Definition 1.1(i). For notational convenience, identify each element of $\mathrm{End}(V)$ with the matrix that represents it with respect to this basis. Thus the matrix A is Hessenberg and the matrix A^* is diagonal. Moreover the diagonal entries of A^* are mutually distinct by Lemma 2.1. Applying Lemma 3.1 with $D = A^*$ and $X = \Delta$, we find Δ is diagonal. For $1 \leq i \leq d$, comparing the $(i, i-1)$-entry of ΔA and $A\Delta$, we find $\Delta_{ii}A_{i,i-1} = A_{i,i-1}\Delta_{i-1,i-1}$. Observe that $A_{i,i-1} \neq 0$ since A is Hessenberg, so $\Delta_{ii} = \Delta_{i-1,i-1}$. Therefore Δ_{ii} is independent of i for $0 \leq i \leq d$. Consequently $\Delta \in \mathbb{K}I$. □

For the rest of this section, let W denote a vector space over $\mathbb{K}$ with dimension $d+1$. Let $\Gamma : V \to W$ denote a $\mathbb{K}$-vector space isomorphism. Then there exists a unique $\mathbb{K}$-algebra isomorphism $\gamma : \mathrm{End}(V) \to \mathrm{End}(W)$ such that $S^\gamma = \Gamma S \Gamma^{-1}$ for all $S \in \mathrm{End}(V)$. Conversely let $\gamma : \mathrm{End}(V) \to \mathrm{End}(W)$ denote a $\mathbb{K}$-algebra isomorphism. By the Skolem-Noether theorem [2, Corollary 9.122] there exists a $\mathbb{K}$-vector space isomorphism $\Gamma : V \to W$ such that $S^\gamma = \Gamma S \Gamma^{-1}$ for all $S \in \mathrm{End}(V)$. Moreover Γ is unique up to multiplication by a nonzero scalar in $\mathbb{K}$.

Definition 3.3. Let A, A^* denote a TH pair on V and let B, B^* denote a TH pair on W. By an *isomorphism of TH pairs* from A, A^* to B, B^* we mean a $\mathbb{K}$-algebra isomorphism $\gamma : \mathrm{End}(V) \to \mathrm{End}(W)$ such that $B = A^\gamma$ and $B^* = A^{*\gamma}$. We say that the TH pairs A, A^* and B, B^* are *isomorphic* whenever there exists an isomorphism of TH pairs from A, A^* to B, B^*.

Lemma 3.4. *Let A, A^* and B, B^* denote isomorphic TH pairs over $\mathbb{K}$. Then the isomorphism of TH pairs from A, A^* to B, B^* is unique.*

Proof: Let γ and γ' denote isomorphisms of TH pairs from A, A^* to B, B^*. We show that $\gamma = \gamma'$. By the comments above Definition 3.3, there exists a $\mathbb{K}$-vector space isomorphism $\Gamma : V \to W$ (resp. $\Gamma' : V \to W$) such that $S^\gamma = \Gamma S \Gamma^{-1}$ (resp. $S^{\gamma'} = \Gamma' S \Gamma'^{-1}$) for all $S \in \mathrm{End}(V)$. Consider the composition $\Delta = \Gamma^{-1}\Gamma'$. Observe that Δ is an invertible element of $\mathrm{End}(V)$. By construction, $\Delta A = A\Delta$ and $\Delta A^* = A^*\Delta$. Therefore $\Delta \in \mathbb{K}I$ by Lemma 3.2. By these comments, there exists $0 \neq \alpha \in \mathbb{K}$ such that $\Delta = \alpha I$. Hence $\Gamma' = \alpha\Gamma$, so $\gamma = \gamma'$. □

Definition 3.5. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V and let $\Psi = (B; \{F_i\}_{i=0}^d; B^*; \{F_i^*\}_{i=0}^d)$ denote a TH system on W. By an *isomorphism of TH systems* from Φ to Ψ we mean a $\mathbb{K}$-algebra isomorphism $\gamma : \mathrm{End}(V) \to \mathrm{End}(W)$ such that

$$B = A^\gamma, \qquad B^* = A^{*\gamma}, \qquad F_i = E_i^\gamma, \qquad F_i^* = E_i^{*\gamma} \quad (0 \leq i \leq d).$$

We say that the TH systems Φ and Ψ are *isomorphic* whenever there exists an isomorphism of TH systems from Φ to Ψ.

Lemma 3.6. *Let Φ and Ψ denote isomorphic TH systems over $\mathbb{K}$. Then the isomorphism of TH systems from Φ to Ψ is unique.*

Proof: Similar to the proof of Lemma 3.4. □

We give another interpretation of isomorphism for TH pairs and TH systems.

Lemma 3.7. *Let A, A^* denote a TH pair on V and let B, B^* denote a TH pair on W. Then the following* (i), (ii) *are equivalent.*

(i) *The TH pairs A, A^* and B, B^* are isomorphic.*

(ii) *There exists a $\mathbb{K}$-vector space isomorphism $\Gamma : V \to W$ such that $B\Gamma = \Gamma A$ and $B^*\Gamma = \Gamma A^*$.*

Moreover assume (i), (ii) *hold. Then Γ is unique up to a multiplication by a nonzero scalar in $\mathbb{K}$.*

Lemma 3.8. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V and let $\Psi = (B; \{F_i\}_{i=0}^d; B^*; \{F_i^*\}_{i=0}^d)$ denote a TH system on W. Then the following* (i), (ii) *are equivalent.*

(i) *The TH systems Φ and Ψ are isomorphic.*

(ii) *There exists a $\mathbb{K}$-vector space isomorphism $\Gamma : V \to W$ such that*

$$B\Gamma = \Gamma A, \qquad B^*\Gamma = \Gamma A^*, \qquad F_i\Gamma = \Gamma E_i, \qquad F_i^*\Gamma = \Gamma E_i^* \quad (0 \leq i \leq d).$$

Moreover assume (i), (ii) *hold. Then Γ is unique up to a multiplication by a nonzero scalar in $\mathbb{K}$.*

4 The classification of TH systems

In [1] we classified the TH systems up to isomorphism. We recall the result in this section.

Definition 4.1. Let Φ denote a TH system. By the *parameter array* of Φ we mean the sequence $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$, where $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$) is the eigenvalue (resp. dual eigenvalue) sequence of Φ and $\{\phi_i\}_{i=1}^d$ is the split sequence of Φ.

Theorem 4.2. [1, Theorem 6.3] *Let*

$$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d) \tag{24}$$

denote a sequence of scalars taken from $\mathbb{K}$. Then there exists a TH system Φ over $\mathbb{K}$ with parameter array (24) if and only if (i)–(iii) *hold below.*

(i) $\theta_i \neq \theta_j$ *if* $i \neq j$ $(0 \leq i, j \leq d)$.

(ii) $\theta_i^* \neq \theta_j^*$ *if* $i \neq j$ $(0 \leq i, j \leq d)$.

(iii) $\phi_i \neq 0$ $(1 \leq i \leq d)$.

Moreover assume (i)–(iii) *hold. Then Φ is unique up to isomorphism of TH systems.*

Definition 4.3. By a *parameter array over $\mathbb{K}$ of diameter d* we mean a sequence of scalars $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ taken from $\mathbb{K}$ that satisfies conditions (i)–(iii) of Theorem 4.2.

Corollary 4.4. *The map which sends a given TH system to its parameter array induces a bijection from the set of isomorphism classes of TH systems over $\mathbb{K}$ of diameter d, to the set of parameter arrays over $\mathbb{K}$ of diameter d.*

Proof: Immediate from Theorem 4.2. □

To illuminate the bijection in Corollary 4.4 we now describe its inverse in concrete terms. Let π denote the bijection in Corollary 4.4.

Proposition 4.5. *Let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ denote a parameter array over $\mathbb{K}$ of diameter d. Let A (resp. A^*) denote the matrix on the left (resp. right) in (8). Observe that A (resp. A^*) is multiplicity-free with eigenvalues $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i^*\}_{i=0}^d$). For $0 \leq i \leq d$ let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) that corresponds to θ_i (resp. θ_i^*). Then $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a TH system over $\mathbb{K}$. Moreover π^{-1} sends $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ to the isomorphism class of Φ.*

Proof: This is proven as part of the proof of [1, Theorem 6.3]. □

5 The affine transformations of a TH system

A given TH system can be modified in several ways to get a new TH system. In this section we describe one way. In the next section we describe another way.

Lemma 5.1. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Let $\alpha, \beta, \alpha^*, \beta^*$ denote scalars in $\mathbb{K}$ with α, α^* nonzero. Then the sequence*

$$(\alpha A + \beta I; \{E_i\}_{i=0}^d; \alpha^* A^* + \beta^* I; \{E_i^*\}_{i=0}^d) \tag{25}$$

is a TH system on V.

Proof: Routine. □

Definition 5.2. Referring to Lemma 5.1, we call the TH system (25) the *affine transformation of Φ associated with $\alpha, \beta, \alpha^*, \beta^*$*.

Definition 5.3. Let Φ and Φ' denote TH systems over $\mathbb{K}$. We say that Φ and Φ' are *affine isomorphic* whenever Φ is isomorphic to an affine transformation of Φ'. Observe that affine isomorphism is an equivalence relation.

Lemma 5.4. *With reference to Lemma 5.1, let $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ denote the parameter array of Φ. Then the parameter array of the TH system (25) is $(\{\alpha\theta_i + \beta\}_{i=0}^d, \{\alpha^*\theta_i^* + \beta^*\}_{i=0}^d, \{\alpha\alpha^*\phi_i\}_{i=1}^d)$.*

Proof: Let Φ' denote the TH system (25). By Definition 2.5, for $0 \leq i \leq d$ the scalar θ_i is the eigenvalue of A associated with E_i, so $\alpha\theta_i + \beta$ is the eigenvalue of $\alpha A + \beta I$ associated with E_i. Thus $\{\alpha\theta_i + \beta\}_{i=0}^d$ is the eigenvalue sequence of Φ'. Similarly $\{\alpha^*\theta_i^* + \beta^*\}_{i=0}^d$ is the dual eigenvalue sequence of Φ'. In (23), if we replace A by $\alpha A + \beta I$ and replace θ_j (resp. θ_j^*) by $\alpha\theta_j + \beta$ (resp. $\alpha^*\theta_j^* + \beta^*$) for $0 \leq j \leq d$, then the left-hand side becomes $\alpha\alpha^*\phi_i$. Therefore $\{\alpha\alpha^*\phi_i\}_{i=1}^d$ is the split sequence of Φ'. □

6 The relatives of a TH system

Let Φ denote a TH system. In the previous section we modified Φ in a certain way to get another TH system. In this section we modify Φ in a different way to obtain two more TH systems. These TH systems are called Φ^* and $\tilde{\Phi}$. We start with Φ^*.

Definition 6.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Observe that $(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$ is a TH system on V, which we denote by Φ^*.

Lemma 6.2. [1, Lemma 6.4] *Let Φ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Then the TH system Φ^* has parameter array $(\{\theta_i^*\}_{i=0}^d, \{\theta_i\}_{i=0}^d, \{\phi_{d-i+1}\}_{i=1}^d)$.*

We now consider $\tilde{\Phi}$. For the rest of this section, let W denote a vector space over $\mathbb{K}$ with dimension $d+1$. For $\mathbb{K}$-algebras $\mathcal{A}$ and $\mathcal{A}'$, by a *$\mathbb{K}$-algebra anti-isomorphism* from $\mathcal{A}$ to $\mathcal{A}'$ we mean a $\mathbb{K}$-vector space isomorphism $\dagger : \mathcal{A} \to \mathcal{A}'$ such that $(RS)^\dagger = S^\dagger R^\dagger$ for all $R, S \in \mathcal{A}$. By a *$\mathbb{K}$-algebra anti-automorphism* of $\mathcal{A}$ we mean a $\mathbb{K}$-algebra anti-isomorphism from $\mathcal{A}$ to $\mathcal{A}$. The anti-automorphisms of $\mathrm{Mat}_{d+1}(\mathbb{K})$ are described as follows. Let R denote an invertible element of $\mathrm{Mat}_{d+1}(\mathbb{K})$. Then there exists a unique $\mathbb{K}$-algebra anti-automorphism $\dagger$ of $\mathrm{Mat}_{d+1}(\mathbb{K})$ such that $S^\dagger = RS^tR^{-1}$ for all $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Conversely, let $\dagger$ denote a $\mathbb{K}$-algebra anti-automorphism of $\mathrm{Mat}_{d+1}(\mathbb{K})$. By the Skolem-Noether theorem [2, Corollary 9.122], there exists an invertible $R \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $S^\dagger = RS^tR^{-1}$ for all $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Moreover R is unique up to a multiplication by a nonzero scalar in $\mathbb{K}$.

Define $Z \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $Z_{ij} = \delta_{i+j,d}$ for $0 \leq i, j \leq d$. Observe that $Z^{-1} = Z$. Define ς to be the $\mathbb{K}$-algebra anti-automorphism of $\mathrm{Mat}_{d+1}(\mathbb{K})$ such that $S^\varsigma = ZS^tZ$ for all $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$. For $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$, S^ς is obtained from S by reflecting about the diagonal connecting the top right corner of S and the bottom left corner of S. In other words, $(S^\varsigma)_{ij} = S_{d-j,d-i}$ for $0 \leq i, j \leq d$. For example,

$$S = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \qquad S^\varsigma = \begin{pmatrix} 9 & 6 & 3 \\ 8 & 5 & 2 \\ 7 & 4 & 1 \end{pmatrix}.$$

Observe that $(S^\varsigma)^\varsigma = S$ for all $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Note that if $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ is Hessenberg then H^ς is Hessenberg.

Lemma 6.3. *Let A denote a multiplicity-free element of $\mathrm{End}(V)$ with eigenvalues $\{\theta_i\}_{i=0}^d$. For $0 \leq i \leq d$, let $E_i \in \mathrm{End}(V)$ denote the primitive idempotent of A corresponding to θ_i. For any anti-isomorphism $\dagger : \mathrm{End}(V) \to \mathrm{End}(W)$, the following* (i), (ii) *hold.*

(i) *$A^\dagger$ is a multiplicity-free element of $\mathrm{End}(W)$ with eigenvalues $\{\theta_i\}_{i=0}^d$.*

(ii) *For $0 \leq i \leq d$, $E_i^\dagger$ is the primitive idempotent of $A^\dagger$ corresponding to θ_i.*

Proof: (i) For $f \in \mathbb{K}[\lambda]$ we have $f(A) = 0$ if and only if $f(A^\dagger) = 0$. Therefore A and $A^\dagger$ have the same minimal polynomial. The minimal polynomial of A is $\prod_{i=0}^d (\lambda - \theta_i)$ so the minimal polynomial of $A^\dagger$ is $\prod_{i=0}^d (\lambda - \theta_i)$. By this and since $\{\theta_i\}_{i=0}^d$ are mutually distinct, $A^\dagger$ is diagonalizable with eigenvalues $\{\theta_i\}_{i=0}^d$. Recall $\dim W = d+1$ so $A^\dagger$ is multiplicity-free. (ii) Apply $\dagger$ to (1). $\square$

Proposition 6.4. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Let $\dagger$ denote an anti-isomorphism from $\mathrm{End}(V)$ to $\mathrm{End}(W)$. Then $(A^\dagger; \{E_{d-i}^\dagger\}_{i=0}^d; A^{*\dagger}; \{E_{d-i}^{*\dagger}\}_{i=0}^d)$ is a TH system on W.*

Proof: Define $\Psi = (A^\dagger; \{E_{d-i}^\dagger\}_{i=0}^d; A^{*\dagger}; \{E_{d-i}^{*\dagger}\}_{i=0}^d)$. In order to show that Ψ is a TH system on W, we show that Ψ satisfies conditions (i)–(v) of Definition 2.2. By Lemma 6.3, Ψ satisfies conditions (i)–(iii). We now show that Ψ satisfies condition (iv). Since $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a TH system, we have

$$E_i A^* E_j = \begin{cases} 0, & \text{if } i-j>1 \\ \neq 0, & \text{if } i-j=1 \end{cases} \qquad (0 \leq i,j \leq d). \tag{26}$$

Applying $\dagger$ to (26), we find

$$E_j^\dagger A^{*\dagger} E_i^\dagger = \begin{cases} 0, & \text{if } i-j>1 \\ \neq 0, & \text{if } i-j=1 \end{cases} \qquad (0 \leq i,j \leq d). \tag{27}$$

Relabelling the indices in (27), we find

$$E_{d-i}^\dagger A^{*\dagger} E_{d-j}^\dagger = \begin{cases} 0, & \text{if } i-j>1 \\ \neq 0, & \text{if } i-j=1 \end{cases} \qquad (0 \leq i,j \leq d).$$

Therefore Ψ satisfies condition (iv). Similarly Ψ satisfies condition (v). Therefore Ψ is a TH system on W. $\square$

Definition 6.5. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V and let $\Psi = (B; \{F_i\}_{i=0}^d; B^*; \{F_i^*\}_{i=0}^d)$ denote a TH system on W. By an *anti-isomorphism of TH systems* from Φ to Ψ we mean a $\mathbb{K}$-algebra anti-isomorphism $\dagger : \mathrm{End}(V) \to \mathrm{End}(W)$ such that

$$B = A^\dagger, \qquad B^* = A^{*\dagger}, \qquad F_i = E_{d-i}^\dagger, \qquad F_i^* = E_{d-i}^{*\dagger} \quad (0 \leq i \leq d).$$

Observe that if $\dagger$ is an anti-isomorphism of TH systems from Φ to Ψ, then $\dagger^{-1}$ is an anti-isomorphism of TH systems from Ψ to Φ. We say that the TH systems Φ and Ψ are *anti-isomorphic* whenever there exists an anti-isomorphism of TH systems from Φ to Ψ.

