THE PERIODIC TWO-DIMENSIONAL μ-b-EQUATION AS AN EPDIFF EQUATION

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ABSTRACT. We introduce a periodic two-dimensional μ-b-equation and a periodic two-dimensional two-component (μ)-Camassa-Holm equation which we study as geodesic flows on the diffeomorphism group of the torus and a semidirect product respectively. The paper explains the derivation of these equations within Arnold's [2] general framework, some analogies to recently discussed related equations and gives a self-contained presentation of the geometric aspects. As an application, we obtain well-posedness results and some explicit curvature computations.

CONTENTS

1. INTRODUCTION

The mathematical theory of ideal fluid motion is an active area of research, coined by an interplay of different mathematical methods, e.g., coming from spectral theory, harmonic analysis or topology. In the present paper, we will use group structures and geometric arguments to discover a new and highly relevant family of equations arising in fluid dynamics. For the motion of an ideal (i.e., incompressible, homogeneous and inviscid) fluid, two physical parameters are of particular importance: the velocity field u and the momentum m of the fluid particles; moreover, there is an operator $\mathbb{A}$ (called the *inertia operator*) which maps u to m in a nice way. In two dimensions is natural to associate with u a vector field $u^1\partial_{x^1} + u^2\partial_{x^2}$ and with m the one-form density $(m_1\,\mathrm{d}x^1 + m_2\,\mathrm{d}x^2) \otimes \mathrm{d}^2x$, where d^2x is the volume element of the domain in $\mathbb{R}^2$ that is filled with the fluid. Let $\mathrm{Diff}(\mathbb{T})$ denote the group of orientation-preserving diffeomorphisms of the torus $\mathbb{T} = \mathbb{S} \times \mathbb{S}$, where

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$\mathbb{R}/\mathbb{Z} \simeq \mathbb{S}$ is the 1-sphere. A natural action of $\mathrm{Diff}(\mathbb{T})$ on the momentum densities is given by

$$\psi(\varphi, m_i \, \mathrm{d}x^i \otimes \mathrm{d}^2x) = |\nabla\varphi| \left[(\nabla\varphi)^T (m \circ \varphi)\right]_i \, \mathrm{d}x^i \otimes \mathrm{d}^2x,$$

where $\nabla\varphi$ is the Jacobian matrix associated with $\varphi \in \mathrm{Diff}(\mathbb{T})$ and $|\nabla\varphi|$ is the Jacobian determinant. Computing the Lie derivative of ψ in the direction of the velocity field $u^i\partial_{x^i}$, i.e.,

$$(\mathcal{L}\psi)(u^i\partial_{x^i}, m_i \, \mathrm{d}x^i \otimes \mathrm{d}^2x) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi(\varphi(t), m_i \, \mathrm{d}x^i \otimes \mathrm{d}^2x),$$

where $\varphi(t)$ is a C^1-path in $\mathrm{Diff}(\mathbb{T})$ with $\varphi(0) = \mathrm{id}$ and $\dot\varphi(0) = u$, one observes, that $\mathcal{L}\psi(u^i\partial_{x^i}, m_i \, \mathrm{d}x^i \otimes \mathrm{d}^2x) = 0$ if and only if the equation

$$m_t = -u \cdot \nabla m - (\nabla u)^T \cdot m - m(\nabla \cdot u) \tag{1}$$

or equivalently the system of equations

$$\frac{\partial m_i}{\partial t} = -u^j \frac{\partial m_i}{\partial x^j} - m_j \frac{\partial u^j}{\partial x^i} - m_i \frac{\partial u^j}{\partial x^j}, \quad i = 1, 2,$$

is satisfied. For the particular choice $\mathbb{A} = (1 - \Delta)\mathrm{diag}(1, 1)$, Eq. (1) is called the two-dimensional Camassa-Holm equation. Note also that $\mathcal{L}\psi(u^i\partial_{x^i}, \cdot) = \mathrm{ad}^*_{u^i\partial_{x^i}}$, where ad^* is the dual operator of the Lie bracket $[u, v] = \nabla v \cdot u - \nabla u \cdot v$ with respect to the right-invariant metric induced by $\mathbb{A}$ on the diffeomorphism group of the torus $\mathbb{T}$, [32].

For the purposes of this paper, we will consider higher order tensor densities of the form

$$m_i \, \mathrm{d}x^i \otimes^b \mathrm{d}^2x := m_i \, \mathrm{d}x^i \otimes \mathrm{d}^2x \otimes \cdots \otimes \mathrm{d}^2x, \quad (b - 1 \text{ factors}),$$

where $b \geq 2$ is an integer. Defining an action ψ_b as above and demanding that the directional derivative $\mathcal{L}\psi_b(u^i\partial_{x^i}, m_i \, \mathrm{d}x^i \otimes^b \mathrm{d}^2x)$ is zero, one obtains the family

$$\frac{\partial m_i}{\partial t} = -u^j \frac{\partial m_i}{\partial x^j} - m_j \frac{\partial u^j}{\partial x^i} - (b-1) m_i \frac{\partial u^j}{\partial x^j}, \quad i = 1, 2, \quad m = \mathbb{A}u. \tag{2}$$

If $\mathbb{A} = (1 - \Delta)\mathrm{diag}(1, 1)$, it is natural to call this equation the two-dimensional periodic b-equation, since it is the analog of the one-dimensional equation [11, 15]

$$m_t = -m_x u - b u_x m, \quad m = (1 - \partial_x^2)u, \tag{3}$$

which becomes the 1D-Camassa-Holm equation [4]

$$u_t + 3uu_x = 2u_x u_{xx} + uu_{xxx} + u_{txx} \tag{4}$$

and the 1D-Degasperis-Procesi equation [10, 11]

$$u_t + 4uu_x = 3u_x u_{xx} + uu_{xxx} + u_{txx} \tag{5}$$

for the particular choices $b = 2, 3$. Note that $b = 2, 3$ are the only parameters for which Eq. (3) is integrable, [21, 22, 23]. Moreover, (4) and (5) are higher order approximations to the governing equations of hydrodynamics capturing typically nonlinear effects like wave-breaking or peakons [5, 9, 13]. An appendix of the paper [32] mentions the equations (1) and the 2-dimensional Degasperis-Procesi equation

$$\frac{\partial m_i}{\partial t} = -u^j \frac{\partial m_i}{\partial x^j} - m_j \frac{\partial u^j}{\partial x^i} - 2m_i \frac{\partial u^j}{\partial x^j}, \quad m_i = u_i - \Delta u_i, \quad i = 1, 2, \tag{6}$$

which are embedded into the family (2) for $b = 2, 3$.

In the present paper, we will consider the operator $\mathbb{A} = (\mu - \Delta)\mathrm{diag}(1, 1)$, where

$$\mu(u) = \int_{\mathbb{T}} u \, \mathrm{d}(x, y)$$

is the mean of a periodic function u. The corresponding family (2) will be referred to as the two-dimensional periodic μ-b-equation. For $b = 2, 3$, we will call the resulting equations the 2D-μ-Camassa-Holm [24, 32] and the 2D-μ-Degasperis-Procesi [32] equation respectively.

We will work out that the case $b = 2$ is of particular interest, and, for this choice of b, we will also consider a two-component extension of the 2D-μ-Camassa-Holm equation, which reads

$$\left\{ \begin{array}{rcl} m_t &=& -u \cdot \nabla m - (\nabla u)^T \cdot m - m(\nabla \cdot u) - (\nabla \rho)^T \cdot \rho, \\ \rho_t &=& -\nabla \rho \cdot u - \rho(\nabla \cdot u). \end{array} \right. \tag{7}$$

Observe that (7) is the analog of the one-dimensional two-component equation

$$\left\{ \begin{array}{rcl} m_t &=& -m_x u - 2u_x m - \rho \rho_x, \\ \rho_t &=& -(\rho u)_x, \end{array} \right. \qquad m = \int_{\mathbb{S}} u \, \mathrm{d}x - u_{xx}, \tag{8}$$

which has been studied in [25, 33, 38]. Note that the corresponding two-component Camassa-Holm equation was the subject of [6, 12, 19, 30]. The purposes of the paper at hand and its organization are as follows:

Section 2 provides some basic facts about diffeomorphism groups and semidirect products and introduces the notation which will be used throughout the paper.

In Section 3 we show that the 2-dimensional μ-b-equation (2) re-expresses geodesic motion on the diffeomorphism group of the torus for any b. For the particular choice $b = 2$, we show that the geodesic flow is weakly Riemannian in the sense that it is compatible with the right-invariant metric on the torus diffeomorphism group induced by the inertia operator $\mathbb{A} = (\mu - \Delta)\mathrm{diag}(1, 1)$. As a byproduct of our approach, we achieve well-posedness results for the 2-dimensional μ-b-equation in the geometric picture and in its initial form on a scale of Sobolev spaces as well as some curvature computations, for $b = 2$.

In Section 4 we prove that the case $b = 2$ is of particular interest, since only for $b = 2$ the 2-dimensional μ-b-equation can be associated with a crucial Riemannian structure. In particular, the 2D-μ-Degasperis-Procesi equation has no inertia operator of type $A \cdot \mathrm{diag}(1, 1)$, where A is a topological isomorphism on the smooth functions on the torus.

