# A Note on Terence Tao's Paper "On the Number of Solutions to $\frac{4}{p} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}$"

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**Abstract.** For the positive integer $n$, let $f(n)$ denote the number of positive integer solutions $(n_1,\, n_2,\, n_3)$ of the Diophantine equation

$$\frac{4}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.$$

For the prime number $p$, $f(p)$ can be split into $f_1(p) + f_2(p)$, where $f_i(p)(i = 1,\, 2)$ counts those solutions with exactly $i$ of denominators$n_1,\, n_2,\, n_3$ divisible by $p$.

Recently Terence Tao proved that

$$\sum_{p<x} f_2(p) \ll x \log^2 x \log\log x.$$

with other results. But actually only the upper bound $x \log^2 x \log\log^2 x$ can be obtained in his discussion. In this note we shall use an elementary method to save a factor $\log\log x$ and recover the above estimate.

## 1. Introduction

For the positive integer $n$, let $f(n)$ denote the number of positive integer solutions $(n_1,\, n_2,\, n_3)$ of the Diophantine equation

$$\frac{4}{n} = \frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3}.$$

Erdös and Straus conjectured that for all $n \geq 2$, $f(n) > 0$. It is still an open problem now although there are some partial results.

In 1970, R. C. Vaughan[2] showed that the number of $n < x$ for which $f(n) = 0$ is at most $x \exp(-c \log^{\frac{2}{3}} x)$, where $x$ is sufficiently large and $c$ is a positive constant.

Recently Terence Tao[1] studied the situation in which $n$ is the prime number $p$. He gave lower bound and upper bound for the mean value of $f(p)$. Precisely, he split $f(p)$ into $f_1(p) + f_2(p)$, where $f_i(p)(i = 1,\, 2)$ counts those solutions with exactly $i$ of denominators $n_1$, $n_2$, $n_3$ divisible by $p$. He proved that

$$x\log^2 x \ll \sum_{p<x} f_2(p) \ll x\log^2 x \log\log x \tag{1}$$

and

$$x\log^2 x \ll \sum_{p<x} f_1(p) \ll x\exp(\frac{c\log x}{\log\log x}),$$

where $p$ denotes the prime number, $x$ is sufficiently large and $c$ is a positive constant. Then he conjectured that for $i = 1, 2$,

$$\sum_{p<x} f_i(p) \ll x\log^2 x. \tag{2}$$

But actually Terence Tao[1] only proved

$$\sum_{p<x} f_2(p) \ll x\log^2 x \log\log^2 x, \tag{3}$$

since there was an error in his discussion. In this note we shall use an elementary method to save a factor $\log\log x$ and recover the upper bound in the right side of (1).

**Theorem.** Let $p$ denote the prime number. Then for sufficiently large $x$, we have

$$\sum_{p<x} f_2(p) \ll x\log^2 x \log\log x.$$

**2. The proof of Theorem**

**Lemma 1.** If $\varphi(n)$ is the Euler totient function, then

$$\varphi(n) \gg \frac{n}{g(n)}.$$

Here

$$g(n) = \prod_{p|n}(1+\frac{1}{p}) = \sum_{d|n}\frac{\mu^2(d)}{d},$$

where $\mu(d)$ is the Möbius functions.

**Proof.** We know that

$$\varphi(n) = n\prod_{p|n}(1-\frac{1}{p}).$$

Then

$$\varphi(n) = n\frac{\prod_{p|n}(1-\frac{1}{p^2})}{\prod_{p|n}(1+\frac{1}{p})} \geq \frac{n}{g(n)}\prod_{p}(1-\frac{1}{p^2}) \gg \frac{n}{g(n).}$$

It is easy to see

$$g(n) = \sum_{d|n}\frac{\mu^2(d)}{d}.$$

**Lemma 2.** If $x \geq 1$, then

$$\sum_{x<n\leq 2x}\frac{1}{\varphi(n)} \ll 1.$$

**Proof.** By Lemma 1, we have

$$\begin{aligned}\sum_{x<n\leq 2x}\frac{1}{\varphi(n)} &\ll \sum_{x<n\leq 2x}\frac{g(n)}{n}\\ &= \sum_{x<n\leq 2x}\frac{1}{n}\sum_{d|n}\frac{\mu^2(d)}{d}\\ &= \sum_{d\leq 2x}\frac{\mu^2(d)}{d}\sum_{\substack{x<n\leq 2x\\ d|n}}\frac{1}{n}\\ &= \sum_{d\leq 2x}\frac{\mu^2(d)}{d^2}\sum_{\frac{x}{d}<l\leq\frac{2x}{d}}\frac{1}{l}\\ &\ll \sum_{d\leq 2x}\frac{\mu^2(d)}{d^2} \ll 1.\end{aligned}$$