Lemma 6.6. *Let Φ denote a TH system over $\mathbb{K}$. Then there exists a TH system Ψ over $\mathbb{K}$ such that Φ and Ψ are anti-isomorphic. Moreover Ψ is unique up to isomorphism of TH systems.*

Proof: We first show that Ψ exists. Write $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ and assume V is the vector space underlying Φ. By elementary linear algebra, there exists a $\mathbb{K}$-algebra anti-automorphism $\dagger$ of $\mathrm{End}(V)$. Define $\Psi = (A^\dagger; \{E_{d-i}^\dagger\}_{i=0}^d; A^{*\dagger}; \{E_{d-i}^{*\dagger}\}_{i=0}^d)$. By Proposition 6.4, Ψ is a TH system on V. By Definition 6.5, Φ and Ψ are anti-isomorphic. We have shown that Ψ exists. Next we show that Ψ is unique. Suppose that Ψ' is a TH system on W such that Φ and Ψ' are anti-isomorphic. We show that Ψ and Ψ' are isomorphic. Let $\dagger'$ denote an anti-isomorphism of TH systems from Φ to Ψ'. Then the composition $\dagger'\dagger^{-1}$ is an isomorphism of TH systems from Ψ to Ψ'. Therefore Ψ and Ψ' are isomorphic. $\square$

Lemma 6.7. *Let Φ and Ψ denote anti-isomorphic TH systems over $\mathbb{K}$. Then the anti-isomorphism of TH systems from Φ to Ψ is unique.*

Proof: Let $\dagger$ and $\dagger'$ denote anti-isomorphisms of TH systems from Φ to Ψ. We show that $\dagger = \dagger'$. Observe that the composition $\dagger^{-1}\dagger'$ is an isomorphism of TH systems from Φ to Φ. The map $\dagger^{-1}\dagger'$ is the identity by Lemma 3.6, so $\dagger = \dagger'$. $\square$

Proposition 6.8. *Let Φ denote a TH system over $\mathbb{K}$ with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Let Ψ denote a TH system over $\mathbb{K}$. Then the following* (i), (ii) *are equivalent.*

(i) *Φ and Ψ are anti-isomorphic.*

(ii) *The parameter array of Ψ is $(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\phi_{d-i+1}\}_{i=1}^d)$.*

Proof: (ii)$\Rightarrow$(i) Write $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ and $\Psi = (B; \{F_i\}_{i=0}^d; B^*; \{F_i^*\}_{i=0}^d)$. Assume V (resp. W) is the vector space underlying Φ (resp. Ψ). For notational convenience, fix a Φ-split basis for V (resp. Ψ-split basis for W) and identify each element of $\mathrm{End}(V)$ (resp. $\mathrm{End}(W)$) with the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ that represents it with respect to this basis. By Proposition 2.8,

$$A = \begin{pmatrix} \theta_d & & & & & \mathbf{0} \\ \phi_1 & \theta_{d-1} & & & & \\ & \phi_2 & \theta_{d-2} & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & \phi_d & \theta_0 \end{pmatrix}, \qquad A^* = \begin{pmatrix} \theta_0^* & 1 & & & & \mathbf{0} \\ & \theta_1^* & 1 & & & \\ & & \theta_2^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & \theta_d^* \end{pmatrix}.$$

Moreover

$$B = \begin{pmatrix} \theta_0 & & & & & \mathbf{0} \\ \phi_d & \theta_1 & & & & \\ & \phi_{d-1} & \theta_2 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ \mathbf{0} & & & & \phi_1 & \theta_d \end{pmatrix}, \qquad B^* = \begin{pmatrix} \theta_d^* & 1 & & & & \mathbf{0} \\ & \theta_{d-1}^* & 1 & & & \\ & & \theta_{d-2}^* & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & 1 \\ \mathbf{0} & & & & & \theta_0^* \end{pmatrix}.$$

Recall the $\mathbb{K}$-algebra anti-automorphism ς of $\mathrm{Mat}_{d+1}(\mathbb{K})$ from above Lemma 6.3. Observe that $B = A^\varsigma$ and $B^* = A^{*\varsigma}$. By this and (1) we find $F_i = E_{d-i}^\varsigma$ and $F_i^* = E_{d-i}^{*\varsigma}$ for $0 \leq i \leq d$. Therefore ς is an anti-isomorphism of TH systems from Φ to Ψ, so Φ and Ψ are anti-isomorphic.
(i)$\Rightarrow$(ii) Routine by Theorem 4.2, Lemma 6.6, and the previous part. $\square$

We now discuss the notion of anti-isomorphism for TH pairs.

Definition 6.9. Let A, A^* denote a TH pair on V and let B, B^* denote a TH pair on W. By an *anti-isomorphism of TH pairs* from A, A^* to B, B^* we mean a $\mathbb{K}$-algebra anti-isomorphism $\dagger : \mathrm{End}(V) \to \mathrm{End}(W)$ such that $B = A^\dagger$ and $B^* = A^{*\dagger}$. Observe that if $\dagger$ is an anti-isomorphism of TH pairs from A, A^* to B, B^*, then $\dagger^{-1}$ is an anti-isomorphism of TH pairs from B, B^* to A, A^*. We say that the TH pairs A, A^* and B, B^* are *anti-isomorphic* whenever there exists an anti-isomorphism of TH pairs from A, A^* to B, B^*.

Lemma 6.10. *Let A, A^* denote a TH pair over $\mathbb{K}$. Then there exists a TH pair B, B^* over $\mathbb{K}$ such that A, A^* and B, B^* are anti-isomorphic. Moreover B, B^* is unique up to isomorphism of TH pairs.*

Proof: Similar to the proof of Lemma 6.6. □

Lemma 6.11. *Let A, A^* and B, B^* denote anti-isomorphic TH pairs over $\mathbb{K}$. Then the anti-isomorphism of TH pairs from A, A^* to B, B^* is unique.*

Proof: Similar to the proof of Lemma 6.7. □

We recall some more terms and facts from elementary linear algebra. A map $\langle\ ,\ \rangle : V \times W \to \mathbb{K}$ is called a *bilinear form* whenever the following conditions hold for all $v, v' \in V$, $w, w' \in W$, and $\alpha \in \mathbb{K}$: (i) $\langle v + v', w\rangle = \langle v, w\rangle + \langle v', w\rangle$; (ii) $\langle \alpha v, w\rangle = \alpha\langle v, w\rangle$; (iii) $\langle v, w + w'\rangle = \langle v, w\rangle + \langle v, w'\rangle$; (iv) $\langle v, \alpha w\rangle = \alpha\langle v, w\rangle$. We observe that a scalar multiple of a bilinear form is a bilinear form.

Let $\langle\ ,\ \rangle : V \times W \to \mathbb{K}$ denote a bilinear form. Then the following are equivalent: (i) there exists a nonzero $v \in V$ such that $\langle v, w\rangle = 0$ for all $w \in W$; (ii) there exists a nonzero $w \in W$ such that $\langle v, w\rangle = 0$ for all $v \in V$. The form $\langle\ ,\ \rangle$ is said to be *degenerate* whenever (i), (ii) hold and *nondegenerate* otherwise.

Bilinear forms are related to anti-isomorphisms as follows. Let $\langle\ ,\ \rangle : V \times W \to \mathbb{K}$ denote a nondegenerate bilinear form. Then there exists a unique anti-isomorphism $\dagger : \mathrm{End}(V) \to \mathrm{End}(W)$ such that $\langle Sv, w\rangle = \langle v, S^\dagger w\rangle$ for all $v \in V$, $w \in W$, and $S \in \mathrm{End}(V)$. Conversely, given an anti-isomorphism $\dagger : \mathrm{End}(V) \to \mathrm{End}(W)$ there exists a nonzero bilinear form $\langle\ ,\ \rangle : V \times W \to \mathbb{K}$ such that $\langle Sv, w\rangle = \langle v, S^\dagger w\rangle$ for all $v \in V$, $w \in W$, and $S \in \mathrm{End}(V)$. This bilinear form is nondegenerate, and uniquely determined by $\dagger$ up to multiplication by a nonzero scalar in $\mathbb{K}$. We say that the form $\langle\ ,\ \rangle$ is *associated* with $\dagger$.

Define $\tilde{V}$ to be the dual space of V, consisting of all $\mathbb{K}$-linear transformations from V to $\mathbb{K}$. By elementary linear algebra, $\tilde{V}$ is a vector space over $\mathbb{K}$ and $\dim \tilde{V} = \dim V$. Define a bilinear form $\langle\ ,\ \rangle : V \times \tilde{V} \to \mathbb{K}$ such that $\langle v, f\rangle = f(v)$ for all $v \in V$ and $f \in \tilde{V}$. The form $\langle\ ,\ \rangle$ is nondegenerate. We call $\langle\ ,\ \rangle$ the *canonical bilinear form* between V and $\tilde{V}$. Let $\sigma : \mathrm{End}(V) \to \mathrm{End}(\tilde{V})$ denote the anti-isomorphism associated with $\langle\ ,\ \rangle$. Thus $\langle Sv, f\rangle = \langle v, S^\sigma f\rangle$ for all $v \in V$, $f \in \tilde{V}$, and $S \in \mathrm{End}(V)$. We call σ the *canonical anti-isomorphism* from $\mathrm{End}(V)$ to $\mathrm{End}(\tilde{V})$.

Definition 6.12. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Define $\tilde{\Phi} = (A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^d; A^{*\sigma}; \{E_{d-i}^{*\sigma}\}_{i=0}^d)$, where $\sigma : \mathrm{End}(V) \to \mathrm{End}(\tilde{V})$ is the canonical anti-isomorphism. By Proposition 6.4, $\tilde{\Phi}$ is a TH system on $\tilde{V}$. By Definition 6.5, Φ and $\tilde{\Phi}$ are anti-isomorphic. By Proposition 6.8, $\tilde{\Phi}$ has parameter array $(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\phi_{d-i+1}\}_{i=1}^d)$.

7 The $\mathbb{Z}_2 \times \mathbb{Z}_2$ action

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system. We saw in the previous section that each of the following is a TH system:

$$\begin{aligned} \Phi^* &= (A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d), \\ \tilde{\Phi} &= (A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^d; A^{*\sigma}; \{E_{d-i}^{*\sigma}\}_{i=0}^d). \end{aligned}$$

Viewing $*$, $\sim$ as permutations on the set of all TH systems,

$$*^2 = \sim^2 = 1, \qquad * \sim = \sim *. \tag{28}$$

The group generated by symbols $*$, $\sim$ subject to the relations (28) is the group $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus $*$, $\sim$ induce an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ on the set of all TH systems. Two TH systems will be called *relatives* whenever they are in the same orbit of this $\mathbb{Z}_2 \times \mathbb{Z}_2$ action. The relatives of Φ are as follows:

name	relative
Φ	$(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$
Φ^*	$(A^*; \{E_i^*\}_{i=0}^d; A; \{E_i\}_{i=0}^d)$
$\tilde{\Phi}$	$(A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^d; A^{*\sigma}; \{E_{d-i}^{*\sigma}\}_{i=0}^d)$
$\tilde{\Phi}^*$	$(A^{*\sigma}; \{E_{d-i}^{*\sigma}\}_{i=0}^d; A^\sigma; \{E_{d-i}^\sigma\}_{i=0}^d)$

Corollary 7.1. *Let Φ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Then the parameter arrays of its relatives are as follows:*

name	*parameter array*
Φ	$(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$
Φ^*	$(\{\theta_i^*\}_{i=0}^d, \{\theta_i\}_{i=0}^d, \{\phi_{d-i+1}\}_{i=1}^d)$
$\tilde{\Phi}$	$(\{\theta_{d-i}\}_{i=0}^d, \{\theta_{d-i}^*\}_{i=0}^d, \{\phi_{d-i+1}\}_{i=1}^d)$
$\tilde{\Phi}^*$	$(\{\theta_{d-i}^*\}_{i=0}^d, \{\theta_{d-i}\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$

Proof: Immediate from Lemma 6.2 and Proposition 6.8. □

We will use the following notational convention.

Definition 7.2. Let Φ denote a TH system. For $g \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and for an object f associated with Φ, let f^g denote the corresponding object associated with Φ^g.

We end this section by giving some more characterizations of the split sequence, as promised at the end of Section 2.

Lemma 7.3. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Then the following* (i)–(iii) *hold for $0 \le i \le d$.*

(i) $E_0 \eta_i^*(A^*) E_0 = \dfrac{\phi_d \phi_{d-1} \cdots \phi_{d-i+1}}{(\theta_0 - \theta_1)(\theta_0 - \theta_2) \cdots (\theta_0 - \theta_i)} E_0.$

(ii) $E_d^* \tau_i(A) E_d^* = \dfrac{\phi_d \phi_{d-1} \cdots \phi_{d-i+1}}{(\theta_d^* - \theta_{d-1}^*)(\theta_d^* - \theta_{d-2}^*) \cdots (\theta_d^* - \theta_{d-i}^*)} E_d^*.$

(iii) $E_d \tau_i^*(A^*) E_d = \dfrac{\phi_1 \phi_2 \cdots \phi_i}{(\theta_d - \theta_{d-1})(\theta_d - \theta_{d-2}) \cdots (\theta_d - \theta_{d-i})} E_d.$

Proof: Let Φ denote the TH system in question.
(i) Apply Lemma 2.11 to Φ^*.
(ii) Apply Lemma 2.11 to $\tilde{\Phi}$ and then apply σ^{-1} to each side of the resulting equations.
(iii) Apply (ii) to Φ^*. □

Corollary 7.4. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Then the following* (i)–(iii) *hold for $0 \leq i \leq d$.*

(i) $\phi_d\phi_{d-1}\cdots\phi_{d-i+1} = (\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_i)\mathrm{trace}(\eta_i^*(A^*)E_0)$.

(ii) $\phi_d\phi_{d-1}\cdots\phi_{d-i+1} = (\theta_d^*-\theta_{d-1}^*)(\theta_d^*-\theta_{d-2}^*)\cdots(\theta_d^*-\theta_{d-i}^*)\mathrm{trace}(\tau_i(A)E_d^*)$.

(iii) $\phi_1\phi_2\cdots\phi_i = (\theta_d-\theta_{d-1})(\theta_d-\theta_{d-2})\cdots(\theta_d-\theta_{d-i})\mathrm{trace}(\tau_i^*(A^*)E_d)$.

Moreover each of $\eta_i^*(A^*)E_0, \tau_i(A)E_d^*, \tau_i^*(A^*)E_d$ has nonzero trace.

Proof: Let Φ denote the TH system in question.
(i) Apply Corollary 2.12 to Φ^*.
(ii) In the equation of Lemma 7.3(ii), take the trace of each side and simplify the result using the fact that $\mathrm{trace}(E_d^*) = 1$ and $\mathrm{trace}(E_d^*\tau_i(A)E_d^*) = \mathrm{trace}(\tau_i(A)E_d^*E_d^*) = \mathrm{trace}(\tau_i(A)E_d^*)$.
(iii) Apply (ii) to Φ^*.
The last assertion follows since $\phi_i \neq 0$ for $1 \leq i \leq d$. □

Corollary 7.5. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Then the following* (i)–(iii) *hold for $1 \leq i \leq d$.*

(i) $\phi_i = (\theta_0-\theta_{d-i+1})\mathrm{trace}(\eta_{d-i+1}^*(A^*)E_0)/\mathrm{trace}(\eta_{d-i}^*(A^*)E_0)$.

(ii) $\phi_i = (\theta_d^*-\theta_{i-1}^*)\mathrm{trace}(\tau_{d-i+1}(A)E_d^*)/\mathrm{trace}(\tau_{d-i}(A)E_d^*)$.

(iii) $\phi_i = (\theta_d-\theta_{d-i})\mathrm{trace}(\tau_i^*(A^*)E_d)/\mathrm{trace}(\tau_{i-1}^*(A^*)E_d)$.

Proof: Routine by Corollary 7.4. □

8 The scalars $\{\ell_i\}_{i=0}^d$

Let Φ denote a TH system. In this section we associate with Φ a sequence of scalars $\{\ell_i\}_{i=0}^d$ that will help us describe Φ.

Definition 8.1. Let Φ denote a TH system with dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. For $0 \leq i \leq d$, define

$$\begin{aligned}\ell_i &= \frac{\eta_d^*(\theta_0^*)}{\tau_i^*(\theta_i^*)\eta_{d-i}^*(\theta_i^*)} \\ &= \frac{(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_d^*)}{(\theta_i^*-\theta_0^*)(\theta_i^*-\theta_1^*)\cdots(\theta_i^*-\theta_{i-1}^*)(\theta_i^*-\theta_{i+1}^*)\cdots(\theta_i^*-\theta_{d-1}^*)(\theta_i^*-\theta_d^*)}.\end{aligned}$$

Observe that $\ell_0 = 1$.

Lemma 8.2. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Then for $0 \leq i \leq d$,*

$$\begin{aligned}
\ell_i^* &= \frac{\eta_d(\theta_0)}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)} \\
&= \frac{(\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_d)}{(\theta_i-\theta_0)(\theta_i-\theta_1)\cdots(\theta_i-\theta_{i-1})(\theta_i-\theta_{i+1})\cdots(\theta_i-\theta_{d-1})(\theta_i-\theta_d)}, \\
\tilde{\ell}_i &= \frac{\tau_d^*(\theta_d^*)}{\eta_i^*(\theta_{d-i}^*)\tau_{d-i}^*(\theta_{d-i}^*)} \\
&= \frac{(\theta_d^*-\theta_{d-1}^*)(\theta_d^*-\theta_{d-2}^*)\cdots(\theta_d^*-\theta_0^*)}{(\theta_{d-i}^*-\theta_d^*)(\theta_{d-i}^*-\theta_{d-1}^*)\cdots(\theta_{d-i}^*-\theta_{d-i+1}^*)(\theta_{d-i}^*-\theta_{d-i-1}^*)\cdots(\theta_{d-i}^*-\theta_1^*)(\theta_{d-i}^*-\theta_0^*)}, \\
\tilde{\ell}_i^* &= \frac{\tau_d(\theta_d)}{\eta_i(\theta_{d-i})\tau_{d-i}(\theta_{d-i})} \\
&= \frac{(\theta_d-\theta_{d-1})(\theta_d-\theta_{d-2})\cdots(\theta_d-\theta_0)}{(\theta_{d-i}-\theta_d)(\theta_{d-i}-\theta_{d-1})\cdots(\theta_{d-i}-\theta_{d-i+1})(\theta_{d-i}-\theta_{d-i-1})\cdots(\theta_{d-i}-\theta_1)(\theta_{d-i}-\theta_0)}.
\end{aligned}$$

Moreover $\tilde{\ell}_i = \dfrac{\tau_d^*(\theta_d^*)}{\eta_d^*(\theta_0^*)}\ell_{d-i}$ and $\tilde{\ell}_i^* = \dfrac{\tau_d(\theta_d)}{\eta_d(\theta_0)}\ell_{d-i}^*$.

Proof: Combine Corollary 7.1 and Definition 8.1. □

We give one significance of the sequence $\{\ell_i\}_{i=0}^d$.