Finally, Section 5 is devoted to a study of the two-component system (7) for which we prove that it re-expresses geodesic motion on the semidirect product $\mathrm{Diff}(\mathbb{T}) \circledS H^{s-1}(\mathbb{T})$ of the H^s diffeomorphism group of $\mathbb{T}$ with the Sobolev space

$H^{s-1}(\mathbb{T})$, with respect to a suitable metric. Again, the geometric picture provides some features of the solutions of (7).

Before we start, we will fix the following notation: The Camassa-Holm equation and the Degasperis-Procesi equation will be written as CH and DP. The notation (n,m)CH means that we consider the n-dimensional Camassa-Holm equation with m components. We write $\mu-(n,m)$CH for the associated μ-variant. The notations (n,m)DP and $\mu-(n,m)$DP are accordingly.

2. Preliminaries

Let $\mathbb{S}=\mathbb{R}/\mathbb{Z}$ and denote by $\mathbb{T}=\mathbb{S}^2$ the two-dimensional torus, i.e., $\mathbb{T}$ consists of equivalence classes of pairs of real numbers such that $n=(n_1,n_2)$ and $m=(m_1,m_2)$ are equivalent if and only if n_i-m_i is an integer, for $i\in\{1,2\}$. It is well known that $\mathbb{T}$ is a smooth manifold with trivial tangent bundle $T\mathbb{T}\simeq\mathbb{T}\times\mathbb{R}^2$. We will label the coordinates in $\mathbb{T}$ by $x=x_1$ and $y=x_2$. For any differentiable function $u=(u_1,u_2)$ on $\mathbb{T}$ we write $\nabla u=(\nabla u_1,\nabla u_2)$, where $\nabla=(\partial_x,\partial_y)$ is the nabla operator. Sobolev mappings between two smooth manifolds M of order n and N of order d, for $s>n/2$, are defined as follows: A continuous mapping $f\colon M\to N$ is an element of $H^s(M,N)$ if for any two charts $\psi\colon\mathcal{U}\to U\subset\mathbb{R}^n$ and $\eta\colon\mathcal{V}\to V\subset\mathbb{R}^d$ with $f(\mathcal{U})\subseteq\mathcal{V}$, $\eta\circ f\circ\psi^{-1}\colon U\to V$ is an element of the usual Sobolev space $H^s(U,\mathbb{R}^d)$. Let $H^s(\mathbb{T})=H^s(\mathbb{T},\mathbb{T})$. Furthermore, we introduce the diffeomorphism groups

$$\mathcal{D}^s(\mathbb{T})=\left\{\varphi\in\mathrm{Diff}^1_+(\mathbb{T});\ \varphi\in H^s(\mathbb{T})\right\},\quad s>2,$$

where $\mathrm{Diff}^1_+(\mathbb{T})$ denotes the orientation-preserving C^1-diffeomorphisms of $\mathbb{T}$; precisely, $\varphi\in\mathrm{Diff}^1_+(\mathbb{T})$ if and only if $\varphi\in C^1(\mathbb{T};\mathbb{T})$ is a diffeomorphism and $|\nabla\varphi|=\det\nabla\varphi>0$. Elements of $\mathcal{D}^s(\mathbb{T})$ are referred to as orientation-preserving H^s diffeomorphisms. It is shown in [20] that $\mathcal{D}^s(\mathbb{T})$ is open in $H^s(\mathbb{T})$ and hence is a C^∞-Hilbert manifold. Moreover, $\mathcal{D}^s(\mathbb{T})$ is a topological group, i.e., the group product $(\varphi,\psi)\mapsto\varphi\circ\psi$ and the inversion map $\varphi\mapsto\varphi^{-1}$ are continuous maps $\mathcal{D}^s(\mathbb{T})\times\mathcal{D}^s(\mathbb{T})\to\mathcal{D}^s(\mathbb{T})$ and $\mathcal{D}^s(\mathbb{T})\to\mathcal{D}^s(\mathbb{T})$ respectively. For $s\to\infty$, the groups $\mathcal{D}^s(\mathbb{T})$ approximate the Lie group

$$\mathrm{Diff}^\infty(\mathbb{T})=\left\{\varphi\in\mathrm{Diff}^1_+(\mathbb{T});\ \varphi\in C^\infty(\mathbb{T};\mathbb{T})\right\}=\bigcap_{s>2}\mathcal{D}^s(\mathbb{T}),$$

a C^∞-Fréchet manifold on which multiplication and inversion are smooth maps. Let $\varphi\in\mathcal{D}^s(\mathbb{T})$ and let $\gamma(t)\subset\mathcal{D}^s(\mathbb{T})$ be a C^1-curve starting at φ. Then the vector $\dot\gamma(0)$ is an element of $H^s(\mathbb{T})$ and hence

$$T_\varphi\mathcal{D}^s(\mathbb{T})\simeq H^s(\mathbb{T})\simeq\left\{u\cdot\nabla;\ u\in H^s(\mathbb{T})\right\};$$

the latter space denotes the H^s vector fields on the torus. We further have the identification

$$T\mathcal{D}^s(\mathbb{T})=\bigcup_{\varphi\in\mathcal{D}^s(\mathbb{T})}\{\varphi\}\times T_\varphi\mathcal{D}^s(\mathbb{T})\simeq\mathcal{D}^s(\mathbb{T})\times H^s(\mathbb{T})$$

and $T_\varphi \mathcal{D}^s(\mathbb{T})^* \simeq H^s(\mathbb{T})$ for any $\varphi \in \mathcal{D}^s(\mathbb{T})$. Let $I_\varphi \colon \mathcal{D}^s(\mathbb{T}) \to \mathcal{D}^s(\mathbb{T})$ denote the inner automorphism $\psi \mapsto \varphi \circ \psi \circ \varphi^{-1}$. We let

$$\mathrm{Ad}_\varphi v = (D_{\mathrm{id}} I_\varphi) v = [(\nabla\varphi)\cdot v] \circ \varphi^{-1}, \quad \forall (\varphi, v) \in \mathcal{D}^s(\mathbb{T}) \times H^s(\mathbb{T}),$$

and

$$\mathrm{ad}_u v = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}_{\varphi(t)} v = (\nabla u)\cdot v - (\nabla v)\cdot u,$$

where $\varphi(t)$ is a C^1-curve in $\mathcal{D}^s(\mathbb{T})$ starting at id with $\dot{\varphi}(0) = u$ and $v \in H^s(\mathbb{T})$. The map $\mathrm{ad}\colon H^s(\mathbb{T}) \to \mathrm{End}(H^s(\mathbb{T}))$ satisfies the Lie bracket properties and we write

$$[u, v] = (\nabla u)\cdot v - (\nabla v)\cdot u \tag{9}$$

in the following. Let $\mathbb{A}$ be some topological isomorphism of $H^s(\mathbb{T})$ such that

$$\langle u, v\rangle = \int_{\mathbb{T}} u \mathbb{A} v \,\mathrm{d}(x, y)$$

is a scalar product on $H^s(\mathbb{T})$. Some easy computations show that the adjoint operators Ad_φ^* and ad_u^* (which is computed with respect to the scalar product $\langle\cdot,\cdot\rangle$) are given explicitly by

$$\mathrm{Ad}_\varphi^* w = (\nabla\varphi)^T (w\circ\varphi)|\nabla\varphi|, \quad w \in H^s(\mathbb{T}),$$

and

$$\mathrm{ad}_u^* w = \mathbb{A}^{-1}\left\{(\nabla u)^T \cdot \mathbb{A}w + \nabla(\mathbb{A}w)\cdot u + (\nabla\cdot u)\mathbb{A}w\right\}, \quad w \in H^s(\mathbb{T}). \tag{10}$$

Finally, let $R_\psi, L_\psi \colon \mathcal{D}^s(\mathbb{T}) \to \mathcal{D}^s(\mathbb{T})$, $R_\psi\colon \eta \to \eta\circ\psi$, $L_\psi \colon \eta \to \psi\circ\eta$ denote the right and left translation map on $\mathcal{D}^s(\mathbb{T})$ respectively. It is easy to derive that

$$(D_\varphi R_{\varphi^{-1}}) v = v \circ \varphi^{-1}, \quad (D_\varphi L_{\varphi^{-1}}) v = (\nabla\varphi)^{-1} v, \quad \forall v \in T_\varphi \mathcal{D}^s(\mathbb{T}) \simeq H^s(\mathbb{T}). \tag{11}$$

The formulas (11) will be of great importance for the issues of this paper since we will consider right-invariant vector fields and metrics on the group $\mathcal{D}^s(\mathbb{T})$. In case of right invariance, the sign of the commutator bracket (9) and the operators depending linearly on it will have to be changed.