**Lemma 3.** Let $p$ denote the prime number. Then the functions $f_2(p)$ is equal to three times the number of triples $(a,\, b,\, c)$ of positive integers such that

$$(a,\, b) = 1, \quad c|a+b, \quad 4ab|p+c.$$

One can see Proposition 1.2 of [1].

By some transformation, Terence Tao[1] got

$$\sum_{p<x} f_2(p) \ll \sum_{1\leq i\leq\frac{1}{2}\log_2 x}\ \sum_{i\leq j\leq\log_2 x-i}\frac{x}{1+\log_2 x-i-j}\cdot$$
$$\cdot\sum_{2^i<a\leq 2^{i+1}}\ \sum_{\substack{2^j<b\leq 2^{j+1}\\ (a,b)=1}}\frac{d(a+b)}{\varphi(a)\varphi(b)}.$$

Here $d(n)$ is the divisor function. It is necessary to keep the condition $(a,\, b) = 1$.

Now we consider the estimate for the sum

$$\sum_{V<a\le 2V} \frac{1}{\varphi(a)} \sum_{\substack{W<b\le 2W\\(a,\,b)=1}} \frac{d(a+b)}{\varphi(b)}, \tag{4}$$

where $1 \le V \le W \le x$.

Let

$$S(a,\, W) = \sum_{\substack{W<b\le 2W\\(a,\,b)=1}} \frac{d(a+b)}{\varphi(b)}. \tag{5}$$

Then Lemma 1 yields that

$$\begin{aligned}
S(a,\, W) &\ll \sum_{\substack{W<b\le 2W\\(a,\,b)=1}} d(a+b)\cdot\frac{g(b)}{b}\\
&\ll \frac{1}{W}\sum_{\substack{W<b\le 2W\\(a,\,b)=1}} d(a+b)g(b)\\
&= \frac{1}{W}\sum_{\substack{W+a<k\le 2W+a\\(k,\,a)=1}} d(k)g(k-a)\\
&= \frac{1}{W}\sum_{\substack{W+a<rl\le 2W+a\\(rl,\,a)=1}} g(rl-a)\\
&\le \frac{2}{W}\sum_{\substack{r\le\sqrt{2W+a}\\(r,\,a)=1}} \sum_{\substack{l\\W+a<rl\le 2W+a\\(l,\,a)=1}} g(rl-a)\\
&\ll \frac{1}{W}\sum_{\substack{r\le\sqrt{2W+a}\\(r,\,a)=1}} \sum_{\substack{W<n\le 2W\\n\equiv -a(\operatorname{mod} r)\\(n,\,a)=1}} g(n)\\
&\le \frac{1}{W}\sum_{\substack{r\le\sqrt{2W+a}\\(r,\,a)=1}} \sum_{\substack{W<n\le 2W\\n\equiv -a(\operatorname{mod} r)}} g(n).
\end{aligned}$$

Since $(r, a) = 1$, $n \equiv -a (\bmod r) \Longrightarrow (n, r) = 1$. Then

$$\begin{aligned}
\sum_{\substack{W < n \le 2W \\ n \equiv -a (\bmod r)}} g(n) &= \sum_{\substack{W < n \le 2W \\ n \equiv -a (\bmod r) \\ (n, r) = 1}} g(n) \\
&= \sum_{\substack{W < n \le 2W \\ n \equiv -a (\bmod r) \\ (n, r) = 1}} \sum_{d | n} \frac{\mu^2(d)}{d} \\
&= \sum_{\substack{d \le 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \sum_{\substack{W < n \le 2W \\ n \equiv -a (\bmod r) \\ (n, r) = 1 \\ d | n}} 1 \\
&= \sum_{\substack{d \le 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \sum_{\substack{\frac{W}{d} < k \le \frac{2W}{d} \\ (k, r) = 1 \\ dk \equiv -a (\bmod r)}} 1 \\
&= \sum_{\substack{d \le 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \sum_{\substack{\frac{W}{d} < k \le \frac{2W}{d} \\ (k, r) = 1 \\ k \equiv -\bar{d} a (\bmod r)}} 1 \\
&\le \sum_{\substack{d \le 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \sum_{\substack{\frac{W}{d} < k \le \frac{2W}{d} \\ k \equiv -\bar{d} a (\bmod r)}} 1,
\end{aligned}$$