Lemma 8.3. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system. Then $E_d E_i^* E_0 = \ell_i E_d E_0^* E_0$ for $0 \leq i \leq d$.*

Proof: Let Φ denote the TH system in question and assume V is the underlying vector space. For notational convenience, fix a Φ-split basis for V and identify each element of $\mathrm{End}(V)$ with the matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ that represents it with respect to this basis. We show that $E_d(E_i^* - \ell_i E_0^*)E_0 = 0$. By (9) the entries of all but the first column of E_d are zero and the entries of all but the last row of E_0 are zero. Therefore for $0 \leq m, n \leq d$, the (m,n)-entry of $E_d(E_i^* - \ell_i E_0^*)E_0$ is

$$(E_d)_{m0}(E_i^* - \ell_i E_0^*)_{0d}(E_0)_{dn}. \tag{29}$$

By (10) the middle factor in (29) is 0, so (29) is 0. Therefore $E_d(E_i^* - \ell_i E_0^*)E_0 = 0$ and the result follows. □

Corollary 8.4. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system. Then the following* (i)–(iii) *hold for $0 \leq i \leq d$.*

(i) $E_d^* E_i E_0^* = \ell_i^* E_d^* E_0 E_0^*$.

(ii) $E_d E_i^* E_0 = \tilde{\ell}_{d-i} E_d E_d^* E_0$.

(iii) $E_d^* E_i E_0^* = \tilde{\ell}_{d-i}^* E_d^* E_d E_0^*$.

Proof: Let Φ denote the TH system in question.
(i) Apply Lemma 8.3 to Φ^*.
(ii) Apply Lemma 8.3 to $\tilde{\Phi}$ and then apply σ^{-1} to each side of the resulting equations.
(iii) Apply (ii) to Φ^*. $\square$

Definition 8.5. Let Φ denote a TH system of diameter d. We associate with Φ a diagonal matrix $L \in \mathrm{Mat}_{d+1}(\mathbb{K})$ with (i,i)-entry ℓ_i for $0 \leq i \leq d$.

9 The scalar ν

Let Φ denote a TH system. In this section we associate with Φ a scalar ν that will help us describe Φ.

Definition 9.1. Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system. By [1, Lemma 7.5], $\mathrm{trace}(E_0E_0^*)$ is nonzero. Let ν denote the reciprocal of $\mathrm{trace}(E_0E_0^*)$.

We give one significance of the scalar ν.

Lemma 9.2. [1, Lemma 7.4] *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system. Then both*

$$\nu E_0E_0^*E_0 = E_0, \qquad \nu E_0^*E_0E_0^* = E_0^*.$$

Lemma 9.3. *Let Φ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ and let ν denote the scalar from Definition 9.1. Then*

$$\begin{aligned} \nu &= \frac{(\theta_0-\theta_1)(\theta_0-\theta_2)\cdots(\theta_0-\theta_d)(\theta_0^*-\theta_1^*)(\theta_0^*-\theta_2^*)\cdots(\theta_0^*-\theta_d^*)}{\phi_1\phi_2\cdots\phi_d}, \\ \tilde{\nu} &= \frac{(\theta_d-\theta_{d-1})(\theta_d-\theta_{d-2})\cdots(\theta_d-\theta_0)(\theta_d^*-\theta_{d-1}^*)(\theta_d^*-\theta_{d-2}^*)\cdots(\theta_d^*-\theta_0^*)}{\phi_1\phi_2\cdots\phi_d}. \end{aligned} \tag{30}$$

Moreover $\nu^* = \nu$ and $\tilde{\nu}^* = \tilde{\nu}$.

Proof: Line (30) holds by [1, Lemma 7.6]. The remaining assertions follow from Corollary 7.1. $\square$

Lemma 9.4. *Let $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system. Then both*

$$\tilde{\nu} E_dE_d^*E_d = E_d, \qquad \tilde{\nu} E_d^*E_dE_d^* = E_d^*.$$

Proof: Let Φ denote the TH system in question. Apply Lemma 9.2 to $\tilde{\Phi}$ and then apply σ^{-1} to each side of the resulting equations. $\square$

10 Anti-isomorphic TH systems and the bilinear form

Let Φ denote a TH system on V and let $\tilde{\Phi}$ denote the relative of Φ from Definition 6.12. Recall that Φ and $\tilde{\Phi}$ are anti-isomorphic. In this section, we discuss further the relationship between Φ and $\tilde{\Phi}$. Recall the canonical bilinear form $\langle\ ,\ \rangle : V \times \tilde{V} \to \mathbb{K}$ from above Definition 6.12. Let U (resp. $\tilde{U}$) denote a subspace of V (resp. $\tilde{V}$). We say that U and $\tilde{U}$ are *orthogonal* whenever $\langle x, y\rangle = 0$ for all $x \in U$ and $y \in \tilde{U}$. Let $\{V_i\}_{i=0}^d$ (resp. $\{\tilde{V}_i\}_{i=0}^d$) denote a decomposition of V (resp. $\tilde{V}$). We say that $\{V_i\}_{i=0}^d$ and $\{\tilde{V}_i\}_{i=0}^d$ are *dual* whenever V_i and $\tilde{V}_j$ are orthogonal for $0 \leq i, j \leq d$, $i \neq j$. By elementary linear algebra, for any decomposition $\{V_i\}_{i=0}^d$ (resp. $\{\tilde{V}_i\}_{i=0}^d$) of V (resp. $\tilde{V}$) there exists a unique decomposition $\{\tilde{V}_i\}_{i=0}^d$ (resp. $\{V_i\}_{i=0}^d$) of $\tilde{V}$ (resp. V) such that $\{V_i\}_{i=0}^d$ and $\{\tilde{V}_i\}_{i=0}^d$ are dual. Let $\{v_i\}_{i=0}^d$ (resp. $\{\tilde{v}_i\}_{i=0}^d$) denote a basis for V (resp. $\tilde{V}$). We say that $\{v_i\}_{i=0}^d$ and $\{\tilde{v}_i\}_{i=0}^d$ are *dual* whenever $\langle v_i, \tilde{v}_j\rangle = \delta_{ij}$ for $0 \leq i, j \leq d$. By elementary linear algebra, for any basis $\{v_i\}_{i=0}^d$ (resp. $\{\tilde{v}_i\}_{i=0}^d$) for V (resp. $\tilde{V}$) there exists a unique basis $\{\tilde{v}_i\}_{i=0}^d$ (resp. $\{v_i\}_{i=0}^d$) for $\tilde{V}$ (resp. V) such that $\{v_i\}_{i=0}^d$ and $\{\tilde{v}_i\}_{i=0}^d$ are dual. Given any sequence $\{\alpha_i\}_{i=0}^d$, by the *inversion* of $\{\alpha_i\}_{i=0}^d$ we mean the sequence $\{\alpha_{d-i}\}_{i=0}^d$.

Recall the Φ-standard decomposition $\{E_i^* V\}_{i=0}^d$ from above (3). Observe by Definition 6.12 that $\{E_{d-i}^{*\sigma}\tilde{V}\}_{i=0}^d$ is the $\tilde{\Phi}$-standard decomposition. We now compare these two decompositions.

Lemma 10.1. *With reference to Definition 6.12, the following* (i), (ii) *are inverted dual.*

(i) *The Φ-standard decomposition of V.*

(ii) *The $\tilde{\Phi}$-standard decomposition of $\tilde{V}$.*

Proof: For distinct i, j $(0 \leq i, j \leq d)$ we show that $E_i^* V$ and $E_j^{*\sigma}\tilde{V}$ are orthogonal. Let $u \in E_i^* V$ and $v \in E_j^{*\sigma}\tilde{V}$. Simplify the equation $\langle A^* u, v\rangle = \langle u, A^{*\sigma} v\rangle$ using $A^* u = \theta_i^* u$ and $A^{*\sigma} v = \theta_j^* v$ to obtain $(\theta_i^* - \theta_j^*)\langle u, v\rangle = 0$. Now $\langle u, v\rangle = 0$ since $\theta_i^* \neq \theta_j^*$. Therefore $E_i^* V$ and $E_j^{*\sigma}\tilde{V}$ are orthogonal and the result follows. □

Let $0 \neq \xi_0 \in E_0 V$ and recall the Φ-standard basis $\{E_i^* \xi_0\}_{i=0}^d$ for V from above (3). Let $0 \neq \tilde{\xi}_d \in E_d^\sigma \tilde{V}$ and observe by Definition 6.12 that $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$ is a $\tilde{\Phi}$-standard basis for $\tilde{V}$. These two bases are related as follows.

Proposition 10.2. *With reference to Definition 6.12, let $0 \neq \xi_0 \in E_0 V$ and $0 \neq \tilde{\xi}_d \in E_d^\sigma \tilde{V}$. Then for $0 \leq i, j \leq d$,*

$$\langle E_i^* \xi_0, E_j^{*\sigma}\tilde{\xi}_d\rangle = \delta_{ij}\ell_i \langle E_0^* \xi_0, \tilde{\xi}_d\rangle,$$

where ℓ_i is from Definition 8.1.

Proof: Using the definition of σ from above Definition 6.12 along with Lemma 8.3, we find

$$\begin{aligned}
\langle E_i^* \xi_0, E_j^{*\sigma}\tilde{\xi}_d\rangle &= \langle E_i^* E_0 \xi_0, E_j^{*\sigma} E_d^\sigma \tilde{\xi}_d\rangle \\
&= \langle E_j^* E_i^* E_0 \xi_0, E_d^\sigma \tilde{\xi}_d\rangle \\
&= \delta_{ij}\langle E_i^* E_0 \xi_0, E_d^\sigma \tilde{\xi}_d\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \delta_{ij}\langle E_d E_i^* E_0 \xi_0, \tilde{\xi}_d\rangle \\
&= \delta_{ij}\ell_i\langle E_d E_0^* E_0 \xi_0, \tilde{\xi}_d\rangle \\
&= \delta_{ij}\ell_i\langle E_0^* E_0 \xi_0, E_d^\sigma \tilde{\xi}_d\rangle \\
&= \delta_{ij}\ell_i\langle E_0^* \xi_0, \tilde{\xi}_d\rangle.
\end{aligned}
$$

□

Corollary 10.3. *With reference to Definition 6.12, let $\{v_i\}_{i=0}^d$ (resp. $\{w_i\}_{i=0}^d$) denote a basis for V (resp. $\tilde{V}$). Suppose that $\{v_i\}_{i=0}^d$ and $\{w_i\}_{i=0}^d$ are inverted dual. Then the following* (i), (ii) *are equivalent.*

(i) $\{\ell_i v_i\}_{i=0}^d$ *is a Φ-standard basis for V.*

(ii) $\{w_i\}_{i=0}^d$ *is a $\tilde{\Phi}$-standard basis for $\tilde{V}$.*

Proof: Use Proposition 10.2. □

By Definition 6.1, the sequence $\{E_i V\}_{i=0}^d$ is the Φ^*-standard decomposition. By Definition 6.12, the sequence $\{E_{d-i}^\sigma \tilde{V}\}_{i=0}^d$ is the $\tilde{\Phi}^*$-standard decomposition. We now compare these two decompositions.

Lemma 10.4. *With reference to Definition 6.12, the following* (i), (ii) *are inverted dual.*

(i) *The Φ^*-standard decomposition of V.*

(ii) *The $\tilde{\Phi}^*$-standard decomposition of $\tilde{V}$.*

Proof: Apply Lemma 10.1 to Φ^*. □

Let $0 \neq \xi_0^* \in E_0^* V$ and observe by Definition 6.1 that $\{E_i \xi_0^*\}_{i=0}^d$ is a Φ^*-standard basis for V. Let $0 \neq \tilde{\xi}_d^* \in E_d^{*\sigma} \tilde{V}$ and observe by Definition 6.12 that $\{E_{d-i}^\sigma \tilde{\xi}_d^*\}_{i=0}^d$ is a $\tilde{\Phi}^*$-standard basis for $\tilde{V}$. These two bases are related as follows.

Proposition 10.5. *With reference to Definition 6.12, let $0 \neq \xi_0^* \in E_0^* V$ and $0 \neq \tilde{\xi}_d^* \in E_d^{*\sigma} \tilde{V}$. Then for $0 \leq i, j \leq d$,*

$$\langle E_i \xi_0^*, E_j^\sigma \tilde{\xi}_d^*\rangle = \delta_{ij} \ell_i^* \langle E_0 \xi_0^*, \tilde{\xi}_d^*\rangle,$$

where ℓ_i^* is from Lemma 8.2.

Proof: Apply Proposition 10.2 to Φ^*. □

Corollary 10.6. *With reference to Definition 6.12, let $\{v_i\}_{i=0}^d$ (resp. $\{w_i\}_{i=0}^d$) denote a basis for V (resp. $\tilde{V}$). Suppose that $\{v_i\}_{i=0}^d$ and $\{w_i\}_{i=0}^d$ are inverted dual. Then the following* (i), (ii) *are equivalent.*

(i) $\{\ell_i^* v_i\}_{i=0}^d$ *is a Φ^*-standard basis for V.*

(ii) $\{w_i\}_{i=0}^d$ *is a $\tilde{\Phi}^*$-standard basis for $\tilde{V}$.*

Proof: Apply Corollary 10.3 to Φ^*. $\square$

Let $\{U_i\}_{i=0}^d$ denote the Φ-split decomposition of V. Recall from (3) that for $0 \leq i \leq d$,

$$U_i = (E_0^*V + E_1^*V + \cdots + E_i^*V) \cap (E_0V + E_1V + \cdots + E_{d-i}V). \tag{31}$$

Let $\{\tilde{U}_i\}_{i=0}^d$ denote the $\tilde{\Phi}$-split decomposition of $\tilde{V}$. Combining Definition 6.12 and (31), we find that for $0 \leq i \leq d$,

$$\tilde{U}_i = (E_d^{*\sigma}\tilde{V} + E_{d-1}^{*\sigma}\tilde{V} + \cdots + E_{d-i}^{*\sigma}\tilde{V}) \cap (E_d^{\sigma}\tilde{V} + E_{d-1}^{\sigma}\tilde{V} + \cdots + E_i^{\sigma}\tilde{V}). \tag{32}$$

We now compare these two decompositions.

Lemma 10.7. *With reference to Definition 6.12, the following* (i), (ii) *are inverted dual.*

(i) *The Φ-split decomposition of V.*

(ii) *The $\tilde{\Phi}$-split decomposition of $\tilde{V}$.*

Proof: We use the notation from above this lemma. For $0 \leq i, j \leq d$ with $i+j \neq d$, we show that U_i and $\tilde{U}_j$ are orthogonal. We consider two cases: $i+j < d$ and $i+j > d$. First suppose that $i+j < d$. Abbreviate $M = E_0^*V + E_1^*V + \cdots + E_i^*V$ and $N = E_d^{*\sigma}\tilde{V} + E_{d-1}^{*\sigma}\tilde{V} + \cdots + E_{d-j}^{*\sigma}\tilde{V}$. Observe that $U_i \subseteq M$ by (31) and $\tilde{U}_j \subseteq N$ by (32). Moreover M and N are orthogonal by our assumption and Lemma 10.1. Therefore U_i and $\tilde{U}_j$ are orthogonal. Next suppose that $i+j > d$. Abbreviate $S = E_0V + E_1V + \cdots + E_{d-i}V$ and $T = E_d^{\sigma}\tilde{V} + E_{d-1}^{\sigma}\tilde{V} + \cdots + E_j^{\sigma}\tilde{V}$. Observe $U_i \subseteq S$ by (31) and $\tilde{U}_j \subseteq T$ by (32). Moreover S and T are orthogonal by our assumption and Lemma 10.4. Therefore U_i and $\tilde{U}_j$ are orthogonal and the result follows. $\square$

Let $0 \neq \xi_0 \in E_0V$ and recall from (7) that the sequence

$$\eta_{d-i}^*(A^*)\xi_0 \qquad\qquad (0 \leq i \leq d) \tag{33}$$

is a Φ-split basis for V. Let $0 \neq \tilde{\xi}_d \in E_d^{\sigma}\tilde{V}$ and observe by Definition 6.12 that the sequence

$$\tau_{d-i}^*(A^{*\sigma})\tilde{\xi}_d \qquad\qquad (0 \leq i \leq d) \tag{34}$$

is a $\tilde{\Phi}$-split basis for $\tilde{V}$. The bases (33), (34) are related as follows.

Proposition 10.8. *With reference to Definition 6.12, let $0 \neq \xi_0 \in E_0V$ and $0 \neq \tilde{\xi}_d \in E_d^{\sigma}\tilde{V}$. Then for $0 \leq i, j \leq d$,*

$$\langle \eta_i^*(A^*)\xi_0, \tau_j^*(A^{*\sigma})\tilde{\xi}_d \rangle = \delta_{i+j,d}\, \eta_d^*(\theta_0^*)\langle E_0^*\xi_0, \tilde{\xi}_d \rangle.$$

Proof: First suppose $i + j \neq d$. Then the result holds by Lemma 10.7 and the comments above this proposition. Next suppose that $i + j = d$. Using (2), Proposition 10.2, and the definition of σ from above Definition 6.12, we find

$$\begin{aligned}
\langle \eta_i^*(A^*)\xi_0, \tau_j^*(A^{*\sigma})\tilde{\xi}_d \rangle &= \langle \eta_{d-j}^*(A^*)\xi_0, \tau_j^*(A^{*\sigma})\tilde{\xi}_d \rangle \\
&= \langle \tau_j^*(A^*)\eta_{d-j}^*(A^*)\xi_0, \tilde{\xi}_d \rangle \\
&= \tau_j^*(\theta_j^*)\eta_{d-j}^*(\theta_j^*)\langle E_j^*\xi_0, \tilde{\xi}_d \rangle \\
&= \tau_j^*(\theta_j^*)\eta_{d-j}^*(\theta_j^*)\langle E_j^*E_j^*\xi_0, \tilde{\xi}_d \rangle \\
&= \tau_j^*(\theta_j^*)\eta_{d-j}^*(\theta_j^*)\langle E_j^*\xi_0, E_j^{*\sigma}\tilde{\xi}_d \rangle \\
&= \tau_j^*(\theta_j^*)\eta_{d-j}^*(\theta_j^*)\ell_j\langle E_0^*\xi_0, \tilde{\xi}_d \rangle \\
&= \eta_d^*(\theta_0^*)\langle E_0^*\xi_0, \tilde{\xi}_d \rangle.
\end{aligned}$$

□

Corollary 10.9. *With reference to Definition 6.12, let* $\{v_i\}_{i=0}^d$ *(resp.* $\{w_i\}_{i=0}^d$*) denote a basis for* V *(resp.* $\tilde{V}$*). Suppose that* $\{v_i\}_{i=0}^d$ *and* $\{w_i\}_{i=0}^d$ *are inverted dual. Then the following* (i), (ii) *are equivalent.*

(i) $\{v_i\}_{i=0}^d$ *is a* Φ*-split basis for* V*.*

(ii) $\{w_i\}_{i=0}^d$ *is a* $\tilde{\Phi}$*-split basis for* $\tilde{V}$*.*

Proof: Use Proposition 10.8. □

At the end of of Section 17, we give the relationship between a Φ-standard (resp. Φ^*-standard) basis for V and a $\tilde{\Phi}^*$-standard (resp. $\tilde{\Phi}$-standard) basis for $\tilde{V}$. This relationship is of a different type than the ones in the present section.