Let G be a Lie group and V be a vector space. If G acts on the right on V, one defines

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_2 + v_1 g_2)$$

and with this product, $G \times V$ becomes a Lie group (the *semidirect product of G and V*) which is denoted as $G \circledS V$. It is easy to see that $(e, 0)$ is the neutral element, where e denotes the neutral element of G, and that (g, v) has the inverse $(g^{-1}, -vg^{-1})$. To obtain the Lie bracket on the Lie algebra $\mathfrak{g} \circledS V$, we consider the inner automorphism

$$I_{(g,v)}(h, w) = (g, v)(h, w)(g, v)^{-1} = (ghg^{-1}, -vg^{-1} + (w + vh)g^{-1}).$$

Writing $v\xi$ for the induced infinitesimal action of $\mathfrak{g}$ on V, i.e., the map

$$V \times \mathfrak{g} \mapsto V, \quad (v, \xi) \mapsto v\xi := \left.\frac{\mathrm{d}}{\mathrm{d}t} v g(t)\right|_{t=0},$$

$g(t)$ being a curve in G starting from e in the direction of ξ, we obtain

$$\mathrm{Ad}_{(g,v)}(\xi, w) = (\mathrm{Ad}_g \xi, (w + v\xi) g^{-1}), \tag{12}$$

$$\mathrm{ad}_{(\eta,v)}(\xi,w)=(\mathrm{ad}_\eta\xi, v\xi-w\eta) \tag{13}$$

and hence

$$[(\xi_1,v_1),(\xi_2,v_2)]=\mathrm{ad}_{(\xi_2,v_2)}(\xi_1,v_1)=([\xi_1,\xi_2],v_2\xi_1-v_1\xi_2).$$

For our purposes, we consider the semidirect product $\mathcal{D}^s(\mathbb{T})ⓈH^{s-1}(\mathbb{T})$; the structure of our equation motivates to enforce the second component to have one order less regularity than the first, cf. also [12]. Although $\mathcal{D}^s(\mathbb{T})$ is not a Lie group, we will use the above terminology for the convenience of the reader. The group product in the product group is defined by

$$(\varphi_1,f_1)(\varphi_2,f_2):=(\varphi_1\circ\varphi_2,f_2+f_1\varphi_2)$$

where $\circ$ denotes the group product in $\mathcal{D}^s(\mathbb{T})$ (i.e., composition) and $f\varphi:=f\circ\varphi$ is a right action of $\mathcal{D}^s(\mathbb{T})$ on the functions $\mathbb{T}\to\mathbb{R}^2$. The neutral element is $(\mathrm{id}_{\mathbb{T}},0)$ and (φ,f) has the inverse $(\varphi^{-1},-f\circ\varphi^{-1})$. The above calculations show that

$$\mathrm{Ad}_{(\varphi,f)}(u,\rho)=(\mathrm{Ad}_\varphi u,[(\nabla f)\cdot u+\rho]\circ\varphi^{-1}),$$

$$\mathrm{ad}_{(v,g)}(u,\rho)=(\mathrm{ad}_v u,(\nabla g)\cdot u-(\nabla\rho)\cdot v)$$

and

$$[(u_1,u_2),(v_1,v_2)]=([u_1,v_1],(\nabla v_2)\cdot u_1-(\nabla u_2)\cdot v_1),$$

where $[u_1,v_1]=(\nabla v_1)\cdot u_1-(\nabla u_1)\cdot v_1$ is the Lie bracket on $\mathcal{D}^s(\mathbb{T})$. Observe that $T(\mathcal{D}^s(\mathbb{T})ⓈH^{s-1}(\mathbb{T}))\simeq\mathcal{D}^s(\mathbb{T})ⓈH^{s-1}(\mathbb{T})\times H^s(\mathbb{T})\times H^{s-1}(\mathbb{T})$ and that

$$(D_{(\varphi,f)}R_{(\varphi,f)^{-1}})v=v\circ\varphi^{-1},\quad (D_{(\varphi,f)}L_{(\varphi,f)^{-1}})v=((\nabla\varphi)^{-1}v_1,v_2-\nabla f(\nabla\varphi)^{-1}v_1),$$

for any $v\in T_{(\varphi,f)}\mathcal{D}^s(\mathbb{T})ⓈH^{s-1}(\mathbb{T})\simeq H^s(\mathbb{T})\times H^{s-1}(\mathbb{T})$. For further details about semidirect product groups we refer the reader to [18, 19].

3. The μ-b-equation on the diffeomorphism group of the torus

Let $\mathbb{A}=\mathrm{diag}(\mu-\Delta,\mu-\Delta)$. We can rewrite the 2D-$\mu$-$b$-equation (2) as

$$u_t=-\mathbb{A}^{-1}\{u\cdot\nabla(\mathbb{A}u)+(\nabla u)^T\mathbb{A}u+(b-1)\mathbb{A}u(\nabla\cdot u)\}. \tag{14}$$

Introducing the quadratic operators

$$B(u,v)=-\mathbb{A}^{-1}\{u\cdot\nabla(\mathbb{A}v)+(\nabla u)^T\mathbb{A}v+(b-1)\mathbb{A}v(\nabla\cdot u)\}$$

and

$$\Gamma(u,v)=\frac{1}{2}\left((\nabla u)\cdot v+(\nabla v)\cdot u+B(u,v)+B(v,u)\right), \tag{15}$$

we find that

$$u_t+\nabla u\cdot u=\Gamma(u,u). \tag{16}$$

Note that $\Gamma\colon H^s(\mathbb{T})\times H^s(\mathbb{T})\to H^s(\mathbb{T})$ which follows from the identity

$$u\cdot\nabla(\mathbb{A}v)-\mathbb{A}(\nabla v\cdot u)=-\mu(\nabla v\cdot u)+\nabla v\cdot\Delta u+2\nabla v_x\cdot u_x+2\nabla v_y\cdot u_y.$$

In the special case $b = 2$, we find that $B(u, v) = -\mathrm{ad}_u^* v$ with ad^* as in (10). For $u \in H^s(\mathbb{T})$ and $s > 3$, let φ be the solution of the initial value problem

$$\begin{cases} \varphi_t(t,z) &= u(t,\varphi(t,z)), \quad z \in \mathbb{T}, t > 0, \\ \varphi(0,z) &= z, \quad z \in \mathbb{T} \end{cases}$$

in $\mathcal{D}^s(\mathbb{T})$ on some time interval $[0, T)$. We will use the short hand notation $\varphi_t = u \circ \varphi$ to emphasize that φ_t is the right-invariant vector field on $\mathcal{D}^s(\mathbb{T})$ with value u at id; equivalently, φ is a local flow for the vector field $u \cdot \nabla$ on $\mathbb{T}$. Next, we introduce the mapping

$$\Gamma_\varphi \colon \mathcal{D}^s(\mathbb{T}) \times H^s(\mathbb{T}) \times H^s(\mathbb{T}) \to H^s(\mathbb{T}), \quad (\varphi, U, V) \mapsto \Gamma(U \circ \varphi^{-1}, V \circ \varphi^{-1}) \circ \varphi$$

which is the right-invariant extension of the map Γ on the group $\mathcal{D}^s(\mathbb{T})$. We will call the map Γ_φ the *Christoffel operator* for the 2D-μ-b equation. It follows that Eq. (16) is equivalent to

$$\varphi_{tt} = \Gamma_\varphi(\varphi_t, \varphi_t) \tag{17}$$

and our purpose is to show that Eq. (17) is a geodesic equation for the Lagrangian variable $\varphi \in \mathcal{D}^s(\mathbb{T})$. To this end, we define a smooth torsion-free affine connection $\bar\nabla$ on $\mathcal{D}^s(\mathbb{T})$ and prove that (17) is indeed the geodesic equation corresponding to the connection $\bar\nabla$. Let $\mathfrak{X}(\mathcal{D}^s(\mathbb{T}))$ denote the space of smooth vector fields on $\mathcal{D}^s(\mathbb{T})$. For any $X, Y \in \mathfrak{X}(\mathcal{D}^s(\mathbb{T}))$, we let

$$(\bar\nabla_X Y)(\varphi) = DY(\varphi) \cdot X(\varphi) - \Gamma_\varphi(X(\varphi), Y(\varphi)), \tag{18}$$

and it is easy to check that $\bar\nabla \colon \mathfrak{X}(\mathcal{D}^s(\mathbb{T})) \times \mathfrak{X}(\mathcal{D}^s(\mathbb{T})) \to \mathfrak{X}(\mathcal{D}^s(\mathbb{T}))$ satisfies

(i) $\bar\nabla_{fX+gY} Z = f \bar\nabla_X Z + g \bar\nabla_Y Z$,
(ii) $\bar\nabla_X (Y + Z) = \bar\nabla_X Y + \bar\nabla_X Z$,
(iii) $\bar\nabla_X (fY) = f \bar\nabla_X Y + X(f) Y$,
(iv) $\bar\nabla_X Y - \bar\nabla_Y X = [X, Y]$,

for all $X, Y, Z \in \mathfrak{X}(D^s(\mathbb{T}))$ and all $f, g \in C^\infty(\mathcal{D}^s(\mathbb{T}); \mathbb{R})$. Moreover, the map $\varphi \mapsto (\varphi, (\bar\nabla_X Y)(\varphi))$, $\mathcal{D}^s(\mathbb{T}) \to T\mathcal{D}^s(\mathbb{T})$ is smooth for any $X, Y \in \mathfrak{X}(\mathcal{D}^s(\mathbb{T}))$, which can be concluded from the following well-known line of arguments: To verify that the map $\mathcal{D}^s(\mathbb{T}) \to \mathcal{L}^2_{\mathrm{sym}}(H^s(\mathbb{T}); H^s(\mathbb{T}))$, $\varphi \mapsto \Gamma_\varphi$ is smooth, we fix $U, V \in H^s(\mathbb{T})$ and observe that

$$G(\varphi, U, V) = (\varphi, \Gamma_\varphi(U, V)) = (\varphi, (\mathbb{A}^{-1} P(U \circ \varphi^{-1}, V \circ \varphi^{-1})) \circ \varphi),$$

where P is a polynomial operator involving the map μ. Introducing the maps

$$\tilde{\mathbb{A}}(\varphi, W) = (\varphi, R_\varphi \circ \mathbb{A} \circ R_{\varphi^{-1}} W), \quad \tilde P(\varphi, W) = (\varphi, R_\varphi \circ P \circ R_{\varphi^{-1}} W)$$

we see that $G = \tilde{\mathbb{A}}^{-1} \circ \tilde P$. Moreover $\tilde{\mathbb{A}}$ and $\tilde P$ are smooth maps, since $\mathbb{A}$ and P are polynomial operators which depend smoothly on μ and $H^s(\mathbb{T}; \mathbb{R})$ is a Banach algebra for $s > 1$, [1]. It follows from

$$D\tilde{\mathbb{A}} = \begin{pmatrix} \mathrm{id} & 0 \\ * & R_\varphi \circ \mathbb{A} \circ R_{\varphi^{-1}} \end{pmatrix}$$

that $D\tilde{\mathbb{A}}_{(\varphi, W)} \colon H^s(\mathbb{T}) \times H^s(\mathbb{T}) \to H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ is a bijective bounded linear map for any (φ, W). In view of the open mapping theorem and the inverse mapping theorem, $\tilde{\mathbb{A}}^{-1}$ is a smooth map and hence also G is smooth. Since the existence of a smooth connection on a Banach manifold immediately achieves well-posedness of the geodesic flow, we conclude from the above the following