where $\bar{d}$ is an integer such that $\bar{d} d \equiv 1 (\bmod r)$.

We have

$$\sum_{\substack{\frac{W}{d} < k \le \frac{2W}{d} \\ k \equiv -\bar{d} a (\bmod r)}} 1 \ll \frac{W}{dr} + 1.$$

Thus

$$\begin{aligned}
\sum_{\substack{W < n \le 2W \\ n \equiv -a (\bmod r)}} g(n) &\ll \sum_{\substack{d \le 2W \\ (d, r) = 1}} \frac{\mu^2(d)}{d} \left(\frac{W}{dr} + 1\right) \\
&\le \frac{W}{r} \sum_{d \le 2W} \frac{\mu^2(d)}{d^2} + \sum_{d \le 2W} \frac{\mu^2(d)}{d} \\
&\ll \frac{W}{r} + \log 2W.
\end{aligned}$$

It follows that

$$\begin{aligned} S(a,\,W) &\ll \frac{1}{W}\sum_{\substack{r\le\sqrt{2W+a}\\(r,\,a)=1}}(\frac{W}{r}+\log 2W)\\ &\le \frac{1}{W}\sum_{r\le 2\sqrt{W}}(\frac{W}{r}+\log 2W)\\ &\ll \log 2W. \end{aligned}$$

By Lemma 2, we have

$$\begin{aligned} &\sum_{V<a\le 2V}\frac{1}{\varphi(a)}\sum_{\substack{W<b\le 2W\\(a,\,b)=1}}\frac{d(a+b)}{\varphi(b)}\\ \ll &\sum_{V<a\le 2V}\frac{1}{\varphi(a)}\cdot\log x\\ \ll &\log x. \end{aligned}$$

Therefore

$$\sum_{p<x} f_2(p) \ll x\log x \sum_{1\le i\le\frac{1}{2}\log_2 x}\ \sum_{i\le j\le \log_2 x-i}\frac{1}{1+\log_2 x-i-j}. \tag{6}$$

We have

$$\begin{aligned} &\sum_{1\le i\le\frac{1}{2}\log_2 x}\ \sum_{i\le j\le \log_2 x-i}\frac{1}{1+\log_2 x-i-j}\\ \le &\sum_{1\le i\le\frac{1}{2}\log_2 x}\ \sum_{1\le h\le \log_2 x-2i+2}\frac{1}{h}\\ \ll &\sum_{1\le i\le\frac{1}{2}\log_2 x}\log(\log_2 x-2i+4)\\ \ll &\sum_{1\le k\le\frac{1}{2}\log_2 x+1}\log(2k+8)\\ \ll &\log x\log\log x. \end{aligned}$$

So far the proof of Theorem is finished.

Similar discussion can yield

$$\sum_{1\le i\le\frac{1}{2}\log_2 x}\ \sum_{i\le j\le \log_2 x-i}\frac{1}{1+\log_2 x-i-j}\gg \log x\log\log x.$$

In [1],

$$\sum_{1\le i\le \frac{1}{2}\log_2 x}\ \sum_{i\le j\le \log_2 x-i}\frac{1}{1+\log_2 x-i-j}\ll \log x$$

is proved, where a factor $\log\log x$ is lost. From the above discussion, it seems reasonable to conjecture

$$x\log^2 x\log\log x\ll \sum_{p<x} f_2(p). \tag{7}$$

## References


[1] Terence Tao, *On the number of solutions to* $\frac{4}{p}=\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}$, available at http://arxiv.org/abs/1107.1010

[2] R. C. Vaughan, *On a problem of Erdös, Straus and Schinzel*, Mathematika, **17**(1970), 193-198.



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