11 The transition matrices for a TH system

Let Φ denote a TH system. In this section we consider several transition matrices associated with Φ. First we clarify our terms. Let $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ denote bases for V. By the *transition matrix* from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$, we mean the matrix $T \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $v_j = \sum_{i=0}^d T_{ij} u_i$ for $0 \leq j \leq d$.

Definition 11.1. [1, Definition 10.6] Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Let $0 \neq \xi_0 \in E_0 V$ and $0 \neq \xi_0^* \in E_0^* V$. Recall the Φ-standard basis $\{E_i^* \xi_0\}_{i=0}^d$ for V and the Φ^*-standard basis $\{E_i \xi_0^*\}_{i=0}^d$ for V. Let $P \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{E_i \xi_0^*\}_{i=0}^d$ to $\{E_i^* \xi_0\}_{i=0}^d$, with ξ_0, ξ_0^* chosen so that $\xi_0^* = E_0^* \xi_0$.

Theorem 11.2. [1, Theorem 10.8] *Let* Φ *denote a TH system with parameter array* $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ *and let* P *denote the matrix from Definition 11.1. For* $0 \leq i, j \leq d$*, the* (i,j)*-entry of* P *is equal to* ℓ_j *times*

$$\sum_{h=0}^{d} \frac{(\theta_i - \theta_d)(\theta_i - \theta_{d-1}) \cdots (\theta_i - \theta_{d-h+1})(\theta_j^* - \theta_0^*)(\theta_j^* - \theta_1^*) \cdots (\theta_j^* - \theta_{h-1}^*)}{\phi_1 \phi_2 \cdots \phi_h}, \tag{35}$$

where ℓ_j *is from Definition 8.1.*

Corollary 11.3. *With reference to Definition 11.1, for* $0 \leq i \leq d$ *both*

$$P_{i0} = 1, \qquad P_{di} = \ell_i,$$

where ℓ_i *is from Definition 8.1.*

Proof: Use Theorem 11.2. □

The following definition is motivated by Definition 11.1 and Corollary 11.3.

Definition 11.4. We call the matrix P from Definition 11.1 the *west normalized transition matrix* of Φ.

Motivated by the sum (35), we make a definition. Let λ, μ denote commuting indeterminates. Let $\mathbb{K}[\lambda, \mu]$ denote the $\mathbb{K}$-algebra consisting of the polynomials in λ, μ that have all coefficients in $\mathbb{K}$.

Definition 11.5. Let Φ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Define $p \in \mathbb{K}[\lambda, \mu]$ by

$$p = \sum_{h=0}^{d} \frac{\eta_h(\lambda)\tau_h^*(\mu)}{\phi_1\phi_2\cdots\phi_h}, \tag{36}$$

where $\{\tau_i^*\}_{i=0}^d$ and $\{\eta_i\}_{i=0}^d$ are from Notation 2.7. We call p the *two-variable polynomial* of Φ.

Example 11.6. With reference to Definition 11.5, assume $d = 2$. Then

$$p = 1 + \frac{(\lambda-\theta_2)(\mu-\theta_0^*)}{\phi_1} + \frac{(\lambda-\theta_2)(\lambda-\theta_1)(\mu-\theta_0^*)(\mu-\theta_1^*)}{\phi_1\phi_2}.$$

Remark 11.7. With reference to Definition 11.5, for $0 \leq i, j \leq d$ the scalar $p(\theta_i, \theta_j^*)$ is the sum (35).

Definition 11.8. Let Φ denote a TH system with parameter array $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$. Let $\mathcal{P} \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the matrix with (i,j)-entry $p(\theta_i, \theta_j^*)$ for $0 \leq i, j \leq d$. Observe that by Theorem 11.2 and Remark 11.7, the matrix P from Definition 11.1 is equal to $\mathcal{P}L$, where L is the matrix from Definition 8.5.

Example 11.9. With reference to Definition 11.8, assume $d = 2$. Then

$$\mathcal{P} = \begin{pmatrix} 1 & 1 + \frac{(\theta_0-\theta_2)(\theta_1^*-\theta_0^*)}{\phi_1} & 1 + \frac{(\theta_0-\theta_2)(\theta_2^*-\theta_0^*)}{\phi_1} + \frac{(\theta_0-\theta_2)(\theta_0-\theta_1)(\theta_2^*-\theta_0^*)(\theta_2^*-\theta_1^*)}{\phi_1\phi_2} \\ 1 & 1 + \frac{(\theta_1-\theta_2)(\theta_1^*-\theta_0^*)}{\phi_1} & 1 + \frac{(\theta_1-\theta_2)(\theta_2^*-\theta_0^*)}{\phi_1} \\ 1 & 1 & 1 \end{pmatrix}.$$

Corollary 11.10. *With reference to Definition 11.8, for $0 \leq i \leq d$ both*

$$\mathcal{P}_{i0} = 1, \qquad \mathcal{P}_{di} = 1.$$

Proof: Routine. □

We now interpret the matrix $\mathcal{P}$ as a transition matrix. Let $\{u_i\}_{i=0}^d$ denote the inverted dual of a $\tilde{\Phi}$-standard basis for $\tilde{V}$. By Corollary 10.3, $\{\ell_i u_i\}_{i=0}^d$ is a Φ-standard basis for V, where ℓ_i is from Definition 8.1. Recall the canonical bilinear form $\langle\ ,\ \rangle : V \times \tilde{V} \to \mathbb{K}$ from above Definition 6.12.

Corollary 11.11. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Let $0 \neq \xi_0^* \in E_0^*V$ and $0 \neq \tilde{\xi}_d \in E_d^\sigma \tilde{V}$. Note by Lemma 10.1 that $\langle \xi_0^*, \tilde{\xi}_d \rangle \neq 0$. Recall the Φ^*-standard basis $\{E_i\xi_0^*\}_{i=0}^d$ for V and the $\tilde{\Phi}$-standard basis $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$ for $\tilde{V}$. Let $\mathcal{P}$ denote the matrix from Definition 11.8. Then $\alpha\mathcal{P}$ is the transition matrix from $\{E_i\xi_0^*\}_{i=0}^d$ to the inverted dual of $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$, where α is the reciprocal of $\langle \xi_0^*, \tilde{\xi}_d \rangle$. In particular if we choose $\xi_0^*, \tilde{\xi}_d$ so that $\langle \xi_0^*, \tilde{\xi}_d \rangle = 1$, then $\mathcal{P}$ is the transition matrix from $\{E_i\xi_0^*\}_{i=0}^d$ to the inverted dual of $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$.*

Proof: Choose $\xi_0 \in E_0V$ so that $\xi_0^* = E_0^*\xi_0$, and recall the Φ-standard basis $\{E_i^*\xi_0\}_{i=0}^d$. The matrix P from Definition 11.1 is the transition matrix from $\{E_i\xi_0^*\}_{i=0}^d$ to $\{E_i^*\xi_0\}_{i=0}^d$. Now by Definition 11.8, $\alpha\mathcal{P}$ is the transition matrix from $\{E_i\xi_0^*\}_{i=0}^d$ to $\{\alpha\ell_i^{-1}E_i^*\xi_0\}_{i=0}^d$. Moreover by Proposition 10.2, $\{\alpha\ell_i^{-1}E_i^*\xi_0\}_{i=0}^d$ is the inverted dual of $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$. Therefore $\alpha\mathcal{P}$ is the transition matrix from $\{E_i\xi_0^*\}_{i=0}^d$ to the inverted dual of $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$. □

The following definition is motivated by Corollary 11.10 and Corollary 11.11.

Definition 11.12. We call the matrix $\mathcal{P}$ from Definition 11.8 the *west-south normalized transition matrix* of Φ.

12 The transition matrices $P, \mathcal{P}$ and their relatives

Let Φ denote a TH system. In the previous section we discussed two closely related transition matrices $P, \mathcal{P}$ associated with Φ. In this section we find the relationship between $P, \mathcal{P}$ and their relatives. There are two types of relations; one type is best expressed in terms of P and its relatives, while the other is best expressed in terms of $\mathcal{P}$ and its relatives.

Proposition 12.1. [1, Proposition 10.9] *With reference to Definition 11.1, both*

$$PP^* = P^*P = \nu I, \qquad \tilde{P}\tilde{P}^* = \tilde{P}^*\tilde{P} = \tilde{\nu} I,$$

where $\nu, \tilde{\nu}$ are from Definition 9.1.

Let $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ denote bases for V. Let $T \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. By elementary linear algebra, T^t is the transition matrix from the dual of $\{v_i\}_{i=0}^d$ to the dual of $\{u_i\}_{i=0}^d$. Therefore T^ς is the transition matrix from the inverted dual of $\{v_i\}_{i=0}^d$ to the inverted dual of $\{u_i\}_{i=0}^d$.

Proposition 12.2. *With reference to Definition 11.8, both*

$$\mathcal{P}^\varsigma = \tilde{\mathcal{P}}^*, \qquad (\mathcal{P}^*)^\varsigma = \tilde{\mathcal{P}}. \tag{37}$$

Proof: Choose $0 \neq \xi_0^* \in E_0^*V$ and $0 \neq \tilde{\xi}_d \in E_d^\sigma \tilde{V}$ so that $\langle \xi_0^*, \tilde{\xi}_d \rangle = 1$. Recall the Φ^*-standard basis $\{E_i\xi_0^*\}_{i=0}^d$ for V and the $\tilde{\Phi}$-standard basis $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$ for $\tilde{V}$. By Corollary 11.11, $\mathcal{P}$ is the transition matrix from $\{E_i\xi_0^*\}_{i=0}^d$ to the inverted dual of $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$. Moreover by Corollary 11.11 applied to $\tilde{\Phi}^*$, $\tilde{\mathcal{P}}^*$ is the transition matrix from $\{E_{d-i}^{*\sigma}\tilde{\xi}_d\}_{i=0}^d$ to the inverted dual of $\{E_i\xi_0^*\}_{i=0}^d$. By these comments and the ones above this proposition we obtain the equation on the left in (37). The equation on the right in (37) is similarly obtained. □

The following lemma could be used to give another proof of Proposition 12.2.

Lemma 12.3. *With reference to Definition 11.5, both*

$$p(\mu, \lambda) = \tilde{p}^*(\lambda, \mu), \qquad p^*(\mu, \lambda) = \tilde{p}(\lambda, \mu).$$

Proof: Combining Corollary 7.1 and Definition 11.5, we find

$$\tilde{p}^*(\lambda,\mu) = \sum_{h=0}^{d} \frac{\tau_h^*(\lambda)\eta_h(\mu)}{\phi_1\phi_2\cdots\phi_h}.$$

Therefore $p(\mu,\lambda) = \tilde{p}^*(\lambda,\mu)$. The proof for the other claim is similar. □

We will continue our discussion of the transition matrices $P, \mathcal{P}$ in Section 17. In Sections 13–16 we recall some linear algebra that will be needed in the discussion.

13 Vandermonde matrices and systems

Let Φ denote a TH system. In Section 17 we will show that each of the transition matrices $P, \mathcal{P}$ of Φ has a certain structure said to be double Vandermonde. To prepare for that, over the next few sections we discuss some linear algebra related to Vandermonde matrices.

Definition 13.1. Let n denote a nonnegative integer. Let $\{f_i\}_{i=0}^n$ denote a sequence of polynomials in $\mathbb{K}[\lambda]$. We say that $\{f_i\}_{i=0}^n$ is *graded* whenever

(i) $f_0 = 1$;

(ii) the degree of f_i is equal to i for $0 \leq i \leq n$.

Definition 13.2. A matrix $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ is called *west Vandermonde* whenever the following (i), (ii) hold.

(i) There exists a sequence of mutually distinct scalars $\{\theta_i\}_{i=0}^d$ taken from $\mathbb{K}$ and a graded sequence of polynomials $\{f_i\}_{i=0}^d$ in $\mathbb{K}[\lambda]$ such that

$$X_{ij} = X_{i0} f_j(\theta_i) \qquad (0 \leq i, j \leq d). \tag{38}$$

(ii) $X_{i0} \neq 0$ for $0 \leq i \leq d$.

With reference to Definition 13.2, assume X is west Vandermonde. As we will see, the polynomials $\{f_i\}_{i=0}^d$ are uniquely determined by the sequence of scalars $\{\theta_i\}_{i=0}^d$ but the sequence $\{\theta_i\}_{i=0}^d$ is not unique. To facilitate our discussion of this issue, we introduce the following term.

Definition 13.3. Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west Vandermonde matrix. Let $\{\theta_i\}_{i=0}^d$ denote a sequence of scalars taken from $\mathbb{K}$. We say that X and $\{\theta_i\}_{i=0}^d$ are *compatible* whenever

(i) $\theta_i \neq \theta_j$ if $i \neq j$ $\qquad (0 \leq i, j \leq d)$;

(ii) there exists a graded sequence of polynomials $\{f_i\}_{i=0}^d$ in $\mathbb{K}[\lambda]$ that satisfies (38).

Observe that if X and $\{\theta_i\}_{i=0}^d$ are compatible, then X and $\{\alpha\theta_i + \beta\}_{i=0}^d$ are compatible for any $\alpha, \beta \in \mathbb{K}$ with $\alpha \neq 0$.

Lemma 13.4. *Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west Vandermonde matrix. Let $\{\theta_i\}_{i=0}^d$ denote a sequence of scalars taken from $\mathbb{K}$. Then the following* (i), (ii) *are equivalent provided $d \geq 1$.*

(i) *X and $\{\theta_i\}_{i=0}^d$ are compatible.*

(ii) *There exists $a, b \in \mathbb{K}$ with $a \neq 0$ such that $\theta_i = aX_{i1}/X_{i0} + b$ for $0 \leq i \leq d$.*

Proof: (i) $\Rightarrow$ (ii) By Definition 13.3, there exists a polynomial $f_1 \in \mathbb{K}[\lambda]$ of degree 1 such that $X_{i1} = X_{i0} f_1(\theta_i)$ for $0 \leq i \leq d$. Write $f_1 = \alpha\lambda + \beta$ for some $\alpha, \beta \in \mathbb{K}$ with $\alpha \neq 0$. Thus $X_{i1} = X_{i0}(\alpha\theta_i + \beta)$ for $0 \leq i \leq d$. Rearranging terms, we find that there exists $a, b \in \mathbb{K}$ with $a \neq 0$ such that $\theta_i = aX_{i1}/X_{i0} + b$ for $0 \leq i \leq d$.
(ii) $\Rightarrow$ (i) By Definition 13.2, there exists a sequence of scalars $\{\theta'_i\}_{i=0}^d$ taken from $\mathbb{K}$ that is compatible with X. By the previous part, there exists $a', b' \in \mathbb{K}$ with $a' \neq 0$ such that $\theta'_i = a'X_{i1}/X_{i0} + b'$ for $0 \leq i \leq d$. Thus there exists $\alpha, \beta \in \mathbb{K}$ with $\alpha \neq 0$ such that $\theta_i = \alpha\theta'_i + \beta$ for $0 \leq i \leq d$. Now X and $\{\theta_i\}_{i=0}^d$ are compatible by the observation at the end of Definition 13.3. □

Lemma 13.5. *Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west Vandermonde matrix. Let $\{\theta_i\}_{i=0}^d$ denote a sequence of scalars taken from $\mathbb{K}$ that is compatible with X. Let $\{\theta'_i\}_{i=0}^d$ denote a sequence of scalars taken from $\mathbb{K}$. Then the following* (i), (ii) *are equivalent.*

(i) *X and $\{\theta'_i\}_{i=0}^d$ are compatible.*

(ii) *There exists $\alpha, \beta \in \mathbb{K}$ with $\alpha \neq 0$ such that $\theta'_i = \alpha\theta_i + \beta$ for $0 \leq i \leq d$.*

Proof: (i) $\Rightarrow$ (ii) Routine by Lemma 13.4.
(ii) $\Rightarrow$ (i) This is the observation at the end of Definition 13.3. □

Lemma 13.6. *Let $\{\theta_i\}_{i=0}^d$ denote a sequence of mutually distinct scalars taken from $\mathbb{K}$. Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west Vandermonde matrix that is compatible with $\{\theta_i\}_{i=0}^d$. Then there exists a unique graded sequence of polynomials $\{f_i\}_{i=0}^d$ in $\mathbb{K}[\lambda]$ that satisfies (38).*

Proof: By Definition 13.3 there exists a graded sequence of polynomials $\{f_i\}_{i=0}^d$ in $\mathbb{K}[\lambda]$ that satisfies (38). We show that this sequence is unique. Suppose that $\{f'_i\}_{i=0}^d$ is a graded sequences of polynomials in $\mathbb{K}[\lambda]$ that satisfies (38). We show that $f'_i = f_i$ for $0 \leq i \leq d$. Let i be given and define $g_i = f'_i - f_i$. Using (38), we find $g_i(\theta_j) = 0$ for $0 \leq j \leq d$. Since $\{\theta_i\}_{i=0}^d$ are mutually distinct and g_i has degree at most i, it follows that $g_i = 0$. Therefore $f'_i = f_i$. We have shown that the sequence $\{f_i\}_{i=0}^d$ is unique. □

Lemma 13.7. *Let $\{\theta_i\}_{i=0}^d$ denote a sequence of mutually distinct scalars taken from $\mathbb{K}$. Let $\{f_i\}_{i=0}^d$ denote a graded sequence of polynomials in $\mathbb{K}[\lambda]$. Let $\{c_i\}_{i=0}^d$ denote a sequence of nonzero scalars taken from $\mathbb{K}$. Define $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $X_{ij} = c_i f_j(\theta_i)$ for $0 \leq i, j \leq d$. Then X is west Vandermonde and compatible with $\{\theta_i\}_{i=0}^d$. Moreover $\{f_i\}_{i=0}^d$ are the corresponding polynomials from Lemma 13.6.*

Proof: Routine. □

Lemma 13.8. *Let $\{\theta_i\}_{i=0}^d$ denote a sequence of mutually distinct scalars taken from $\mathbb{K}$. Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west Vandermonde matrix that is compatible with $\{\theta_i\}_{i=0}^d$ and let $\{f_i\}_{i=0}^d$ denote the corresponding polynomials from Lemma 13.6. Let $X' \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Then the following* (i), (ii) *are equivalent.*

(i) *X' is a west Vandermonde matrix that is compatible with $\{\theta_i\}_{i=0}^d$ and $\{f_i\}_{i=0}^d$ are the corresponding polynomials from Lemma 13.6.*

(ii) *There exists an invertible diagonal matrix $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $X' = DX$.*

Proof: (i) $\Rightarrow$ (ii) Using (38), we find that for $0 \leq i, j \leq d$,

$$\frac{X'_{ij}}{X'_{i0}} = \frac{X_{ij}}{X_{i0}} = f_j(\theta_i). \tag{39}$$

Define a diagonal matrix $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ with (i,i)-entry X'_{i0}/X_{i0} for $0 \leq i \leq d$. Observe that D is invertible. Moreover $X' = DX$ by (39).
(ii) $\Rightarrow$ (i) Since $X' = DX$ we find that for $0 \leq i, j \leq d$,

$$X'_{ij} = X'_{i0}\frac{X_{ij}}{X_{i0}}.$$

By this and (38), we find that $X'_{ij} = X'_{i0} f_j(\theta_i)$ and (i) follows. □

Lemma 13.9. *Let $\{\theta_i\}_{i=0}^d$ denote a sequence of mutually distinct scalars taken from $\mathbb{K}$. Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west Vandermonde matrix that is compatible with $\{\theta_i\}_{i=0}^d$ and let $\{f_i\}_{i=0}^d$ denote the corresponding polynomials from Lemma 13.6. Let $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote an invertible diagonal matrix. Then XD is a west Vandermonde matrix that is compatible with $\{\theta_i\}_{i=0}^d$ and $\{D_{ii} f_i / D_{00}\}_{i=0}^d$ are the corresponding polynomials from Lemma 13.6.*

Proof: Routine using (38). □

Definition 13.10. By a *west Vandermonde system* in $\mathrm{Mat}_{d+1}(\mathbb{K})$ we mean a sequence $(X, \{\theta_i\}_{i=0}^d)$ such that

(i) X is a west Vandermonde matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$;

(ii) $\{\theta_i\}_{i=0}^d$ is a sequence of mutually distinct scalars taken from $\mathbb{K}$ that is compatible with X.