Theorem 1. *Let Γ be the 2D-μ-b-Christoffel map defined in* (15)*. Fix u_0 in some open neighborhood $U \subset H^s(\mathbb{T})$ containing zero, for some $s > 3$. Then the initial value problem*

$$\left\{ \begin{array}{rcl} \varphi_{tt} &=& \Gamma(\varphi_t, \varphi_t), \\ \varphi_t(0) &=& u_0, \\ \varphi(0) &=& \mathrm{id} \end{array} \right. \tag{19}$$

for the geodesic flow corresponding to the 2D-μ-b-equation on $\mathcal{D}^s(\mathbb{T})$ has a unique solution $\varphi \in C^2([0,T);\mathcal{D}^s(\mathbb{T}))$, for some $T > 0$, which depends smoothly on time and the initial value, i.e., the map $(t, u_0) \mapsto \varphi$, $[0,T) \times U \to \mathcal{D}^s(\mathbb{T})$ is smooth.

Since $\mathcal{D}^s(\mathbb{T})$ is a topological group, we may set $u = \varphi_t \circ \varphi^{-1}$ to obtain a solution of (2) with the desired regularity properties.

Corollary 2. *Fix $s > 3$. There exists an open neighborhood $U \subset H^s(\mathbb{T})$ of zero, such that for any $u_0 \in U$ there is $T > 0$ and a unique solution*

$$u \in C([0,T); H^s(\mathbb{T})) \cap C^1([0,T); H^{s-1}(\mathbb{T}))$$

to the 2D-μ-b-equation (2) *with continuous dependence on time and the initial value, i.e., the mapping*

$$u_0 \mapsto u, \quad U \to C([0,T); H^s(\mathbb{T})) \cap C^1([0,T); H^{s-1}(\mathbb{T}))$$

is continuous.

In what follows, we will fix $b = 2$, since in this case we can establish that $\bar{\nabla}$ is a Riemannian connection in the sense that it is compatible with a suitable metric. With $\mathbb{A} = (\mu - \Delta)\mathrm{diag}(1,1)$ we define

$$\langle u, v \rangle = \int_{\mathbb{T}} u \mathbb{A} v \, \mathrm{d}(x,y), \quad u, v \in H^s(\mathbb{T}).$$

It follows from a standard Fourier representation argument that $\mathbb{A}$ is a topological isomorphism $H^s(\mathbb{T}) \to H^{s-2}(\mathbb{T})$ and it is easy to derive that $\langle \cdot, \cdot \rangle$ is a scalar product on $H^s(\mathbb{T})$. Again, we extend $\langle \cdot, \cdot \rangle$ to a right-invariant inner product on $\mathcal{D}^s(\mathbb{T})$ by setting

$$\langle X, Y \rangle_\varphi = \left\langle X(\varphi) \circ \varphi^{-1}, Y(\varphi) \circ \varphi^{-1} \right\rangle, \quad \forall X, Y \in \mathfrak{X}(\mathcal{D}^s(\mathbb{T})). \tag{20}$$

Observe that $(\mathcal{D}^s(\mathbb{T}), \langle \cdot, \cdot \rangle)$ is a Riemannian manifold since the mapping $\varphi \mapsto Q(\varphi; U, V) = \langle U, V \rangle_\varphi$ is smooth for any $U, V \in H^s(\mathbb{T})$; in view of the change of variables $(\tilde{x}, \tilde{y}) = \varphi^{-1}(x,y)$, this follows immediately from the representation

$$\begin{aligned} Q(\varphi; U, V) \quad &= \quad \sum_{i=1}^{2} \Big\{ \mu(U_i |\nabla\varphi|) \mu(V_i |\nabla\varphi|) \\ &\qquad + \int_{\mathbb{T}} \left\{ [\nabla U_i \cdot (\nabla\varphi)^{-1}] \cdot [\nabla V_i \cdot (\nabla\varphi)^{-1}] \right\} |\nabla\varphi| \, \mathrm{d}(\tilde{x}, \tilde{y}) \Big\}. \end{aligned}$$

Indeed, the connection $\bar{\nabla}$ preserves the metric $\langle\cdot,\cdot\rangle$ in the usual sense

$$X\langle Y,Z\rangle = \left\langle \bar{\nabla}_X Y, Z\right\rangle + \left\langle \bar{\nabla}_X Z, Y\right\rangle, \quad \forall X,Y,Z\in\mathfrak{X}(\mathcal{D}^s(\mathbb{T})).$$

Using the identity

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} Y_i(\varphi+\varepsilon X(\varphi))\circ(\varphi+\varepsilon X(\varphi))^{-1} = [DY_i(\varphi)\cdot X(\varphi)]\circ\varphi^{-1} - \nabla v_i\cdot u, \tag{21}$$

where $X,Y\in\mathfrak{X}(\mathcal{D}^s(\mathbb{T}))$ and $u = X(\varphi)\circ\varphi^{-1}$, $v = Y(\varphi)\circ\varphi^{-1}$, we get that

$$\begin{aligned}(X\langle Y,Z\rangle)(\varphi) \quad &= \quad \int_{\mathbb{T}} \left\{[DY(\varphi)\cdot X(\varphi)]\circ\varphi^{-1} - \nabla v\cdot u\right\}\mathbb{A}w\,\mathrm{d}(x,y) \\ &\quad + \int_{\mathbb{T}} \left\{[DZ(\varphi)\cdot X(\varphi)]\circ\varphi^{-1} - \nabla w\cdot u\right\}\mathbb{A}v\,\mathrm{d}(x,y)\end{aligned}$$

with $w = Z(\varphi)\circ\varphi^{-1}$ accordingly. On the other hand

$$\langle\nabla_X Y, Z\rangle_\varphi = \int_{\mathbb{T}} \left\{[DY(\varphi)\cdot X(\varphi)]\circ\varphi^{-1} - \Gamma(u,v)\right\}\mathbb{A}w\,\mathrm{d}(x,y)$$

so that it remains to check the identity

$$\int_{\mathbb{T}}[(\nabla v\cdot u)\mathbb{A}w + (\nabla w\cdot u)\mathbb{A}v]\,\mathrm{d}(x,y) = \int_{\mathbb{T}}[\Gamma(u,v)\mathbb{A}w + \Gamma(u,w)\mathbb{A}v]\,\mathrm{d}(x,y). \tag{22}$$

In view of the definition of Γ and the fact that the adjoint of ∇u with respect to the L_2 inner product is $(\nabla u)^T$, the right hand side of Eq. (22) equals

$$\begin{aligned}&-\frac{1}{2}\int_{\mathbb{T}}\{[(u\cdot\nabla)\mathbb{A}v]\cdot w + [(v\cdot\nabla)\mathbb{A}u]\cdot w + [(\nabla v)^T\mathbb{A}u]\cdot w + (\nabla\cdot v)(\mathbb{A}u)\cdot w \\ &\quad +(\nabla\cdot u)(\mathbb{A}v)\cdot w - [\mathbb{A}(\nabla v\cdot u)]\cdot w + (u\cdot\nabla)(\mathbb{A}w)\cdot v + (w\cdot\nabla)(\mathbb{A}u)\cdot v \\ &\quad +[(\nabla w)^T\mathbb{A}u]\cdot v + (\nabla\cdot w)(\mathbb{A}u)\cdot v + (\nabla\cdot u)(\mathbb{A}w)\cdot v - [\mathbb{A}(\nabla w\cdot u)]\cdot v\}\,\mathrm{d}(x,y).\end{aligned} \tag{23}$$

Using the identity

$$\int_{\mathbb{T}}\{(u\cdot\nabla)\mathbb{A}v + (\mathbb{A}v)(\nabla\cdot u)\}w\,\mathrm{d}(x,y) = -\int_{\mathbb{T}}\mathbb{A}v(u\cdot\nabla)w\,\mathrm{d}(x,y) \tag{24}$$

and $(u\cdot\nabla)v = \nabla v\cdot u$ we can rewrite (23) as

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{T}}\{2(\nabla v\cdot u)\mathbb{A}w + 2(\nabla w\cdot u)\mathbb{A}v + (\nabla w\cdot v)\mathbb{A}u - (\nabla v)^T\mathbb{A}u\cdot w \\ &\qquad +(\nabla v\cdot w)\mathbb{A}u - (\nabla w)^T\mathbb{A}u\cdot v\}\,\mathrm{d}(x,y).\end{aligned} \tag{25}$$

Simplifying these terms it follows that the expression (25) reduces to

$$\int_{\mathbb{T}}\{(\nabla v\cdot u)\mathbb{A}w + (\nabla w\cdot u)\mathbb{A}v\}\,\mathrm{d}(x,y)$$

and this is the left hand side of Eq. (22).

Proposition 3. *For $b=2$, the pair $(\mathcal{D}^s(\mathbb{T}),\langle\cdot,\cdot\rangle)$, $s>3$, is a Riemannian manifold and the connection $\bar{\nabla}$ defined in* (18) *preserves the metric on $\mathcal{D}^s(\mathbb{T})$.*

We will address the question of compatibility with a Riemannian structure in the following section where we give a rigorous proof that the case $b = 2$ is distinguished. Note that the geodesic flow is indeed length minimizing in the sense that it minimizes the functional

$$L(\gamma) = \int_J \langle \gamma_t(t), \gamma_t(t) \rangle_{\gamma(t)}^{1/2} \, \mathrm{d}t.$$

In fact, the arguments are just a repetition of the general approach for Riemannian metrics on Banach manifolds, cf. [29], and they have been worked out by Lenells [31] for the $(1,1)$CH. Exchanging $\mathbb{S}$ with $\mathbb{T}$, we can proceed as in [31] to derive the following theorems.

Theorem 4 (Existence of normal neighborhoods)**.** *Let $\varphi_0 \in \mathcal{D}^s(\mathbb{T})$. Given an open neighborhood $\mathcal{V} = \mathcal{U}_0 \times B_\varepsilon(0)$ of $(\varphi_0, 0) \in T\mathcal{D}^s(\mathbb{T})$, there is an open neighborhood $\mathcal{W} \subset \mathcal{U}_0$ of φ_0 in $\mathcal{D}^s(\mathbb{T})$ such that any two points $\varphi, \psi \in \mathcal{W}$ can by joined by a unique geodesic lying in $\mathcal{U}_0$, and such that for any $\varphi \in \mathcal{W}$, the exponential map $\exp_\varphi$ maps the open set in $T_\varphi\mathcal{D}^s(\mathbb{T})$ represented by $(\varphi, B_\varepsilon(0))$ diffeomorphically onto an open set $\mathcal{U}(\varphi)$ containing $\mathcal{W}$.*

The following theorem is proved by applying the Gauss lemma for the right-invariant metric $\langle \cdot, \cdot \rangle_\varphi$ on $\mathcal{D}^s(\mathbb{T})$.

Theorem 5. *Let $(\mathcal{V}, \mathcal{W})$, $\mathcal{V} = \mathcal{U}_0 \times B_\varepsilon(0)$ constitute a normal neighborhood of an element $\varphi_0 \in \mathcal{D}^s(\mathbb{T})$. Let $\alpha\colon [0,1] \to \mathcal{D}^s(\mathbb{T})$ be the unique geodesic joining two points $\varphi, \psi \in \mathcal{W}$. Then, for any piecewise C^1-path $\gamma\colon [0,1] \to \mathcal{D}^s(\mathbb{T})$ joining φ and ψ, it holds that*

$$L(\alpha) \leq L(\gamma).$$

If equality holds, then a reparametrization of γ is equal to α.

Conversely, it holds globally that any length-minimizing curve is a geodesic.

Theorem 6. *If $\alpha\colon [0,1] \to \mathcal{D}^s(\mathbb{T})$ is a piecewise C^1-path parametrized by arc-length such that $L(\alpha) \leq L(\gamma)$ for all paths γ in $\mathcal{D}^s(\mathbb{T})$ joining $\alpha(0)$ and $\alpha(1)$, then α is a geodesic.*

The existence of a smooth connection $\bar{\nabla}$ on a Banach manifold M immediately implies the existence of a smooth curvature tensor R defined by

$$R(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z,$$

where X, Y, Z are vector fields on M, cf. [29]. In the case of the $\mu - (2,1)$CH equation, since there exists a metric $\langle \cdot, \cdot \rangle$, we can also define an (unnormalized) sectional curvature S by

$$S(X,Y) := \langle R(X,Y)Y, X \rangle.$$

In this section, we will derive a convenient formula for S and use it to determine large subspaces of positive curvature for the 2-dimensional μ-Camassa-Holm equation.

Proposition 7. *Let $s > 3$. Let R be the curvature tensor on $\mathcal{D}^s(\mathbb{T})$ associated with the $\mu-(2,1)$CH. Then $S(u,v) := \langle R(u,v)v, u\rangle$ is given at the identity by*

$$S(u,v) = \langle \Gamma(u,v), \Gamma(u,v)\rangle - \langle \Gamma(u,u), \Gamma(v,v)\rangle + R(u,v), \quad u, v \in T_{\mathrm{id}}\mathcal{D}^s(\mathbb{T}), \tag{26}$$

where

$$\begin{aligned} R(u,v) \;=\; & \langle \nabla u \cdot u, \nabla v \cdot v\rangle - \langle \nabla u \cdot v, \nabla u \cdot v\rangle + \langle \nabla v \cdot u, \nabla u \cdot v\rangle - \langle \nabla v \cdot u, \nabla v \cdot u\rangle \\ & + \langle [\nabla(\nabla u \cdot u)] \cdot v, v\rangle - \langle [\nabla(\nabla u \cdot v)] \cdot v, u\rangle + \langle [\nabla(\nabla v \cdot u)] \cdot v, u\rangle \\ & - \langle [\nabla(\nabla v \cdot u)] \cdot u, v\rangle - \langle \nabla v(\nabla u \cdot u), v\rangle - \langle \nabla u(\nabla v \cdot v), u\rangle \\ & + \langle \nabla v(\nabla v \cdot u), u\rangle + \langle \nabla u(\nabla v \cdot u), v\rangle . \end{aligned}$$

Proof. Let $U, V, W \in T_\varphi\mathcal{D}^s(\mathbb{T})$ be three tangent vectors at a point $\varphi \in \mathcal{D}^s(\mathbb{T})$. The curvature tensor R is given locally by

$$\begin{aligned} R_\varphi(U,V)W = {} & D_1\Gamma_\varphi(W,U)V - D_1\Gamma_\varphi(W,V)U \\ & + \Gamma_\varphi(\Gamma_\varphi(W,V),U) - \Gamma_\varphi(\Gamma_\varphi(W,U),V) \end{aligned}$$

where Γ is the $\mu-(2,1)$CH Christoffel map defined in (15) and D_1 denotes differentiation with respect to φ:

$$D_1\Gamma_\varphi(W,U)V = \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \Gamma_{\varphi+\varepsilon V}(W,U).$$

By right invariance,

$$R_\varphi(X,Y)Z \circ \varphi^{-1} = R_{\mathrm{id}}(u,v)w,$$

where $u = X(\varphi)\circ\varphi^{-1}$, $v = Y(\varphi)\circ\varphi^{-1}$ and $w = Z(\varphi)\circ\varphi^{-1}$, it suffices to discuss the curvature tensor at the identity. To simplify our notation, we will omit the index id of the geometric objects under discussion in the following. First, we note that

$$D_1\Gamma(w,u)v = -\Gamma(\nabla w \cdot v, u) - \Gamma(\nabla u \cdot v, w) + \nabla\Gamma(w,u)\cdot v.$$

Thus,

$$\begin{aligned} S(u,v) = \; & \langle \Gamma(\Gamma(v,v),u), u\rangle - \langle \Gamma(\Gamma(v,u),v), u\rangle \\ & + \langle \nabla\Gamma(v,u)\cdot v, u\rangle - \langle \nabla\Gamma(v,v)\cdot u, u\rangle \\ & + \langle -\Gamma(\nabla v\cdot v, u) - \Gamma(\nabla u \cdot v, v) + 2\Gamma(\nabla v \cdot u, v), u\rangle . \end{aligned} \tag{27}$$

A lengthy but tedious computation similar to the calculation in the proof of Proposition 5.1 in [12] shows that Eq. (27) becomes

$$\begin{aligned} S(u,v) = \; & \langle \Gamma(u,v), \Gamma(u,v)\rangle - \langle \Gamma(u,u), \Gamma(v,v)\rangle \\ & + \langle \nabla u \cdot u, \Gamma(v,v)\rangle - \langle \nabla u \cdot v, \Gamma(v,u)\rangle \\ & + \langle -\Gamma(\nabla v\cdot v, u) - \Gamma(\nabla u \cdot v, v) + 2\Gamma(\nabla v \cdot u, v), u\rangle \end{aligned} \tag{28}$$

and by the definition of Γ, we obtain (26) after a lengthy calculation. □

Remark 8. Identities similar to (26) have been derived for the $(1,1)$CH and the $(1,2)$CH in [35, 36, 12]. The curvature formula presented in [12] is formally identical to Eq. (26) with $R \equiv 0$; simply the metric and the Christoffel operator have to be exchanged by their 1D two-component analogs. It is not known to the author whether $R \equiv 0$ in (26), but it seems to be impossible to handle the large number of terms in a calculation by hand; it is rather worthwhile using computer software to simplify the terms in $R(u,v)$, but this will not be part of this analytical paper. Observe that identities of type $S(u,v) = \langle \Gamma(u,v), \Gamma(u,v)\rangle - \langle \Gamma(u,u), \Gamma(v,v)\rangle$ do *not* follow from the general theory of right-invariant metrics on Banach manifolds; the example of the $\mu-(1,1)$CH in [24] shows that the unnormalized sectional curvature S for this equation contains an additional nontrivial term.