Definition 13.11. Let $(X, \{\theta_i\}_{i=0}^d)$ denote a west Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. In Lemma 13.6 we associated $(X, \{\theta_i\}_{i=0}^d)$ with some polynomials $\{f_i\}_{i=0}^d$. For convenience, let $f_{d+1} = \prod_{i=0}^d (\lambda - \theta_i)$. We call $\{f_i\}_{i=0}^{d+1}$ the *polynomials of* $(X, \{\theta_i\}_{i=0}^d)$.

Definition 13.12. Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$. Let $X' \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a matrix that is obtained by rotating X clockwise 90 degrees. We call X *south Vandermonde* whenever X' is west Vandermonde.

The above notions regarding west Vandermonde matrices carry over to south Vandermonde matrices.

We end this section with a comment.

Lemma 13.13. *Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west or south Vandermonde matrix. Then X is invertible.*

Proof: First assume that X is west Vandermonde. Perform invertible row and column operations on X so that the resulting matrix X' has (i,j)-entry θ_i^j for $0 \leq i, j \leq d$. The determinant of X' is equal to $\prod_{0 \leq i < j \leq d}(\theta_j - \theta_i)$. The $\{\theta_i\}_{i=0}^d$ are mutually distinct so this determinant is nonzero. Therefore X' is invertible so X is invertible. The case of south Vandermonde is similar. □

14 Hessenberg matrices and graded sequences of polynomials

Recall the notion of a Hessenberg matrix from Section 1. In the next section we discuss the role Vandermonde matrices play in the diagonalization of Hessenberg matrices. To prepare for that, in this section we discuss the relationship between Hessenberg matrices and graded sequences of polynomials.

Lemma 14.1. *Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a Hessenberg matrix. Then the minimal polynomial of H is equal to the characteristic polynomial of H.*

Proof: Using the Hessenberg shape of H, we find $I, H, H^2, \ldots, H^d$ are linearly independent. Therefore the minimal polynomial of H has degree $d+1$. The result follows. □

Given a Hessenberg matrix H, we are interested in finding the polynomial in Lemma 14.1.

Notation 14.2. Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a Hessenberg matrix. We denote by c_H the product $\prod_{i=1}^{d} H_{i,i-1}$. Observe that c_H is nonzero.

Definition 14.3. Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a Hessenberg matrix. Define a sequence of polynomials $\{f_i\}_{i=0}^{d+1}$ in $\mathbb{K}[\lambda]$ such that

(i) $f_0 = 1$;

(ii) $\lambda f_j = \sum_{i=0}^{j+1} H_{ij} f_i$ for $0 \leq j \leq d-1$;

(iii) $\lambda f_d = c_H^{-1} f_{d+1} + \sum_{i=0}^{d} H_{id} f_i$, where c_H is from Notation 14.2.

We call $\{f_i\}_{i=0}^{d+1}$ the *polynomials of H*.

Definition 14.4. A graded sequence of polynomials $\{f_i\}_{i=0}^{d+1}$ in $\mathbb{K}[\lambda]$ is called *standard* whenever f_{d+1} is monic.

Lemma 14.5. *Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a Hessenberg matrix with polynomials $\{f_i\}_{i=0}^{d+1}$. Then the following* (i)–(iii) *hold.*

(i) *For $0 \leq i \leq d$, f_i has degree i with λ^i coefficient $(\prod_{j=1}^{i} H_{j,j-1})^{-1}$.*

(ii) *f_{d+1} is monic with degree $d+1$.*

(iii) *The sequence $\{f_i\}_{i=0}^{d+1}$ is graded and standard.*

Proof: Routine. □

Let I denote the identity matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$. For $0 \leq i \leq d$, let ϵ_i denote the i^{th} column of I. Observe that $\{\epsilon_i\}_{i=0}^{d}$ is a basis for the vector space $\mathbb{K}^{d+1}$.

Lemma 14.6. *Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a Hessenberg matrix. Then there exists a unique standard graded sequence of polynomials $\{f_i\}_{i=0}^{d+1}$ in $\mathbb{K}[\lambda]$ such that $f_i(H)\epsilon_0 = \epsilon_i$ for $0 \leq i \leq d$ and $f_{d+1}(H)\epsilon_0 = 0$. The $\{f_i\}_{i=0}^{d+1}$ are the polynomials of H from Definition 14.3.*

Proof: Concerning existence, let $\{f_i\}_{i=0}^{d+1}$ denote the polynomials of H. By Lemma 14.5(iii) the sequence $\{f_i\}_{i=0}^{d+1}$ is graded and standard. We show that $f_i(H)\epsilon_0 = \epsilon_i$ for $0 \leq i \leq d$ and $f_{d+1}(H)\epsilon_0 = 0$. Abbreviate $v_i = f_i(H)\epsilon_0$ for $0 \leq i \leq d+1$ and note that $v_0 = \epsilon_0$. From Definition 14.3, we have

$$Hv_j = \sum_{i=0}^{j+1} H_{ij} v_i \qquad (0 \leq j \leq d), \tag{40}$$

where $H_{d+1,d} = c_H^{-1}$. By the definition of $\{\epsilon_i\}_{i=0}^{d}$, we have

$$H\epsilon_j = \sum_{i=0}^{j+1} H_{ij} \epsilon_i \qquad (0 \leq j \leq d), \tag{41}$$

where $\epsilon_{d+1} = 0$. Comparing (40), (41) and using $v_0 = \epsilon_0$, we find $v_i = \epsilon_i$ for $0 \leq i \leq d+1$. Therefore $f_i(H)\epsilon_0 = \epsilon_i$ for $0 \leq i \leq d$ and $f_{d+1}(H)\epsilon_0 = 0$. Concerning uniqueness, let $\{f_i'\}_{i=0}^{d+1}$ denote a standard graded sequence of polynomials in $\mathbb{K}[\lambda]$ such that $f_i'(H)\epsilon_0 = \epsilon_i$ for $0 \leq i \leq d$ and $f_{d+1}'(H)\epsilon_0 = 0$. We show that $f_i' = f_i$ for $0 \leq i \leq d+1$. Let i be given and define $g_i = f_i' - f_i$. Observe that $g_i(H)\epsilon_0 = 0$. Thus $g_i(H)\epsilon_j = g_i(H)f_j(H)\epsilon_0 = f_j(H)g_i(H)\epsilon_0 = 0$ for $0 \leq j \leq d$, so $g_i(H) = 0$. Therefore the minimal polynomial of H divides g_i. The polynomial g_i has degree at most d, and the minimal polynomial of H has degree $d+1$ by Lemma 14.1. Therefore $g_i = 0$ so $f_i' = f_i$. □

Corollary 14.7. *Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a Hessenberg matrix with polynomials $\{f_i\}_{i=0}^{d+1}$. Then f_{d+1} is both the minimal polynomial and the characteristic polynomial of H.*

Proof: Using Lemma 14.6 we find that $f_{d+1}(H)\epsilon_i = f_{d+1}(H)f_i(H)\epsilon_0 = f_i(H)f_{d+1}(H)\epsilon_0 = 0$ for $0 \leq i \leq d$. Therefore $f_{d+1}(H) = 0$. The result follows by Lemma 14.1 and Lemma 14.5(ii). □

So far, given a Hessenberg matrix we obtain a graded sequence of polynomials. Now turning things around, given a graded sequence of polynomials we obtain a Hessenberg matrix.

Definition 14.8. Let $\{f_i\}_{i=0}^{d+1}$ denote a graded sequence of polynomials in $\mathbb{K}[\lambda]$. Observe that for $0 \leq j \leq d$, λf_j is in the span of $\{f_i\}_{i=0}^{j+1}$. So for $0 \leq j \leq d$, there exists a unique sequence of scalars $\{c_{ij}\}_{i=0}^{j+1}$ taken from $\mathbb{K}$ such that $\lambda f_j = \sum_{i=0}^{j+1} c_{ij} f_i$. We call the scalar c_{ij} the (i,j)-*connection coefficient* for the given graded sequence of polynomials.

Definition 14.9. Let $\{f_i\}_{i=0}^{d+1}$ denote a graded sequence of polynomials in $\mathbb{K}[\lambda]$. By the *connection coefficient matrix* of $\{f_i\}_{i=0}^{d+1}$, we mean the Hessenberg matrix $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $H_{ij} = c_{ij}$ for $0 \leq i,j \leq d$, $i - j \leq 1$. The scalars c_{ij} are from Definition 14.8. Observe that the scalar $c_{d+1,d}$ plays no role in the definition of H.

Lemma 14.10. *Let $\{f_i\}_{i=0}^{d+1}$ (resp. $\{f_i'\}_{i=0}^{d+1}$) denote a graded sequence of polynomials in $\mathbb{K}[\lambda]$ with connection coefficient matrix H (resp. H'). Then the following* (i), (ii) *are equivalent.*

(i) $H = H'$.

(ii) $f_i = f_i'$ *for* $0 \leq i \leq d$ *and there exists* $0 \neq c \in \mathbb{K}$ *such that* $f_{d+1} = cf_{d+1}'$.

Proof: Routine by Definition 14.9. □

Lemma 14.11. *Let* $\{f_i\}_{i=0}^{d+1}$ *denote a standard graded sequence of polynomials in* $\mathbb{K}[\lambda]$ *and let* $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *denote a Hessenberg matrix. Then the following* (i), (ii) *are equivalent.*

(i) $\{f_i\}_{i=0}^{d+1}$ *are the polynomials of* H.

(ii) H *is the connection coefficient matrix of* $\{f_i\}_{i=0}^{d+1}$.

Proof: Routine. □

15 A Vandermonde matrix as a transition matrix

In this section we discuss the role that Vandermonde matrices play in the diagonalization of a Hessenberg matrix.

Lemma 15.1. *Let* $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *denote a Hessenberg matrix with polynomials* $\{f_i\}_{i=0}^{d+1}$. *Then the following* (i)–(iii) *are equivalent.*

(i) H *is diagonalizable.*

(ii) H *is multiplicity-free.*

(iii) f_{d+1} *has* $d+1$ *distinct roots in* $\mathbb{K}$.

Proof: Recall that the polynomial f_{d+1} has degree $d+1$ and it is the minimal polynomial of H by Corollary 14.7. By elementary linear algebra a matrix in $\mathrm{Mat}_{d+1}(\mathbb{K})$ is diagonalizable if and only if its minimal polynomial has distinct roots in $\mathbb{K}$. The result follows. □

Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a multiplicity-free Hessenberg matrix and let $\{\theta_i\}_{i=0}^{d}$ denote an ordering of the eigenvalues of H. Let $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the diagonal matrix with (i,i)-entry θ_i for $0 \leq i \leq d$. By elementary linear algebra, there exists an invertible $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $H = X^{-1}DX$. We comment on the uniqueness of X. Suppose that $Y \in \mathrm{Mat}_{d+1}(\mathbb{K})$ is invertible and $H = Y^{-1}DY$. Then $X^{-1}DX = Y^{-1}DY$ so $DYX^{-1} = YX^{-1}D$. Therefore YX^{-1} is diagonal by Lemma 3.1. By construction YX^{-1} is invertible. By these comments there exists an invertible diagonal matrix $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $Y = \Delta X$.

Lemma 15.2. *Let* $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *denote a multiplicity-free Hessenberg matrix with polynomials* $\{f_i\}_{i=0}^{d+1}$. *Let* $\{\theta_i\}_{i=0}^{d}$ *denote an ordering of the eigenvalues of* H *and let* $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ *denote the diagonal matrix with* (i,i)*-entry* θ_i *for* $0 \leq i \leq d$. *For* $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$, *the following* (i), (ii) *are equivalent.*

(i) X *is invertible and* $H = X^{-1}DX$.

(ii) $(X, \{\theta_i\}_{i=0}^{d})$ *is a west Vandermonde system with polynomials* $\{f_i\}_{i=0}^{d+1}$.

Proof: (i) $\Rightarrow$ (ii) Observe that $\{\theta_i\}_{i=0}^d$ are mutually distinct since H is multiplicity-free. We show that

$$X_{ij} = X_{i0} f_j(\theta_i) \qquad (0 \leq i, j \leq d). \tag{42}$$

Since $H = X^{-1}DX$ we have $f_j(H) = X^{-1} f_j(D) X$ so $X f_j(H) = f_j(D) X$. Hence $\epsilon_i^t X f_j(H) \epsilon_0 = \epsilon_i^t f_j(D) X \epsilon_0$. Simplify this equation using $f_j(H)\epsilon_0 = \epsilon_j$ from Lemma 14.6 together with matrix multiplication to obtain (42). By (42) and since X is invertible we find $X_{i0} \neq 0$ for $0 \leq i \leq d$. By Corollary 14.7 we have $f_{d+1} = \prod_{i=0}^d (\lambda - \theta_i)$. By these comments $(X, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system with polynomials $\{f_i\}_{i=0}^{d+1}$.
(ii) $\Rightarrow$ (i) By Lemma 13.13 X is invertible. We now show that $H = X^{-1}DX$. For $0 \leq i \leq d$, evaluate the equations in Definition 14.3(ii),(iii) at $\lambda = \theta_i$. In the resulting equations multiply each side by X_{i0} and simplify using (38) and Corollary 14.7 to obtain $\theta_i X_{ij} = \sum_{n=0}^d H_{nj} X_{in}$ for $0 \leq j \leq d$. Therefore $DX = XH$ so $H = X^{-1}DX$. □

Corollary 15.3. *Let $(X, \{\theta_i\}_{i=0}^d)$ denote a west Vandermonde system with polynomials $\{f_i\}_{i=0}^{d+1}$. Let $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the diagonal matrix with (i,i)-entry θ_i for $0 \leq i \leq d$. Then $X^{-1}DX$ is the connection coefficient matrix of $\{f_i\}_{i=0}^{d+1}$. Moreover $X^{-1}DX$ is multiplicity-free and Hessenberg.*

Proof: Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the connection coefficient matrix of $\{f_i\}_{i=0}^{d+1}$. Then H is Hessenberg by Definition 14.9. We show that $H = X^{-1}DX$ and that H is multiplicity-free. By Lemma 14.11 the $\{f_i\}_{i=0}^{d+1}$ are the polynomials of H. Since $f_{d+1} = \prod_{i=0}^d (\lambda - \theta_i)$ and $\{\theta_i\}_{i=0}^d$ are mutually distinct, we find using Corollary 14.7 that H is multiplicity-free. Now by Lemma 15.2 we find $H = X^{-1}DX$. The result follows. □

We have been discussing west Vandermonde systems. We now obtain analogous results for south Vandermonde systems.

Definition 15.4. Let $\{f_i\}_{i=0}^{d+1}$ denote a standard graded sequence of polynomials in $\mathbb{K}[\lambda]$. Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the connection coefficient matrix of $\{f_i\}_{i=0}^{d+1}$. Recall from Definition 14.9 that H is Hessenberg, so H^ς is Hessenberg. The polynomials of H^ς will be denoted by $\{f_i^\varsigma\}_{i=0}^{d+1}$. The two polynomial sequences $\{f_i\}_{i=0}^{d+1}$ and $\{f_i^\varsigma\}_{i=0}^{d+1}$ are said to be *associated*. Note that $f_{d+1}^\varsigma = f_{d+1}$ by Corollary 14.7.

Lemma 15.5. *Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a multiplicity-free Hessenberg matrix with polynomials $\{f_i\}_{i=0}^{d+1}$. Let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of H and let $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the diagonal matrix with (i,i)-entry θ_i for $0 \leq i \leq d$. For $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$, the following* (i), (ii) *are equivalent.*

(i) *X is invertible and $H = XDX^{-1}$.*

(ii) *$(X, \{\theta_i\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{f_i^\varsigma\}_{i=0}^{d+1}$.*

Proof: (i) $\Rightarrow$ (ii) In the equation $H = XDX^{-1}$, apply ς to each side to obtain $H^\varsigma = (X^\varsigma)^{-1} D^\varsigma X^\varsigma$. By this and Lemma 15.2 the sequence $(X^\varsigma, \{\theta_{d-i}\}_{i=0}^d)$ is a west Vandermonde system with polynomials $\{f_i^\varsigma\}_{i=0}^{d+1}$. Therefore $(X, \{\theta_i\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{f_i^\varsigma\}_{i=0}^{d+1}$.