Curvature computations have a long tradition in the geometric theory of partial differential equations, see, e.g., [16, 34], and we will now contribute some further examples concerning Eq. (1). For our purposes, two-dimensional subspaces on which $S > 0$ are of particular interest since the positivity of S is related to stability properties of the geodesic flow, cf. [3]. Large subspaces of positive curvature for $(1,1)$CH and $(1,2)$CH have been found in [35, 36, 12] by evaluating the sectional curvature on trigonometric functions. Here we extend this discussion for the $\mu-(2,1)$CH equation.

Example 9. Let $\{e_1, e_2\}$ be the canonical basis of $\mathbb{R}^2$, let $k_1, k_2 \in 2\pi\mathbb{N}$ and let

$$v = \begin{pmatrix} \sin(k_1 x)\sin(k_2 y) \\ \sin(k_1 x)\sin(k_2 y) \end{pmatrix}.$$

In the following, we will make use of the identity

$$(\mu+\Delta)^{-1}\sin(\alpha x)\cos(\beta y) = \frac{\sin(\alpha x)\cos(\beta y)}{\alpha^2+\beta^2}$$

and the trigonometric formulas

$$\int_{\mathbb{S}} \cos^2(\alpha x)\,\mathrm{d}x = \int_{\mathbb{S}} \sin^2(\alpha x)\,\mathrm{d}x = \frac{1}{2}, \quad \int_{\mathbb{S}} \cos(\alpha x)\sin(\beta x)\,\mathrm{d}x = 0, \quad \forall \alpha, \beta \in 2\pi\mathbb{N}.$$

We claim that $S(e_i, v) > 0$ for $i = 1, 2$. First we observe that for general $w \in C^\infty(\mathbb{T};\mathbb{T})$, we have that $R(e_i, w) = 0$, $i = 1, 2$; this follows from a short computation using integration by parts. Hence

$$S(e_i, v) = \langle \Gamma(e_i, v), \Gamma(e_i, v)\rangle .$$

We leave it to the reader to perform explicitly the easy calculations leading to the formulas

$$\begin{aligned} S(e_1, v) &= \frac{1}{8}\frac{2k_1^2 + k_2^2}{k_1^2 + k_2^2}, \\ S(e_2, v) &= \frac{1}{8}\frac{2k_2^2 + k_1^2}{k_1^2 + k_2^2}. \end{aligned}$$

It follows that, for any $k_1, k_2 \in 2\pi\mathbb{N}$, $S > 0$ on the spaces $\operatorname{span}\{e_i, v\}$, $i = 1, 2$.

4. The special role of the case $b = 2$

In Section 3 we have seen that the $\mu-(2,1)$CH re-expresses geodesic motion on $\mathcal{D}^s(\mathbb{T})$ for any value of b and that the geodesic flow is related to a Riemannian metric in the special case $b = 2$. However, such a correspondence with a Riemannian structure of type $\operatorname{diag}(A, A)$, $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{T}))$ works *only* if $b = 2$. This result is stated in the following theorem. In accordance with the results of [14, 26, 27] we are working with the μ-b-equation on the Fréchet-Lie group of smooth orientation-preserving diffeomorphisms of the torus.

Theorem 10. *Let $b \geq 2$ be an integer and $\mathbb{L} = \operatorname{diag}(\mu - \Delta, \mu - \Delta)$. Suppose that there is a regular inertia operator $\mathbb{A} = \operatorname{diag}(A, A)$, $A \in \mathcal{L}_{\text{is}}^{\text{sym}}(C^\infty(\mathbb{T}))$, such that the 2D-$\mu$-$b$-equation*

$$m_t = -u \cdot \nabla m - (\nabla u)^T m - (b-1)m(\nabla \cdot u), \quad m = \mathbb{L}u,$$

is the Euler equation on $\operatorname{Diff}^\infty(\mathbb{T})$ with respect to the right-invariant metric $\rho_{\mathbb{A}}$ defined by $\mathbb{A}$. Then $b = 2$ and $\mathbb{A} = \mathbb{L}$.

Proof. We assume that, for a given $b \geq 2$ and $A \in \mathcal{L}_{\text{is}}^{\text{sym}}(C^\infty(\mathbb{T}))$, the 2D-$\mu$-$b$-equation is the Euler equation on the torus diffeomorphism group with respect to $\rho_{\mathbb{A}}$, i.e.,

$$u_t = -\mathbb{A}^{-1}\{u \cdot \nabla(\mathbb{A}u) + (\nabla u)^T \mathbb{A}u + \mathbb{A}u(\nabla \cdot u)\}.$$

The 2D-μ-b-equation reads

$$(\mathbb{L}u)_t = -u \cdot \nabla(\mathbb{L}u) - (\nabla u)^T(\mathbb{L}u) - (b-1)(\mathbb{L}u)(\nabla \cdot u).$$

Using that $(\mathbb{L}u)_t = \mathbb{L}u_t$ and resolving both equations with respect to u_t we get that

$$\begin{aligned} \mathbb{A}^{-1}\left\{u \cdot \nabla(\mathbb{A}u) + (\nabla u)^T \mathbb{A}u + \mathbb{A}u(\nabla \cdot u)\right\} &= \\ \mathbb{L}^{-1}\left\{u \cdot \nabla(\mathbb{L}u) + (\nabla u)^T(\mathbb{L}u) + (b-1)(\mathbb{L}u)(\nabla \cdot u)\right\}. \end{aligned} \tag{29}$$

We now conclude that $\operatorname{span}\{\mathbf{1}\}$, $\mathbf{1} = e_1 + e_2$, is an invariant subspace of $\mathbb{A}$. We let $A1 = \lambda$ and have $\mathbb{A}\mathbf{1} = \lambda\mathbf{1}$. Evaluating Eq. (29) for e_1 and e_2 shows that $\nabla\lambda = 0$. The structure of (29) suggests that we have the freedom to scale the operator $\mathbb{A}$ so that we can assume that $\mathbf{1}$ is a fixed point for $\mathbb{A}$. We next replace u by $u + \sigma\mathbf{1}$ in (29) and divide (29) by σ, to obtain for $\sigma \to \infty$ that

$$\mathbb{A}^{-1}\left[(\nabla u)^T + \nabla(\mathbb{A}u) + (\nabla \cdot u)\right]\mathbf{1} = \mathbb{L}^{-1}\left[(\nabla u)^T + \nabla(\mathbb{L}u) + (b-1)(\nabla \cdot u)\right]\mathbf{1}. \tag{30}$$

Let $n = (n_1, n_2) \in (2\pi\mathbb{Z})^2 \backslash \{(0,0)\}$, and write $z = (x, y)$ for the variable on $\mathbb{T}$. We will consider the functions $u_n = e^{\mathrm{i}nz}\mathbf{1}$ in the following for which we have $\mathbb{L}u_n = n^2 u_n$ and $\mathbb{L}^{-1}u_n = n^{-2}u_n$, $n^2 = n_1^2 + n_2^2$. Set $v_n = \mathbb{A}u_n$. An explicit calculation of the left hand and right hand side of (30) shows that we have the identity

$$\nabla v_n \cdot \mathbf{1} - \mathrm{i}\alpha_n v_n = -\mathrm{i}\beta_n u_n \tag{31}$$

where

$$\begin{aligned} \alpha_n &= \operatorname{diag}\left((b+1)\frac{n_1}{n^2} + (b-1)\frac{n_2}{n^2} + n_1 + n_2, (b+1)\frac{n_2}{n^2} + (b-1)\frac{n_1}{n^2} + n_1 + n_2\right), \\ \beta_n &= \operatorname{diag}(3n_1 + n_2, 3n_2 + n_1). \end{aligned}$$

Assume that $n_1 \neq n_2$ first. Since $v_n = Ae^{\mathrm{i}nz}\mathbf{1}$ we get that $Ae^{\mathrm{i}nz} = n^2 e^{\mathrm{i}nz}$. For $n_1 = n_2$ we see that the function $v_n = \frac{2}{b}n^2 e^{\mathrm{i}nz}\mathbf{1}$ constitutes a solution to (31). We insert u_n for $n_1 = n_2 \neq 0$ into Eq. (29) to get that $b = 2$. Since $\{e^{\mathrm{i}nz};\, n \in (2\pi\mathbb{Z})^2\}$ is a basis for $C^\infty(\mathbb{T})$, it follows that $\mathbb{A} = \mathbb{L}$. This completes the proof of the theorem. ∎

Corollary 11. *The geodesic flow for the* $\mu-(2,1)DP$ *equation on the diffeomorphism group of the torus is not related to any right-invariant metric on* $\mathrm{Diff}^\infty(\mathbb{T})$ *with inertia operator* $\mathrm{diag}(A,A)$*,* $A \in \mathcal{L}_{\mathrm{is}}^{\mathrm{sym}}(C^\infty(\mathbb{T}))$*.*

5. A two-component generalization for $b = 2$

The geodesic equation for the right-invariant H^1-metric on the group $\mathrm{Diff}(\mathbb{S})$ is the $(1,1)$CH equation (4), [28]. On the semidirect product of the H^s circle diffeomorphism group with $H^{s-1}(\mathbb{S})$, equipped with the right-invariant metric induced by the operator $\mathrm{diag}(1-\partial_x^2, 1)$, one obtains $(1,2)$CH as the associated geodesic equation, [12]. The analogs to these results for the $\mu-(1,1)$CH and the $\mu-(1,2)$CH are obtained on the manifolds $\mathrm{Diff}(\mathbb{S})$ and $\mathrm{Diff}(\mathbb{S}) \circledS H^{s-1}(\mathbb{S})$ with the inertia operators $\mu - \partial_x^2$ and $\mathrm{diag}(\mu - \partial_x^2, 1)$, [24, 25]. Since $\mu-(2,1)$CH is obtained as geodesic flow on $\mathcal{D}^s(\mathbb{T})$ with the metric (20), it is natural to consider the group $G = \mathcal{D}^s(\mathbb{T}) \circledS H^{s-1}(\mathbb{T})$ with the metric defined by the inertia operator

$$\mathbb{B} = \begin{pmatrix} \mu-\Delta & 0 & 0 & 0 \\ 0 & \mu-\Delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathrm{id}_{\mathbb{T}} \end{pmatrix},$$

precisely, for any two smooth vector fields $X = (X_1, \ldots, X_4)$ and $Y = (Y_1, \ldots, Y_4)$ on G, with $\tilde{X} = X(\varphi) \circ \varphi^{-1}$ and $\tilde{Y} = Y(\varphi) \circ \varphi^{-1}$, we define

$$\langle X, Y \rangle_{(\varphi,f)} = \sum_{i=1}^{2} \left\{ \mu(\tilde{X}_i)\mu(\tilde{Y}_i) + \int_{\mathbb{T}} \nabla \tilde{X}_i \cdot \nabla \tilde{Y}_i \,\mathrm{d}(x,y) \right\} + \sum_{i=3}^{4} \int_{\mathbb{T}} \tilde{X}_i \tilde{Y}_i \,\mathrm{d}(x,y). \tag{32}$$

We can rewrite (32) as

$$\begin{aligned} \langle X, Y \rangle_{(\varphi,f)} &= \sum_{i=1}^{2} \mu\left(X_i(\varphi)|\nabla\varphi|\right) \mu\left(Y_i(\varphi)|\nabla\varphi|\right) \\ &\quad + \sum_{i=1}^{2} \int_{\mathbb{T}} \left(\nabla[X_i(\varphi)](\nabla\varphi)^{-1}\right) \cdot \left(\nabla[Y_i(\varphi)](\nabla\varphi)^{-1}\right) |\nabla\varphi| \,\mathrm{d}(x,y) \\ &\quad + \sum_{i=3}^{4} \int_{\mathbb{T}} X_i(\varphi) Y_i(\varphi) |\nabla\varphi| \,\mathrm{d}(x,y). \end{aligned}$$

The dual operators for the maps (12) and (13) with respect to the L_2 pairing and the metric (32) are given by

$$\mathrm{Ad}^*_{(\varphi,f)}(v,\eta) = \begin{pmatrix} (\nabla\varphi)^T(v\circ\varphi)|\nabla\varphi| + (\nabla f)^T(\eta\circ\varphi)|\nabla\varphi| \\ (\eta\circ\varphi)|\nabla\varphi| \end{pmatrix}$$

and

$$\mathrm{ad}^*_{(u_1,u_2)}(v_1,v_2) = \begin{pmatrix} -\mathbb{A}^{-1}(u_1 \cdot \nabla(\mathbb{A}v_1) + \mathbb{A}v_1(\nabla\cdot u_1) + (\nabla u_1)^T \mathbb{A} v_1 + (\nabla u_2)^T v_2) \\ -\nabla v_2 \cdot u_1 - (\nabla \cdot u_1) v_2 \end{pmatrix}$$

respectively. Note that the Euler equation $(u_t, \rho_t) = \mathrm{ad}^*_{(u,\rho)}(u,\rho)$ is the $\mu-(2,2)$CH equation (7). In Lagrangian coordinates $(\varphi, f) \in G$, the $\mu-(2,2)$CH reads

$$(\varphi_{tt}, f_{tt}) = \Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t))$$

and the map Γ is the right-invariant Christoffel map on G defined by

$$\Gamma_{(\mathrm{id},0)}((u,\rho),(v,\eta)) = \begin{pmatrix} \Gamma^0_{\mathrm{id}}(u,v) \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \mathbb{A}^{-1}[(\nabla\rho)^T\eta + (\nabla\eta)^T\rho] \\ \rho(\nabla\cdot v) + \eta(\nabla\cdot u) \end{pmatrix}, \tag{33}$$

with Γ^0_{id} as in (15), and

$$\Gamma_{(\varphi,f)}(U,V) = \Gamma_{(\mathrm{id},0)}(U\circ\varphi^{-1}, V\circ\varphi^{-1})\circ\varphi.$$

The geometric approach to Eq. (7) also provides a conservation law.

Lemma 12. *Let (u,ρ) be a solution of the $\mu-(2,2)$CH equation. Then we have the conservation law*

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} (\nabla\varphi)^T(m\circ\varphi)|\nabla\varphi| + (\nabla f)^T(\rho\circ\varphi)|\nabla\varphi| \\ (\rho\circ\varphi)|\nabla\varphi| \end{pmatrix} = 0.$$

Proof. It follows from $\varphi_t = u\circ\varphi$ that $\frac{\mathrm{d}}{\mathrm{d}t}(\nabla\varphi)^T = (\nabla\varphi)^T(\nabla u\circ\varphi)^T$ and that $\frac{\mathrm{d}}{\mathrm{d}t}(m\circ\varphi) = (m_t + \nabla m\cdot u)\circ\varphi$. Next, we observe that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}|\nabla\varphi| &= \dot\varphi_{1x}\varphi_{2y} + \varphi_{1x}\dot\varphi_{2y} - \dot\varphi_{2x}\varphi_{1y} - \varphi_{2x}\dot\varphi_{1y} \\
&= [(\nabla u_1)\circ\varphi]\varphi_x\varphi_{2y} + [(\nabla u_2)\circ\varphi]\varphi_y\varphi_{1x} \\
&\quad -[(\nabla u_2)\circ\varphi]\varphi_x\varphi_{1y} - [(\nabla u_1)\circ\varphi]\varphi_y\varphi_{2x} \\
&= (u_{1x}\circ\varphi)\varphi_{1x}\varphi_{2y} + (u_{1y}\circ\varphi)\varphi_{2x}\varphi_{2y} + (u_{2x}\circ\varphi)\varphi_{1y}\varphi_{1x} \\
&\quad +(u_{2y}\circ\varphi)\varphi_{2y}\varphi_{1x} - (u_{2x}\circ\varphi)\varphi_{1x}\varphi_{1y} - (u_{2y}\circ\varphi)\varphi_{2x}\varphi_{1y} \\
&\quad -(u_{1x}\circ\varphi)\varphi_{1y}\varphi_{2x} - (u_{1y}\circ\varphi)\varphi_{2y}\varphi_{2x} \\
&= |\nabla\varphi|(\nabla\cdot u)\circ\varphi.
\end{aligned}$$

Consequently we find that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\circ\varphi)|\nabla\varphi| = \{[\rho_t + \nabla\rho\cdot u + \rho(\nabla\cdot u)]\circ\varphi\}|\nabla\varphi| = 0$$

and that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\{(\nabla\varphi)^T(m\circ\varphi)|\nabla\varphi| + (\nabla f)^T(\rho\circ\varphi)|\nabla\varphi|\} \\
&= (\nabla\varphi)^T\{[(\nabla u)^T m + m_t + \nabla m\cdot u + m(\nabla\cdot u) + (\nabla\rho)^T\rho]\circ\varphi\}|\nabla\varphi| \\
&= 0.
\end{aligned}$$

This achieves the proof of our lemma. □

Remark 13. From the above lemma it is clear why we do not consider the group $\mathcal{D}^s(\mathbb{T})$Ⓢ$H^s(\mathbb{T})$: Since $(\rho\circ\varphi)|\nabla\varphi| = \rho_0$ it follows that ρ cannot have more regularity than the first order derivative of φ. So if φ is an element of $\mathcal{D}^s(\mathbb{T})$, it follows that the second component solution ρ cannot be better than H^{s-1}.

Our approach gets in line with Arnold's [2] famous theory which has already been successful for the $(1,1)$CH [7, 8, 27], $(1,2)$CH [12, 17, 19], $\mu-(1,1)$CH [24, 32] and $\mu-(1,2)$CH [25, 33, 38]. Since Arnold's formalism is based on a direct comparison of geometric quantities with physical quantities for the rigid body motion in $\mathbb{R}^3$, see [3], we now give the analogs for the $\mu-(2,2)$CH in the following tabular. More details and an extensive presentation of related equations in this context can be found in, e.g., [12].