(ii) $\Rightarrow$ (i) By assumption $(X, \{\theta_i\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{f_i^\varsigma\}_{i=0}^{d+1}$. Therefore $(X^\varsigma, \{\theta_{d-i}\}_{i=0}^d)$ is a west Vandermonde system with polynomials $\{f_i^\varsigma\}_{i=0}^{d+1}$. Applying Lemma 15.2 we find that X^ς is invertible and $H^\varsigma = (X^\varsigma)^{-1} D^\varsigma X^\varsigma$. Applying ς we find that X is invertible and $H = XDX^{-1}$. □

Corollary 15.6. *Let $(X, \{\theta_i\}_{i=0}^d)$ denote a south Vandermonde system with polynomials $\{f_i\}_{i=0}^{d+1}$. Let $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the diagonal matrix with (i,i)-entry θ_i for $0 \leq i \leq d$. Then XDX^{-1} is the connection coefficient matrix of $\{f_i^\varsigma\}_{i=0}^{d+1}$. Moreover XDX^{-1} is multiplicity-free and Hessenberg.*

Proof: Similar to the proof of Corollary 15.3. □

16 The inverse of a Vandermonde matrix

Let $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote a west or south Vandermonde matrix. In Lemma 13.13 we showed that X is invertible. In this section we discuss the matrix X^{-1}.

Proposition 16.1. *Let $(X, \{\theta_i\}_{i=0}^d)$ denote a west Vandermonde system with polynomials $\{f_i\}_{i=0}^{d+1}$. Then the following* (i), (ii) *hold.*

(i) *$(X^{-1}, \{\theta_i\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{f_i^\varsigma\}_{i=0}^{d+1}$, where $\{f_i^\varsigma\}_{i=0}^{d+1}$ are the associated polynomials of $\{f_i\}_{i=0}^{d+1}$.*

(ii) *$(X^{-1})_{dj} = \frac{c_H}{\tau_j(\theta_j)\eta_{d-j}(\theta_j)X_{j0}}$ for $0 \leq j \leq d$, where H is the connection coefficient matrix of $\{f_i\}_{i=0}^{d+1}$ and c_H is from Notation 14.2.*

Proof: (i) Let $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the diagonal matrix with (i,i)-entry θ_i for $0 \leq i \leq d$. Observe that H is Hessenberg by Definition 14.9 and that $\{f_i\}_{i=0}^{d+1}$ are the polynomials of H by Lemma 14.11. Since $f_{d+1} = \prod_{i=0}^d (\lambda - \theta_i)$ and $\{\theta_i\}_{i=0}^d$ are mutually distinct, we find using Corollary 14.7 that H is multiplicity-free and $\{\theta_i\}_{i=0}^d$ is an ordering of the eigenvalues of H. Therefore $H = X^{-1}DX$ by Lemma 15.2. Applying Lemma 15.5 to X^{-1}, we find $(X^{-1}, \{\theta_i\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{f_i^\varsigma\}_{i=0}^{d+1}$.
(ii) First assume that $X_{i0} = 1$ for $0 \leq i \leq d$. Let $h \in \mathbb{K}[\lambda]$ denote the polynomial $\sum_{i=0}^d (X^{-1})_{ij} f_i$. In the equation $XX^{-1} = I$, evaluate the j^{th}-column using matrix multiplication to find that $h(\theta_i) = \delta_{ij}$ for $0 \leq i \leq d$. Let $e_j \in \mathbb{K}[\lambda]$ denote the polynomial $\frac{\tau_j \eta_{d-j}}{\tau_j(\theta_j)\eta_{d-j}(\theta_j)}$. Observe that $e_j(\theta_i) = \delta_{ij}$ for $0 \leq i \leq d$. Thus $h(\theta_i) = e_j(\theta_i)$ for $0 \leq i \leq d$. It follows that $h = e_j$ since both h and e_j have degree d. In particular, the leading coefficient of h is equal to the leading coefficient of e_j. By Lemma 14.5(i) the leading coefficient of f_d is c_H^{-1}, so the leading coefficient of h is $(X^{-1})_{dj} c_H^{-1}$. The leading coefficient of e_j is $(\tau_j(\theta_j)\eta_{d-j}(\theta_j))^{-1}$. By these comments $(X^{-1})_{dj} c_H^{-1} = (\tau_j(\theta_j)\eta_{d-j}(\theta_j))^{-1}$ so $(X^{-1})_{dj} = \frac{c_H}{\tau_j(\theta_j)\eta_{d-j}(\theta_j)}$. The result is now proven for the special case in which $X_{i0} = 1$ for $0 \leq i \leq d$. For the general case, apply the special case to the west Vandermonde system $(\Delta^{-1}X, \{\theta_i\}_{i=0}^d)$, where $\Delta \in \mathrm{Mat}_{d+1}(\mathbb{K})$ is the diagonal matrix with (i,i)-entry X_{i0} for $0 \leq i \leq d$. □

Proposition 16.2. *Let $(X, \{\theta_i\}_{i=0}^d)$ denote a south Vandermonde system with polynomials $\{f_i\}_{i=0}^{d+1}$. Then the following* (i), (ii) *hold.*

(i) $(X^{-1}, \{\theta_i\}_{i=0}^d)$ *is a west Vandermonde system with polynomials* $\{f_i^\varsigma\}_{i=0}^{d+1}$*, where* $\{f_i^\varsigma\}_{i=0}^{d+1}$ *are the associated polynomials of* $\{f_i\}_{i=0}^{d+1}$*.*

(ii) $(X^{-1})_{i0} = \frac{c_H}{\tau_i(\theta_i)\eta_{d-i}(\theta_i)X_{di}}$ *for* $0 \leq i \leq d$*, where* H *is the connection coefficient matrix of* $\{f_i\}_{i=0}^{d+1}$ *and* c_H *is from Notation 14.2.*

Proof: Similar to the proof of Proposition 16.1. □

In the next section we return to our discussion of TH systems.

17 The transition matrices $P, \mathcal{P}$ and their Vandermonde structures

We return our attention to TH systems. Let Φ denote a TH system. Recall the transition matrices P and $\mathcal{P}$ of Φ from Definition 11.1 and Definition 11.8. We will show that each of $P, \mathcal{P}$ has a west Vandermonde structure and a south Vandermonde structure. We start by associating with Φ a graded sequence of polynomials.

Definition 17.1. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Let $\{v_i\}_{i=0}^d$ denote a Φ-standard basis for V and let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the matrix representing A with respect to $\{v_i\}_{i=0}^d$. Observe that H is Hessenberg. Let $\{s_i\}_{i=0}^{d+1}$ denote the polynomials of H from Definition 14.3, so that $s_i(A)v_0 = v_i$ for $0 \leq i \leq d$ by Lemma 14.6 and s_{d+1} is the minimal polynomial of A by Corollary 14.7.

The following normalization of the $\{s_i\}_{i=0}^{d+1}$ will be useful.

Definition 17.2. With reference to Definition 17.1, let $\{t_i\}_{i=0}^{d+1}$ denote the sequence of polynomials in $\mathbb{K}[\lambda]$ that satisfies (i), (ii) below.

(i) For $0 \leq i \leq d$, $t_i = s_i/\ell_i$ where ℓ_i is from Definition 8.1.

(ii) $t_{d+1} = s_{d+1}$.

We will show in Corollary 17.10 that $t_i(\theta_d) = 1$ for $0 \leq i \leq d$.

In Definition 17.1 we saw how the polynomials $\{s_i\}_{i=0}^{d+1}$ arise naturally from the action of A on a Φ-standard basis for V. We now discuss the meaning of the polynomials $\{t_i\}_{i=0}^{d+1}$ from this point of view. Let $\{u_i\}_{i=0}^d$ denote the inverted dual of a $\tilde{\Phi}$-standard basis for $\tilde{V}$. By Corollary 10.3, $\{\ell_i u_i\}_{i=0}^d$ is a Φ-standard basis for V, where ℓ_i is from Definition 8.1. Therefore by Definition 17.2, $t_i(A)u_0 = u_i$ for $0 \leq i \leq d$ and t_{d+1} is the minimal polynomial of A.

Our next goal is to show that the polynomials $\{s_i\}_{i=0}^{d+1}$ and $\{\tilde{t}_i\}_{i=0}^{d+1}$ are associated in the sense of Definition 15.4. We will use the following fact. Let $\{v_i\}_{i=0}^d$ denote a basis for V and let $R \in \mathrm{End}(V)$. Let $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the matrix representing R with respect to $\{v_i\}_{i=0}^d$. By elementary linear algebra, S^t is the matrix representing R^σ with respect to the dual of $\{v_i\}_{i=0}^d$, where $\sigma : \mathrm{End}(V) \to \mathrm{End}(\tilde{V})$ is the canonical anti-isomorphism from above Definition 6.12. Therefore the matrix S^ς represents R^σ with respect to the inverted dual of $\{v_i\}_{i=0}^d$.

Lemma 17.3. *With reference to Definition 17.1 and Definition 17.2, for each column of the table below, the two graded sequences of polynomials are associated in the sense of Definition 15.4.*

$\{s_i\}_{i=0}^{d+1}$	$\{s_i^*\}_{i=0}^{d+1}$	$\{\tilde{s}_i\}_{i=0}^{d+1}$	$\{\tilde{s}_i^*\}_{i=0}^{d+1}$
$\{\tilde{t}_i\}_{i=0}^{d+1}$	$\{\tilde{t}_i^*\}_{i=0}^{d+1}$	$\{t_i\}_{i=0}^{d+1}$	$\{t_i^*\}_{i=0}^{d+1}$

Proof: Let $\{v_i\}_{i=0}^d$ denote a Φ-standard basis for V and let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the matrix representing A with respect to $\{v_i\}_{i=0}^d$. By Definition 17.1, $\{s_i\}_{i=0}^{d+1}$ are the polynomials of H. Let $\{w_i\}_{i=0}^d$ denote the inverted dual of $\{v_i\}_{i=0}^d$. Applying the comments below Definition 17.2 to $\tilde{\Phi}$, we find $\tilde{t}_i(A^\sigma)w_0 = w_i$ for $0 \leq i \leq d$ and $\tilde{t}_{d+1}$ is the minimal polynomial of A^σ. Moreover by the comments above the present lemma, the matrix H^ς represents A^σ with respect to $\{w_i\}_{i=0}^d$. Now by Lemma 14.6 the $\{\tilde{t}_i\}_{i=0}^{d+1}$ are the polynomials of H^ς. Therefore $\{s_i\}_{i=0}^{d+1}$ and $\{\tilde{t}_i\}_{i=0}^{d+1}$ are associated by Definition 15.4. We have verified our assertions about the first column of the above table. Our assertions about the remaining columns follow from Definition 7.2. $\square$

We recall some elementary linear algebra. Let $\{u_i\}_{i=0}^d$ and $\{v_i\}_{i=0}^d$ denote bases for V. Let $T \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Pick $A \in \mathrm{End}(V)$ and let $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the matrix that represents A with respect to $\{u_i\}_{i=0}^d$. Then the matrix $T^{-1}ST$ represents A with respect to $\{v_i\}_{i=0}^d$.

We now display a west Vandermonde structure for P.

Proposition 17.4. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let P denote the transition matrix of Φ from Definition 11.1. Then $(P, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system, and the corresponding polynomials are the $\{s_i\}_{i=0}^{d+1}$ from Definition 17.1. For each relative of P we display a west Vandermonde system along with the corresponding polynomials.*

west Vandermonde system	*corresponding polynomials*
$(P, \{\theta_i\}_{i=0}^d)$	$\{s_i\}_{i=0}^{d+1}$
$(P^*, \{\theta_i^*\}_{i=0}^d)$	$\{s_i^*\}_{i=0}^{d+1}$
$(\tilde{P}, \{\theta_{d-i}\}_{i=0}^d)$	$\{\tilde{s}_i\}_{i=0}^{d+1}$
$(\tilde{P}^*, \{\theta_{d-i}^*\}_{i=0}^d)$	$\{\tilde{s}_i^*\}_{i=0}^{d+1}$

Proof: Write $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ and assume V is the vector space underlying Φ. Let $H \in \mathrm{Mat}_{d+1}(\mathbb{K})$ (resp. $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$) denote the matrix representing A with respect to a Φ-standard (resp. Φ^*-standard) basis for V. By construction H is Hessenberg and multiplicity-free with an ordering of the eigenvalues $\{\theta_i\}_{i=0}^d$. By construction D is diagonal with (i,i)-entry θ_i for $0 \leq i \leq d$. By Definition 17.1 $\{s_i\}_{i=0}^{d+1}$ are the polynomials of H. By Definition 11.1 and the comments above this proposition, we have $H = P^{-1}DP$. Therefore by Lemma 15.2 $(P, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system with polynomials $\{s_i\}_{i=0}^{d+1}$. We have verified our assertions about the first row of the above table. Our assertions about the remaining rows follow from Corollary 7.1. $\square$

We now display a south Vandermonde structure for P.

Proposition 17.5. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let P denote the transition matrix of Φ from Definition 11.1. Then $(P, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system, and the corresponding polynomials are the $\{\tilde{t}_i^*\}_{i=0}^{d+1}$ from Definition 17.2. For each relative of P we display a south Vandermonde system along with the corresponding polynomials.*

south Vandermonde system	*corresponding polynomials*
$(P, \{\theta_i^*\}_{i=0}^d)$	$\{\tilde{t}_i^*\}_{i=0}^{d+1}$
$(P^*, \{\theta_i\}_{i=0}^d)$	$\{\tilde{t}_i\}_{i=0}^{d+1}$
$(\tilde{P}, \{\theta_{d-i}^*\}_{i=0}^d)$	$\{t_i^*\}_{i=0}^{d+1}$
$(\tilde{P}^*, \{\theta_{d-i}\}_{i=0}^d)$	$\{t_i\}_{i=0}^{d+1}$

Proof: By Proposition 17.4 $(P^*, \{\theta_i^*\}_{i=0}^d)$ is a west Vandermonde system with polynomials $\{s_i^*\}_{i=0}^{d+1}$. Thus by Proposition 16.1 and Lemma 17.3, $((P^*)^{-1}, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{\tilde{t}_i^*\}_{i=0}^{d+1}$. By this and since $PP^* = \nu I$, $(\nu^{-1}P, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{\tilde{t}_i^*\}_{i=0}^{d+1}$. Therefore by Lemma 13.8 $(P, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system with polynomials $\{\tilde{t}_i^*\}_{i=0}^{d+1}$. We have verified our assertions about the first row of the above table. Our assertions about the remaining rows follow from Corollary 7.1. □

We now turn to the matrix $\mathcal{P}$. Below we display a west Vandermonde structure and a south Vandermonde structure for $\mathcal{P}$. We begin with the west Vandermonde structure.

Corollary 17.6. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\mathcal{P}$ denote the transition matrix of Φ from Definition 11.8. Then $(\mathcal{P}, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system, and the corresponding polynomials are the $\{t_i\}_{i=0}^{d+1}$ from Definition 17.2. For each relative of $\mathcal{P}$ we display a west Vandermonde system along with the corresponding polynomials.*

west Vandermonde system	*corresponding polynomials*
$(\mathcal{P}, \{\theta_i\}_{i=0}^d)$	$\{t_i\}_{i=0}^{d+1}$
$(\mathcal{P}^*, \{\theta_i^*\}_{i=0}^d)$	$\{t_i^*\}_{i=0}^{d+1}$
$(\tilde{\mathcal{P}}, \{\theta_{d-i}\}_{i=0}^d)$	$\{\tilde{t}_i\}_{i=0}^{d+1}$
$(\tilde{\mathcal{P}}^*, \{\theta_{d-i}^*\}_{i=0}^d)$	$\{\tilde{t}_i^*\}_{i=0}^{d+1}$

Proof: Routine by Lemma 13.9 and Proposition 17.4. □

We now display a south Vandermonde structure for $\mathcal{P}$.

Corollary 17.7. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\mathcal{P}$ denote the transition matrix of Φ from Definition 11.8. Then $(\mathcal{P}, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system, and the corresponding polynomials are the $\{\tilde{t}_i^*\}_{i=0}^{d+1}$ from Definition 17.2. For each relative of $\mathcal{P}$ we display a south Vandermonde system along with the corresponding polynomials.*

south Vandermonde system	*corresponding polynomials*
$(\mathcal{P}, \{\theta_i^*\}_{i=0}^d)$	$\{\tilde{t}_i^*\}_{i=0}^{d+1}$
$(\mathcal{P}^*, \{\theta_i\}_{i=0}^d)$	$\{\tilde{t}_i\}_{i=0}^{d+1}$
$(\tilde{\mathcal{P}}, \{\theta_{d-i}^*\}_{i=0}^d)$	$\{t_i^*\}_{i=0}^{d+1}$
$(\tilde{\mathcal{P}}^*, \{\theta_{d-i}\}_{i=0}^d)$	$\{t_i\}_{i=0}^{d+1}$

Proof: Similar to the proof of Corollary 17.6 using Proposition 17.5. □

We have now displayed the west Vandermonde and south Vandermonde structures for P and $\mathcal{P}$. As corollaries to these results, we now obtain some facts involving the polynomials $\{s_i\}_{i=0}^{d+1}$ and $\{t_i\}_{i=0}^{d+1}$.

Corollary 17.8. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let P denote the transition matrix of Φ from Definition 11.1 and let $\mathcal{P}$ denote the corresponding matrix from Definition 11.8. Let $\{s_i\}_{i=0}^{d+1}$ (resp. $\{t_i\}_{i=0}^{d+1}$) denote the polynomials of Φ from Definition 17.1 (resp. Definition 17.2). Then the following* (i), (ii) *hold for $0 \leq i, j \leq d$.*

(i) $P_{ij} = \ell_j t_j(\theta_i) = \ell_j \tilde{t}_{d-i}^*(\theta_j^*) = s_j(\theta_i) = \ell_j \tilde{s}_{d-i}^*(\theta_j^*)/\tilde{\ell}_{d-i}^*$.

(ii) $\mathcal{P}_{ij} = t_j(\theta_i) = \tilde{t}_{d-i}^*(\theta_j^*) = s_j(\theta_i)/\ell_j = \tilde{s}_{d-i}^*(\theta_j^*)/\tilde{\ell}_{d-i}^*$.

Here ℓ_j, $\tilde{\ell}_{d-i}^*$ are from Definition 8.1 and Lemma 8.2 respectively.

Proof: (i) Using Corollary 11.3, (38), and Proposition 17.4, we find $P_{ij} = s_j(\theta_i)$. Similarly using Proposition 17.5 in place of Proposition 17.4 we find $P_{ij} = \ell_j \tilde{t}_{d-i}^*(\theta_j^*)$. The remaining assertions follow using Definition 17.2.
(ii) Use (i) and the fact that $P_{ij} = \mathcal{P}_{ij}\ell_j$ for $0 \leq i, j \leq d$. □

We emphasize one aspect of Corollary 17.8 which is telling us that the $\{s_i\}_{i=0}^{d+1}$ and the $\{t_i\}_{i=0}^{d+1}$ each satisfy a variation on the Askey-Wilson duality [3, Theorems 14.7–14.9].

Corollary 17.9. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\{s_i\}_{i=0}^{d+1}$ (resp. $\{t_i\}_{i=0}^{d+1}$) denote the corresponding polynomials from Definition 17.1 (resp. Definition 17.2). Then the following* (i), (ii) *hold for $0 \leq i, j \leq d$.*

(i) $t_j(\theta_i) = \tilde{t}_{d-i}^*(\theta_j^*)$.

(ii) $s_j(\theta_i)/\ell_j = \tilde{s}_{d-i}^*(\theta_j^*)/\tilde{\ell}_{d-i}^*$, *where ℓ_j, $\tilde{\ell}_{d-i}^*$ are from Definition 8.1 and Lemma 8.2 respectively.*

We have a comment on how the polynomials $\{s_i\}_{i=0}^{d+1}$ and $\{t_i\}_{i=0}^{d+1}$ are normalized.

Corollary 17.10. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\{s_i\}_{i=0}^{d+1}$ (resp. $\{t_i\}_{i=0}^{d+1}$) denote the corresponding polynomials from Definition 17.1 (resp. Definition 17.2). Then the following* (i), (ii) *hold.*

(i) $s_i(\theta_d) = \ell_i$ *for $0 \leq i \leq d$, where ℓ_i is from Definition 8.1.*

(ii) $t_i(\theta_d) = 1$ *for $0 \leq i \leq d$.*

Proof: Use Corollary 11.3 and Corollary 17.8(i). □

The polynomials $\{t_i\}_{i=0}^{d+1}$ are not orthogonal in general; however we do have the following.

Corollary 17.11. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\{t_i\}_{i=0}^{d+1}$ denote the corresponding polynomials from Definition 17.2. Then the following* (i), (ii) *hold.*

(i) $\displaystyle\sum_{n=0}^{d} t_i(\theta_n)\tilde{t}_j(\theta_n)\ell_n^* = \delta_{i+j,d}\,\nu\ell_i^{-1}$ $\qquad (0 \le i, j \le d)$.

(ii) $\displaystyle\sum_{i=0}^{d} t_i(\theta_m)\tilde{t}_{d-i}(\theta_n)\ell_i = \delta_{mn}\,\nu(\ell_m^*)^{-1}$ $\qquad (0 \le m, n \le d)$.

Here $\{\ell_i\}_{i=0}^d$, $\{\ell_i^*\}_{i=0}^d$ *are from Definition 8.1 and Lemma 8.2 respectively, and* ν *is from Definition 9.1.*

Proof: (i) Let P denote the transition matrix of Φ from Definition 11.1. In the equation $P^*P = \nu I$, compare the $(d-j, i)$-entry of each side to obtain $\sum_{n=0}^d P_{d-j,n}^* P_{ni} = \delta_{i+j,d}\nu$. In this equation, evaluate P_{ni} and $P_{d-j,n}^*$ using Corollary 17.8 to obtain $\sum_{n=0}^d \ell_n^* \tilde{t}_j(\theta_n)\ell_i t_i(\theta_n) = \delta_{i+j,d}\nu$. The result follows.
(ii) Let P denote the transition matrix of Φ from Definition 11.1. In the equation $PP^* = \nu I$, compare the (m, n)-entry of each side to obtain $\sum_{i=0}^d P_{mi}P_{in}^* = \delta_{mn}\nu$. In this equation, evaluate P_{mi} and P_{in}^* using Corollary 17.8 to obtain $\sum_{i=0}^d \ell_i t_i(\theta_m)\ell_n^*\tilde{t}_{d-i}(\theta_n) = \delta_{mn}\nu$. The result follows. □

We now give an analogue of Corollary 17.11 that applies to the polynomials $\{s_i\}_{i=0}^{d+1}$.

Corollary 17.12. *Let* Φ *denote a TH system with eigenvalue sequence* $\{\theta_i\}_{i=0}^d$ *and dual eigenvalue sequence* $\{\theta_i^*\}_{i=0}^d$. *Let* $\{s_i\}_{i=0}^{d+1}$ *denote the corresponding polynomials from Definition 17.1. Then the following* (i), (ii) *hold.*

(i) $\displaystyle\sum_{n=0}^{d} s_i(\theta_n)\tilde{s}_j(\theta_n)\ell_n^* = \delta_{i+j,d}\,\nu\tilde{\ell}_j$ $\qquad (0 \le i, j \le d)$.

(ii) $\displaystyle\sum_{i=0}^{d} s_i(\theta_m)\tilde{s}_{d-i}(\theta_n)(\tilde{\ell}_{d-i})^{-1} = \delta_{mn}\,\nu(\ell_m^*)^{-1}$ $\qquad (0 \le m, n \le d)$.

Here $\{\tilde{\ell}_i\}_{i=0}^d$, $\{\ell_i^*\}_{i=0}^d$ *are from Lemma 8.2 and* ν *is from Definition 9.1.*

Proof: Use Definition 17.2(i) and Corollary 17.11. □

We now express the polynomials $\{t_i\}_{i=0}^{d+1}$ and $\{s_i\}_{i=0}^{d+1}$ in terms of the parameter array of Φ. To do this we will use the two-variable polynomial p of Φ from Definition 11.5.

Corollary 17.13. *Let* Φ *denote a TH system with eigenvalue sequence* $\{\theta_i\}_{i=0}^d$ *and dual eigenvalue sequence* $\{\theta_i^*\}_{i=0}^d$. *Let* $\{t_i\}_{i=0}^{d+1}$ *denote the corresponding polynomials from Definition 17.2. Then the following* (i), (ii) *hold.*

(i) *For* $0 \le i \le d$, $t_i = p(\lambda, \theta_i^*)$ *where* p *is from Definition 11.5.*

(ii) $t_{d+1} = \prod_{i=0}^d (\lambda - \theta_i)$.

Proof: Let $\mathcal{P}$ denote the transition matrix of Φ from Definition 11.8. Since $\mathcal{P}_{ij} = p(\theta_i, \theta_j^*)$ for $0 \le i, j \le d$, we find that $(\mathcal{P}, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system with polynomials $\{f_i\}_{i=0}^{d+1}$ where $f_i = p(\lambda, \theta_i^*)$ for $0 \le i \le d$ and $f_{d+1} = \prod_{i=0}^d (\lambda - \theta_i)$. The result follows by Corollary 17.6. □

Example 17.14. With reference to Definition 17.2, assume $d = 2$. Then

$$\begin{aligned} t_0 &= 1, \\ t_1 &= 1 + \frac{(\lambda - \theta_2)(\theta_1^* - \theta_0^*)}{\phi_1}, \\ t_2 &= 1 + \frac{(\lambda - \theta_2)(\theta_2^* - \theta_0^*)}{\phi_1} + \frac{(\lambda - \theta_2)(\lambda - \theta_1)(\theta_2^* - \theta_0^*)(\theta_2^* - \theta_1^*)}{\phi_1 \phi_2}. \end{aligned}$$

Corollary 17.15. *Let Φ denote a TH system with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\{s_i\}_{i=0}^{d+1}$ denote the corresponding polynomials from Definition 17.1. Then the following* (i), (ii) *hold.*

(i) *For $0 \leq i \leq d$, $s_i = \ell_i p(\lambda, \theta_i^*)$ where ℓ_i is from Definition 8.1 and p is from Definition 11.5.*

(ii) $s_{d+1} = \prod_{i=0}^d (\lambda - \theta_i)$.

Proof: Use Definition 17.2 and Corollary 17.13. □

Remark 17.16. In view of Corollary 17.13, one may wonder about the polynomial $p(\theta_i, \lambda)$. By Lemma 12.3 and Corollary 17.13, $p(\theta_i, \lambda) = \tilde{p}^*(\lambda, \theta_i) = \tilde{t}_{d-i}^*$ for $0 \leq i \leq d$.

We now give the results promised at the end of Section 10. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Let $0 \neq \xi_0 \in E_0 V$ and recall the Φ-standard basis $\{E_i^* \xi_0\}_{i=0}^d$ for V from above (3). Let $0 \neq \tilde{\xi}_d^* \in E_d^{*\sigma} \tilde{V}$ and recall the $\tilde{\Phi}^*$-standard basis $\{E_{d-i}^{*\sigma} \tilde{\xi}_d^*\}_{i=0}^d$ for $\tilde{V}$ from above Proposition 10.5. These two bases are related as follows.

Proposition 17.17. *With reference to the TH system Φ in Definition 6.12, let $0 \neq \xi_0 \in E_0 V$ and $0 \neq \tilde{\xi}_d^* \in E_d^{*\sigma} \tilde{V}$. Then for $0 \leq i, j \leq d$,*

$$\langle E_i^* \xi_0, E_j^{*\sigma} \tilde{\xi}_d^* \rangle = \nu^{-1} \ell_i \ell_j^* t_i(\theta_j) \langle \xi_0, \tilde{\xi}_d^* \rangle.$$

Here ℓ_i, ℓ_j^* are from Definition 8.1 and Lemma 8.2 respectively, and ν, t_i are from Definition 9.1 and Definition 17.2 respectively.

Proof: Let P denote the transition matrix of Φ from Definition 11.1. Let $\xi_0^* = E_0^* \xi_0$ and observe by Lemma 9.2 that

$$E_0 \xi_0^* = E_0 E_0^* \xi_0 = E_0 E_0^* E_0 \xi_0 = \nu^{-1} E_0 \xi_0 = \nu^{-1} \xi_0. \tag{43}$$

We may now argue

$$\begin{aligned} \langle E_i^* \xi_0, E_j^{*\sigma} \tilde{\xi}_d^* \rangle &= \sum_{n=0}^d P_{ni} \langle E_n \xi_0^*, E_j^{*\sigma} \tilde{\xi}_d^* \rangle && \text{(by Definition 11.1)} \\ &= \ell_j^* P_{ji} \langle E_0 \xi_0^*, \tilde{\xi}_d^* \rangle && \text{(by Proposition 10.5)} \\ &= \ell_j^* s_i(\theta_j) \langle E_0 \xi_0^*, \tilde{\xi}_d^* \rangle && \text{(by Corollary 17.8)} \\ &= \nu^{-1} \ell_j^* s_i(\theta_j) \langle \xi_0, \tilde{\xi}_d^* \rangle && \text{(by (43))} \\ &= \nu^{-1} \ell_i \ell_j^* t_i(\theta_j) \langle \xi_0, \tilde{\xi}_d^* \rangle && \text{(by Definition 17.2).} \end{aligned}$$

□

Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Let $0 \neq \xi_0^* \in E_0^* V$ and recall the Φ^*-standard basis $\{E_i \xi_0^*\}_{i=0}^d$ for V from above Proposition 10.5. Let $0 \neq \tilde{\xi}_d \in E_d^\sigma \tilde{V}$ and recall the $\tilde{\Phi}$-standard basis $\{E_{d-i}^{*\sigma} \tilde{\xi}_d\}_{i=0}^d$ for $\tilde{V}$ from above Proposition 10.2. These two bases are related as follows.

Proposition 17.18. *With reference to the TH system Φ in Definition 6.12, let $0 \neq \xi_0^* \in E_0^* V$ and $0 \neq \tilde{\xi}_d \in E_d^\sigma \tilde{V}$. Then for $0 \leq i, j \leq d$,*

$$\langle E_i \xi_0^*, E_j^{*\sigma} \tilde{\xi}_d \rangle = \nu^{-1} \ell_i^* \ell_j t_i^*(\theta_j^*) \langle \xi_0^*, \tilde{\xi}_d \rangle.$$

Here ℓ_i^*, ℓ_j are from Lemma 8.2 and Definition 8.1 respectively, and ν, t_i^* are from Definition 9.1 and Definition 17.2 respectively.

Proof: Apply Proposition 17.17 to Φ^*. □

18 TH systems and Vandermonde systems

In the previous sections we discussed TH systems and Vandermonde systems. In this section we give a natural correspondence between these two objects.

Definition 18.1. A matrix $X \in \mathrm{Mat}_{d+1}(\mathbb{K})$ is called *west-south* (or *double*) *Vandermonde* whenever X is both west Vandermonde and south Vandermonde. Assume X is west-south Vandermonde. We say that X is *west* (resp. *south*) *normalized* whenever $X_{i0} = 1$ (resp. $X_{di} = 1$) for $0 \leq i \leq d$. We say that X is *normalized* whenever it is both west normalized and south normalized.

Definition 18.2. By a *west-south* (or *double*) *Vandermonde system* in $\mathrm{Mat}_{d+1}(\mathbb{K})$, we mean a sequence $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ such that $(X, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$ and $(X, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ denote a west-south Vandermonde system. Observe that X is west-south Vandermonde. We say that $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ is *west normalized* (resp. *south normalized*) (resp. *normalized*) whenever X is west normalized (resp. south normalized) (resp. normalized) in the sense of Definition 18.1.

Our main goal in this section is to establish a bijection between the following two sets:

$$\text{the set of isomorphism classes of TH systems over } \mathbb{K} \text{ of diameter } d, \tag{44}$$

$$\text{the set of normalized west-south Vandermonde systems in } \mathrm{Mat}_{d+1}(\mathbb{K}). \tag{45}$$

To do this we define a map ρ from (44) to (45) and a map χ from (45) to (44), and show that they are inverses of each other. We start with an observation. Let Φ denote a TH system over $\mathbb{K}$ of diameter d, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\mathcal{P}$ denote the transition matrix of Φ from Definition 11.8. By Corollary 17.6 the sequence $(\mathcal{P}, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$, and by Corollary 17.7 the sequence $(\mathcal{P}, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. By Corollary 11.10, $\mathcal{P}$ is both west normalized and south normalized. Therefore $(\mathcal{P}, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ is a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

Definition 18.3. We define a map ρ from (44) to (45). We do this as follows. Let Φ denote a TH system over $\mathbb{K}$ of diameter d, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\mathcal{P}$ denote the transition matrix of Φ from Definition 11.8. By the above comment $(\mathcal{P}, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ is a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. The map ρ sends the isomorphism class of Φ to $(\mathcal{P}, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$.

We now define the map χ. We start with the following construction. Let $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ denote a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. We construct a TH system Φ on V as follows. Recall that X is invertible by Lemma 13.13. Therefore there exist bases $\{u_i\}_{i=0}^d$, $\{v_i\}_{i=0}^d$ for V such that X is the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. Define $A \in \mathrm{End}(V)$ such that $Au_i = \theta_i u_i$ for $0 \leq i \leq d$. Define $A^* \in \mathrm{End}(V)$ such that $A^* v_i = \theta_i^* v_i$ for $0 \leq i \leq d$. For $0 \leq i \leq d$ let E_i (resp. E_i^*) denote the primitive idempotent of A (resp. A^*) corresponding to θ_i (resp. θ_i^*). Now define $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$. We claim that Φ is a TH system on V. To prove the claim we show that Φ satisfies conditions (i)–(v) in Definition 2.2. Conditions (i)–(iii) hold by construction. Let $D^* \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the diagonal matrix with (i,i)-entry θ_i^* for $0 \leq i \leq d$. Observe that D^* represents A^* with respect to $\{v_i\}_{i=0}^d$. Hence by the comment above Proposition 17.4, the matrix XD^*X^{-1} represents A^* with respect to $\{u_i\}_{i=0}^d$. Moreover since $(X, \{\theta_i^*\}_{i=0}^d)$ is a south Vandermonde system, XD^*X^{-1} is Hessenberg by Corollary 15.6. By these comments, condition (iv) holds. Let $D \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the diagonal matrix with (i,i)-entry θ_i for $0 \leq i \leq d$. Observe that D represents A with respect to $\{u_i\}_{i=0}^d$. Hence by the comment above Proposition 17.4, the matrix $X^{-1}DX$ represents A with respect to $\{v_i\}_{i=0}^d$. Moreover since $(X, \{\theta_i\}_{i=0}^d)$ is a west Vandermonde system, $X^{-1}DX$ is Hessenberg by Corollary 15.3. By these comments, condition (v) holds. Therefore Φ is a TH system on V. By construction Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$.

Definition 18.4. We define a map χ from (45) to (44). We do this as follows. Let $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ denote a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let Φ denote the corresponding TH system constructed above. The map χ sends $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ to the isomorphism class of Φ.

Our next goal is to show that the maps ρ and χ are inverses of each other. We first recall some elementary linear algebra. Let $\{u_i\}_{i=0}^d$, $\{v_i\}_{i=0}^d$, $\{w_i\}_{i=0}^d$ denote bases for V. Let $T \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$ and let $S \in \mathrm{Mat}_{d+1}(\mathbb{K})$ denote the transition matrix from $\{v_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$. Then TS is the transition matrix from $\{u_i\}_{i=0}^d$ to $\{w_i\}_{i=0}^d$.