	Rigid body	$\mu-(2,2)$CH
configuration space	$SO(3)$	$\mathcal{D}^s(\mathbb{T})\circledS H^{s-1}(\mathbb{T})$
material velocity	$\dot{R}$	(φ_t, f_t)
spatial velocity	$\hat{\omega}=\dot{R}R^{-1}$	$(u,\rho)=(\varphi_t\circ\varphi^{-1}, f_t\circ\varphi^{-1})$
body velocity	$\hat{\Omega}=R^{-1}\dot{R}$	$\begin{pmatrix}U_1\\U_2\end{pmatrix}=\begin{pmatrix}(\nabla\varphi)^{-1}\varphi_t\\ f_t-\nabla f(\nabla\varphi)^{-1}\varphi_t\end{pmatrix}$
inertia operator	I	$\mathbb{B}=\begin{pmatrix}\mathbb{A} & 0\\ 0 & \mathrm{id}_{\mathbb{T}}\end{pmatrix}$
spatial momentum	$\pi=R\Pi$	$(m,\rho)=(\mathbb{A}u,\rho)$
body momentum	$\Pi=I\Omega$	$\begin{pmatrix}m_0\\ \rho_0\end{pmatrix}=\begin{pmatrix}(\nabla\varphi)^T(m\circ\varphi)+(\nabla f)^T(\rho\circ\varphi)\\ (\rho\circ\varphi)\end{pmatrix}\lvert\nabla\varphi\rvert$
spatial velocity (Ad)	$\hat{\omega}=\mathrm{Ad}_R\hat{\Omega}$	$(u,\rho)=\mathrm{Ad}_{(\varphi,f)}(U_1,U_2)$
body momentum (Ad*)	$\Pi=\mathrm{Ad}^*_R\pi$	$(m_0,\rho_0)=\mathrm{Ad}^*_{(\varphi,f)}(m,\rho)$
momentum conservation	$\pi=$ const.	$(m_0,\rho_0)=$ const.

Without giving full proofs, we will state some theorems which can be deduced similarly to what we have developed in Section 3 for the one-component μ-b-equation.

Proposition 14. *Let $G=\mathcal{D}^s(\mathbb{T})\circledS H^{s-1}$, for $s>3$, let $\mathfrak{X}(G)$ denote the space of smooth vector fields on G,*

(34)
$$(\bar{\nabla}_X Y)(\varphi,f)=DY(\varphi,f)\cdot X(\varphi,f)-\Gamma_{(\varphi,f)}(X(\varphi,f),Y(\varphi,f)),\quad \forall X,Y\in\mathfrak{X}(G),$$

and let $\langle\cdot,\cdot\rangle$ be as in (32)*. Then:*

1. (1) *$\bar{\nabla}\colon\mathfrak{X}(G)\times\mathfrak{X}(G)\to\mathfrak{X}(G)$ is a smooth torsion-free affine connection on G, i.e.,*
   - (i) $\bar{\nabla}_{fX+gY}Z=f\bar{\nabla}_X Z+g\bar{\nabla}_Y Z$,
   - (ii) $\bar{\nabla}_X(Y+Z)=\bar{\nabla}_X Y+\bar{\nabla}_X Z$,
   - (iii) $\bar{\nabla}_X(fY)=f\bar{\nabla}_X Y+X(f)Y$,
   - (iv) $\bar{\nabla}_X Y-\bar{\nabla}_Y X=[X,Y]$,

   for all $X,Y,Z\in\mathfrak{X}(G)$ and all $f,g\in C^\infty(G;\mathbb{R})$, and the map $(\varphi,f)\mapsto((\varphi,f),(\bar{\nabla}_X Y)(\varphi,f))$, $G\to TG$ is smooth for any $X,Y\in\mathfrak{X}(G)$.
2. (2) *The right-invariant inner product $\langle\cdot,\cdot\rangle:\mathfrak{X}(G)\times\mathfrak{X}(G)\to\mathbb{R}$ is a weak Riemannian metric on G.*
3. (3) *The connection $\bar{\nabla}$ and the metric $\langle\cdot,\cdot\rangle$ are compatible in the sense that*
$$X\langle Y,Z\rangle=\left\langle\bar{\nabla}_X Y,Z\right\rangle+\left\langle\bar{\nabla}_X Z,Y\right\rangle,\quad\forall X,Y,Z\in\mathfrak{X}(G).$$

Proof. We only comment briefly on the compatibility condition; the assertions (1) and (2) follow very similarly from what we have already done. In view of the compatibility condition achieved in the proof of Proposition 3, the formula (21) and the definition of the $\mu-(2,2)$CH Christoffel map, it is enough to check that

$$-\left\langle\tilde{\Gamma}_1(u,v),\mathbb{A}w_1\right\rangle-\left\langle\tilde{\Gamma}_1(u,w),\mathbb{A}v_1\right\rangle-\langle\Gamma_2(u,v),w_2\rangle$$
$$-\langle\Gamma_2(u,w),v_2\rangle+\langle\nabla v_2\cdot u_1,w_2\rangle+\langle\nabla w_2\cdot u_1,v_2\rangle$$

is zero; here $\tilde{\Gamma}_1(u,v)=-\frac{1}{2}\mathbb{A}^{-1}[(\nabla u_2)^T v_2+(\nabla v_2)^T u_2]$ and Γ_2 is the second component of the Christoffel map (33). That this is indeed true follows easily from the integration by parts formula (24). $\square$

Again, we may draw the following conclusions.

Theorem 15. *Let Γ be the $\mu-(2,2)CH$ Christoffel map defined in* (33). *Fix* (u_0,ρ_0) *in some open neighborhood* $U \subset H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$ *containing* $(0,0)$*, for some* $s > 3$*. Then the initial value problem*

$$\left\{ \begin{array}{rcl} (\varphi_{tt}, f_{tt}) & = & \Gamma_{(\varphi,f)}((\varphi_t, f_t), (\varphi_t, f_t)), \\ (\varphi_t, f_t)(0) & = & (u_0, \rho_0), \\ (\varphi, f)(0) & = & (\mathrm{id}, 0) \end{array} \right. \tag{35}$$

for the geodesic flow corresponding to the $\mu-(2,2)CH$ *equation on* G *has a unique solution* $(\varphi, f) \in C^2([0,T);G)$*, for some* $T > 0$*, which depends smoothly on time and the initial value, i.e., the map* $(t, u_0, \rho_0) \mapsto (\varphi, f)$*,* $[0,T) \times U \to G$ *is smooth.*

Since G is a topological group, we may set $(u,\rho) = (\varphi_t, f_t) \circ \varphi^{-1}$ to obtain a solution of (7) with the desired regularity properties.

Corollary 16. *Fix* $s > 3$*. There exists an open neighborhood* $U \subset H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$ *of zero, such that for any* $(u_0,\rho_0) \in U$ *there is* $T > 0$ *and a unique solution*

$$(u,\rho) \in C([0,T); H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})) \cap C^1([0,T); H^{s-1}(\mathbb{T}) \times H^{s-2}(\mathbb{T}))$$

to the $\mu-(2,2)CH$ *equation* (7) *with continuous dependence on time and the initial value, i.e., the mapping*

$$(u_0,\rho_0) \mapsto (u,\rho),\ U \to C([0,T); H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})) \cap C^1([0,T); H^{s-1}(\mathbb{T}) \times H^{s-2}(\mathbb{T}))$$

is continuous.

We will conclude this section with some remarks on the curvature tensor of G associated with (7). With $\bar{\nabla}$ as in (34), we define the curvature tensor

$$R(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z,$$

and the unnormalized sectional curvature

$$S(X,Y) = \langle R(X,Y)Y, X \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the right-invariant metric (32). The steps of calculation in the proof of Proposition 7 can be carried out similarly for the $\mu-(2,2)$CH to obtain the curvature formula

$$\begin{aligned} S(u,v) = &\ \langle \Gamma(u,v), \Gamma(u,v) \rangle - \langle \Gamma(u,u), \Gamma(v,v) \rangle \\ &+ \langle \nabla u \cdot u_1, \Gamma(v,v) \rangle - \langle \nabla u \cdot v_1, \Gamma(v,u) \rangle \\ &+ \langle -\Gamma(\nabla v \cdot v_1, u) - \Gamma(\nabla u \cdot v_1, v) + 2\Gamma(\nabla v \cdot u_1, v), u \rangle, \end{aligned} \tag{36}$$

where $u = (u_1,u_2), v = (v_1,v_2) \in T_{(\mathrm{id},0)}G \simeq H^s(\mathbb{T}) \times H^{s-1}(\mathbb{T})$ and ∇u is the 4×2-matrix whose rows are ∇u_{11}, ∇u_{12}, ∇u_{21} and ∇u_{22}. Clearly, for $u_2 = v_2 = 0$, (36) reduces to the curvature formula for the $\mu-(2,1)$CH in Proposition 7. To prove that there are subspaces of positive curvature along the direction of the second factor of G, we evaluate S on

$$u = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \sin(kx) \end{pmatrix}, \quad k \in 2\pi\mathbb{N},$$

and see that $S(u,v) \equiv \frac{1}{8}$.

Remark 17. Under the obvious changes, the theory developed in the present section also applies to the $(2,2)$CH equation (with $\mathbb{A} = (1-\Delta)\mathrm{diag}(1,1)$) which has not been discussed under geometric aspects in the literature up to now. However, to have a good structure of the paper, we have decided to develop the geometric picture forthright for the μ-variant of $(2,2)$CH.

Remark 18. It would be nice to discuss two-dimensional (μ)-b-equations also for $b \neq 2$, but since in these cases no metric structure is available, it is not clear how to define, e.g., a $(\mu) - (2,2)$DP equation. In the one-dimensional case, there are integrability checks that suggested candidates for a $(1,2)$DP equation, [37]. It is an open problem and a task for further research to understand the integrability properties of the families discussed in the present paper and to obtain multi-component variants maybe starting from this point of view.

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