Lemma 18.5. *Let $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ denote a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$ and let Φ denote the TH system constructed above Definition 18.4. Then X is the transition matrix of Φ from Definition 11.8.*

Proof: Let $\mathcal{P}$ denote the transition matrix of Φ from Definition 11.8. We show that $\mathcal{P} = X$. In what follows we refer to the construction of Φ above Definition 18.4. By the construction of A (resp. A^*) we find that $u_i \in E_i V$ (resp. $v_i \in E_i^* V$) for $0 \leq i \leq d$. Recall that X is the transition matrix from $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$. By these comments, Definition 11.8, and the comment above this lemma, there exist invertible diagonal matrices $D_1, D_2 \in \mathrm{Mat}_{d+1}(\mathbb{K})$ such that $\mathcal{P} = D_1 X D_2$. The matrices $\mathcal{P}$ and X are west normalized, meaning that $P_{i0} =$

$X_{i0} = 1$ for $0 \leq i \leq d$. The matrices $\mathcal{P}$ and X are south normalized, meaning that $P_{di} = X_{di} = 1$ for $0 \leq i \leq d$. Evaluating the equation $\mathcal{P} = D_1 X D_2$ using these comments, we find that D_1 is a nonzero scalar multiple of the identity and D_2 is the inverse of D_1. Therefore $\mathcal{P} = X$. □

Theorem 18.6. *The map ρ from Definition 18.3 and the map χ from Definition 18.4 are inverses of each other. Moreover each of ρ, χ is bijective.*

Proof: We show that $\rho \circ \chi$ is the identity map on (45) and $\chi \circ \rho$ is the identity map on (44). We first show that $\rho \circ \chi$ is the identity map on (45). Let $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ denote a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Let Φ denote the corresponding TH system constructed above Definition 18.4. The map χ sends $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ to the isomorphism class of Φ. Recall from above Definition 18.4 that Φ has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By Lemma 18.5 X is the transition matrix of Φ from Definition 11.8. By these comments and Definition 18.3 the map ρ sends the isomorphism class of Φ to $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$. Therefore $\rho \circ \chi$ is the identity map on (45).

Next we show that $\chi \circ \rho$ is the identity map on (44). Let Φ denote a TH system over $\mathbb{K}$ of diameter d, with eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. Let $\mathcal{P}$ denote the transition matrix of Φ from Definition 11.8. The map ρ sends the isomorphism class of Φ to $(\mathcal{P}, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$. The map χ sends $(\mathcal{P}, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ to the isomorphism class of Φ', where Φ' is the corresponding TH system constructed above Definition 18.4. Recall from above Definition 18.4 that Φ' has eigenvalue sequence $\{\theta_i\}_{i=0}^d$ and dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. We show that Φ and Φ' are isomorphic. To do this we will invoke Lemma 3.8. Write $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ and $\Phi' = (A'; \{E_i'\}_{i=0}^d; A^{*\prime}; \{E_i^{*\prime}\}_{i=0}^d)$. Let V (resp. V') denote the vector space underlying Φ (resp. Φ'). Let $0 \neq \xi_0^* \in E_0^* V$ (resp. $0 \neq \xi_0^{*\prime} \in E_0^{*\prime} V'$) and recall the Φ^*-standard basis (resp. $\Phi^{*\prime}$-standard basis) $\{E_i \xi_0^*\}_{i=0}^d$ (resp. $\{E_i' \xi_0^{*\prime}\}_{i=0}^d$) for V (resp. V') from above Proposition 10.5. Let $\Gamma : V \to V'$ denote the $\mathbb{K}$-vector space isomorphism which sends $E_i \xi_0^*$ to $E_i' \xi_0^{*\prime}$ for $0 \leq i \leq d$. We show that

$$A'\Gamma = \Gamma A, \qquad A^{*\prime}\Gamma = \Gamma A^*, \qquad E_i'\Gamma = \Gamma E_i, \qquad E_i^{*\prime}\Gamma = \Gamma E_i^* \quad (0 \leq i \leq d). \tag{46}$$

We first show that $E_i'\Gamma = \Gamma E_i$ for $0 \leq i \leq d$. Let i be given. In order to show that $E_i'\Gamma = \Gamma E_i$, we show that $E_i'\Gamma$ and ΓE_i agree at each vector in the Φ^*-standard basis $\{E_j \xi_0^*\}_{j=0}^d$. Observe that for $0 \leq j \leq d$, $E_i'\Gamma E_j \xi_0^* = E_i' E_j' \xi_0^{*\prime} = \delta_{ij} E_i' \xi_0^{*\prime}$ and $\Gamma E_i E_j \xi_0^* = \delta_{ij} \Gamma E_i \xi_0^* = \delta_{ij} E_i' \xi_0^{*\prime}$. Thus $E_i'\Gamma = \Gamma E_i$. Next we show that $A'\Gamma = \Gamma A$. Recall $A = \sum_{i=0}^d \theta_i E_i$. Observe that $A' = \sum_{i=0}^d \theta_i E_i'$ since Φ' has eigenvalue sequence $\{\theta_i\}_{i=0}^d$. By these comments $A'\Gamma = \Gamma A$. Next we show that $E_i^{*\prime}\Gamma = \Gamma E_i^*$ for $0 \leq i \leq d$. Let P (resp. P') denote the transition matrix of Φ (resp. Φ') from Definition 11.1, and let L (resp. L') denote the matrix associated with Φ (resp. Φ') from Definition 8.5. Observe that $L = L'$ by Definition 8.1, since Φ and Φ' have the same dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By Lemma 18.5 $\mathcal{P}$ is the transition matrix of Φ' from Definition 11.8. By these comments and Definition 11.8, we have $P = P'$. Let $0 \neq \xi_0 \in E_0 V$ (resp. $0 \neq \xi_0' \in E_0' V'$) such that $\xi_0^* = E_0^* \xi_0$ (resp. $\xi_0^{*\prime} = E_0^{*\prime} \xi_0'$). Recall the Φ-standard basis (resp. Φ'-standard basis) $\{E_i^* \xi_0\}_{i=0}^d$ (resp. $\{E_i^{*\prime} \xi_0'\}_{i=0}^d$) for V (resp. V') from above (3). By Definition 11.1 and since $P = P'$, P is the transition matrix from $\{E_i \xi_0^*\}_{i=0}^d$ (resp. $\{E_i' \xi_0^{*\prime}\}_{i=0}^d$) to $\{E_i^* \xi_0\}_{i=0}^d$ (resp. $\{E_i^{*\prime} \xi_0'\}_{i=0}^d$). We can now easily show that

$E_i^{*\prime}\Gamma = \Gamma E_i^*$ for $0 \leq i \leq d$. Let i be given. In order to show that $E_i^{*\prime}\Gamma = \Gamma E_i^*$ we show that $E_i^{*\prime}\Gamma$ and ΓE_i^* agree at each vector in the Φ-standard basis $\{E_j^*\xi_0\}_{j=0}^d$. For $0 \leq j \leq d$,

$$E_i^{*\prime}\Gamma E_j^*\xi_0 = E_i^{*\prime}\Gamma \sum_{h=0}^d P_{hj}E_h\xi_0^* = E_i^{*\prime} \sum_{h=0}^d P_{hj}E_h'\xi_0^{*\prime} = E_i^{*\prime}E_j^{*\prime}\xi_0' = \delta_{ij}E_j^{*\prime}\xi_0',$$

$$\Gamma E_i^* E_j^*\xi_0 = \delta_{ij}\Gamma E_j^*\xi_0 = \delta_{ij}\Gamma \sum_{h=0}^d P_{hj}E_h\xi_0^* = \delta_{ij}\sum_{h=0}^d P_{hj}E_h'\xi_0^{*\prime} = \delta_{ij}E_j^{*\prime}\xi_0'.$$

We have now shown that $E_i^{*\prime}\Gamma$ and ΓE_i^* agree at each vector in the Φ-standard basis $\{E_j^*\xi_0\}_{j=0}^d$. Therefore $E_i^{*\prime}\Gamma = \Gamma E_i^*$. Next we show that $A^{*\prime}\Gamma = \Gamma A^*$. Recall $A^* = \sum_{i=0}^d \theta_i^* E_i^*$. Observe that $A^{*\prime} = \sum_{i=0}^d \theta_i^* E_i^{*\prime}$ since Φ' has dual eigenvalue sequence $\{\theta_i^*\}_{i=0}^d$. By these comments $A^{*\prime}\Gamma = \Gamma A^*$. We have now shown (46). Now Φ and Φ' are isomorphic in view of Lemma 3.8. Therefore $\chi \circ \rho$ is the identity map on (44). The result follows. □

Combining Corollary 4.4 and Theorem 18.6, we get a bijection between any two of the following three sets:

- The set of isomorphism classes of TH systems over $\mathbb{K}$ of diameter d.
- The set of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of parameter arrays over $\mathbb{K}$ of diameter d.

19 Reduced TH systems and Vandermonde matrices

In the previous section we explained how double Vandermonde systems correspond with TH systems. In this section we turn our attention to double Vandermonde matrices and explain how these correspond with objects called reduced TH systems.

Definition 19.1. A sequence $(\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is called a *reduced TH system* (or *RTH system*) on V whenever there exist $A, A^* \in \mathrm{End}(V)$ such that $(A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ is a TH system on V. Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system on V. Then $(\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ is an RTH system on V, called the *reduction* of Φ.

Definition 19.2. Let $\Lambda = (\{E_i\}_{i=0}^d; \{E_i^*\}_{i=0}^d)$ denote an RTH system on V. Let W denote a vector space over $\mathbb{K}$ with dimension $d+1$, and let $\Omega = (\{F_i\}_{i=0}^d; \{F_i^*\}_{i=0}^d)$ denote an RTH system on W. By an *isomorphism of RTH systems* from Λ to Ω we mean a $\mathbb{K}$-algebra isomorphism $\gamma : \mathrm{End}(V) \to \mathrm{End}(W)$ such that $F_i = E_i^\gamma$ and $F_i^* = E_i^{*\gamma}$ for $0 \leq i \leq d$. We say that the RTH systems Λ and Ω are *isomorphic* whenever there exists an isomorphism of RTH systems from Λ to Ω.

Proposition 19.3. *Let Φ and Φ' denote TH systems over $\mathbb{K}$. Then the following* (i), (ii) *are equivalent.*

(i) *The reduction of Φ is isomorphic to the reduction of Φ'.*

(ii) Φ *is affine isomorphic to* Φ'.

Proof: (i) $\Rightarrow$ (ii) Let P (resp. P') denote the transition matrix of Φ (resp. Φ') from Definition 11.1. From its definition, we see that P (resp. P') is determined by the primitive idempotents of Φ (resp. Φ'). Hence by our assumption $P = P'$. Let $\{\theta_i\}_{i=0}^d$ (resp. $\{\theta_i'\}_{i=0}^d$) denote the eigenvalue sequence of Φ (resp. Φ'). By Proposition 17.4 P and $\{\theta_i\}_{i=0}^d$ are compatible. Similarly P' and $\{\theta_i'\}_{i=0}^d$ are compatible. Since $P = P'$, we conclude that P is compatible with each of $\{\theta_i\}_{i=0}^d$ and $\{\theta_i'\}_{i=0}^d$. Hence by Lemma 13.5 there exist $\alpha, \beta \in \mathbb{K}$ with $\alpha \neq 0$ such that $\theta_i' = \alpha\theta_i + \beta$ for $0 \leq i \leq d$. Let $\{\theta_i^*\}_{i=0}^d$ (resp. $\{\theta_i^{*\prime}\}_{i=0}^d$) denote the dual eigenvalue sequence of Φ (resp. Φ'). By a similar argument, there exist $\alpha^*, \beta^* \in \mathbb{K}$ with $\alpha^* \neq 0$ such that $\theta_i^{*\prime} = \alpha^*\theta_i^* + \beta^*$ for $0 \leq i \leq d$. It follows that Φ is affine isomorphic to Φ'.
(ii) $\Rightarrow$ (i) Clear. $\square$

Corollary 19.4. *Let* Λ *and* Ω *denote isomorphic RTH systems over* $\mathbb{K}$. *Then the isomorphism of RTH systems from* Λ *to* Ω *is unique.*

Proof: Let γ and γ' denote isomorphisms of RTH systems from Λ to Ω. We show that $\gamma = \gamma'$. Let Φ (resp. Ψ) denote a TH system over $\mathbb{K}$ whose reduction is Λ (resp. Ω). By Proposition 19.3 Φ is affine isomorphic to Ψ. In other words, Φ is isomorphic to an affine transformation Ψ' of Ψ. By construction Φ and Ψ' have the same eigenvalue sequence and dual eigenvalue sequence. By this and the comment (iv) above (1), we find that each of γ and γ' is an isomorphism of TH systems from Φ to Ψ'. Now $\gamma = \gamma'$ in view of Lemma 3.6. The result follows. $\square$

We now give a correspondence between TH systems and reduced TH systems.

Corollary 19.5. *The map which sends a TH system to its reduction induces a bijection from the set of affine isomorphism classes of TH systems over* $\mathbb{K}$ *to the set of isomorphism classes of RTH systems over* $\mathbb{K}$.

Proof: Immediate from Proposition 19.3. $\square$

We now turn our attention to double Vandermonde systems and double Vandermonde matrices.

Lemma 19.6. *Let* $\Omega = (X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ *denote a normalized west-south Vandermonde system in* $\mathrm{Mat}_{d+1}(\mathbb{K})$. *Let* $\alpha, \beta, \alpha^*, \beta^*$ *denote scalars in* $\mathbb{K}$ *with* α, α^* *nonzero. Then the sequence*

$$(X, \{\alpha\theta_i + \beta\}_{i=0}^d; \{\alpha^*\theta_i^* + \beta^*\}_{i=0}^d) \tag{47}$$

is a normalized west-south Vandermonde system in $\mathrm{Mat}_{d+1}(\mathbb{K})$.

Proof: Routine by Lemma 13.5. $\square$

Definition 19.7. Referring to Lemma 19.6, we call (47) the *affine transformation of* Ω *associated with* $\alpha, \beta, \alpha^*, \beta^*$.

Definition 19.8. Let Ω and Ω' denote normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$. We say that Ω and Ω' are *affine related* whenever Ω is an affine transformation of Ω'. Observe that the affine relation is an equivalence relation.

Lemma 19.9. *Let $\Omega = (X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ and $\Omega' = (X', \{\theta_i'\}_{i=0}^d, \{\theta_i^{*\prime}\}_{i=0}^d)$ denote normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$. Then the following* (i), (ii) *are equivalent.*

(i) $X = X'$.

(ii) Ω *is affine related to* Ω'.

Proof: (i) $\Rightarrow$ (ii) Immediate from Lemma 13.5.
(ii) $\Rightarrow$ (i) Clear. □

We now give a correspondence between normalized double Vandermonde systems and normalized double Vandermonde matrices.

Corollary 19.10. *The map which sends a normalized west-south Vandermonde system $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ to the matrix X induces a bijection from the set of affine classes of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$ to the set of normalized west-south Vandermonde matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$.*

Proof: Immediate from Lemma 19.9. □

Next we give a correspondence between affine isomorphism classes of TH systems and affine classes of normalized double Vandermonde systems. Recall the map ρ from Definition 18.3.

Lemma 19.11. *Let $\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$ denote a TH system over $\mathbb{K}$. Let $(X, \{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d)$ denote the image under ρ of the isomorphism class of Φ. Let $\alpha, \beta, \alpha^*, \beta^*$ denote scalars in $\mathbb{K}$ with α, α^* nonzero and consider the TH system (25). Then ρ sends the isomorphism class of (25) to $(X, \{\alpha\theta_i + \beta\}_{i=0}^d, \{\alpha^*\theta_i^* + \beta^*\}_{i=0}^d)$.*

Proof: Immediate from Lemma 5.4. □

Corollary 19.12. *The bijection ρ from Definition 18.3 induces a bijection from the set of affine isomorphism classes of TH systems over $\mathbb{K}$ of diameter d, to the set of affine classes of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$.*

Proof: Immediate from Lemma 19.11. □

We now bring the parameter arrays into the discussion.

Definition 19.13. We define a binary relation on the set of parameter arrays over $\mathbb{K}$ of diameter d. We do this as follows. Let $p = (\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\phi_i\}_{i=1}^d)$ and $p' = (\{\theta_i'\}_{i=0}^d, \{\theta_i^{*\prime}\}_{i=0}^d, \{\phi_i'\}_{i=1}^d)$ denote parameter arrays over $\mathbb{K}$ of diameter d. We say that p and p' are *affine related* whenever there exist scalars $\alpha, \beta, \alpha^*, \beta^*$ in $\mathbb{K}$ with α, α^* nonzero such that the following (i)–(iii) hold.

(i) $\theta_i' = \alpha\theta_i + \beta \qquad (0 \le i \le d)$.

(ii) $\theta_i^{*\prime} = \alpha^*\theta_i^* + \beta^* \qquad (0 \le i \le d)$.

(iii) $\phi_i' = \alpha\alpha^*\phi_i \qquad (1 \le i \le d)$.

Observe that the affine relation is an equivalence relation. By a *reduced parameter array* we mean an equivalence class of this relation.

Corollary 19.14. *The bijection from Corollary 4.4 induces a bijection from the set of affine isomorphism classes of TH systems over $\mathbb{K}$ of diameter d, to the set of reduced parameter arrays over $\mathbb{K}$ of diameter d.*

Proof: Immediate from Lemma 5.4. □

Combining Corollaries 19.5, 19.10, 19.12, 19.14, we get a bijection between any two of the following five sets:

- The set of affine isomorphism classes of TH systems over $\mathbb{K}$ of diameter d.
- The set of isomorphism classes of RTH systems over $\mathbb{K}$ of diameter d.
- The set of affine classes of normalized west-south Vandermonde systems in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of normalized west-south Vandermonde matrices in $\mathrm{Mat}_{d+1}(\mathbb{K})$.
- The set of reduced parameter arrays over $\mathbb{K}$ of diameter d.

20 Acknowledgement

This paper was written while the author was a graduate student at the University of Wisconsin-Madison. The author would like to thank his advisor Paul Terwilliger for his many valuable ideas and suggestions.

References

[1] A. Godjali. Thin Hessenberg pairs. *Linear Algebra Appl.* **432** (2010) 3231–3249; `arXiv:math.RA/0911.4118v1`.

[2] J. J. Rotman. *Advanced modern algebra.* Prentice Hall, Saddle River NJ 2002.

[3] P. Terwilliger. Leonard pairs and the q-Racah polynomials. *Linear Algebra Appl.* **387** (2004) 235–276; `arXiv:math.QA/0306301`.

[4] P. Terwilliger. Two linear transformations each tridiagonal with respect to an eigenbasis of the other. *Linear Algebra Appl.* **330** (2001) 149–203; `arXiv:math.RA/0406555`.

[5] P. Terwilliger. An algebraic approach to the Askey scheme of orthogonal polynomials. Orthogonal polynomials and special functions, 255–330, Lecture Notes in Math., 1883, Springer, Berlin, 2006; `arXiv:math.QA/0408390`.

Ali Godjali
Department of Mathematics
University of Wisconsin
Van Vleck Hall
480 Lincoln Drive
Madison, WI 53706-1388 USA
email: godjali@math.wisc.edu