WEIGHTED ISOPERIMETRIC INEQUALITIES AND APPLICATIONS TO ELLIPTIC EQUATIONS

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ABSTRACT. In this paper we study some weighted isoperimetric inequalities relative to cones of $\mathbb{R}^N$. We give some information on the structure of those measures admitting as isoperimetric set the intersection of a cone with the ball centered at vertex of the cone. For instance we prove that when the cone reduces to the half-space $\mathbb{R}^N_+ = \left\{x \in \mathbb{R}^N : x_N > 0\right\}$ and the measure is factorized, this phenomenon can occur just to measures in the form $d\mu = a x_N^k \exp\left(c\,|x|^2\right) dx$ for some $a > 0$; $k, c \geq 0$. Then we establish an isoperimetric inequality for this class of the measures by proving that the optimal set is actually an half ball centered at the origin. Finally, via symmetrization arguments, we apply this last result to the study to a class of degenerate PDE's.

Key words: Relative isoperimetric inequality, weighted perimeter, degenerate elliptic equations.

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1. INTRODUCTION

This paper deals with weighted relative isoperimetric inequalities in cones of $\mathbb{R}^N$. Let ω be an open smooth subset of $\mathscr{S}^{N-1}$, the unit sphere of $\mathbb{R}^N$, and Ω the cone

$$\Omega = \left\{ x \in \mathbb{R}^N : \frac{x}{|x|} \in \omega \right\}. \tag{1.1}$$

We consider measures of the type $d\nu = f(x)dx$ on Ω, where f is a positive function defined in Ω. According to the usage, for any measurable set $G \subset \Omega$, we define the ν -measure of G

$$\nu(G) = \int_G d\nu = \int_G \phi(x)dx \tag{1.2}$$

and the ν-perimeter of G relative to Ω

$$P_\nu(G,\Omega) = \sup\left\{ \int_G \operatorname{div}\,(\mathbf{v}(x)\phi(x))\,dx :\, \mathbf{v} \in C^1(\Omega,\mathbb{R}^N),\ |\mathbf{v}| \leq 1 \right\}.$$

We also write $P_\nu(G,\mathbb{R}^N) = P_\nu(G)$. Notice that if G is a smooth set, then

$$P_\nu(G,\Omega) = \int_{\partial G \cap \Omega} \phi(x) d\mathscr{H}_{N-1}(x).$$

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The isoperimetric problem reads as

$$I_\nu(c) = \inf\{P_\nu(G,\Omega) : G \subset \Omega,\, \nu(G) = c\}, \quad c > 0. \tag{1.3}$$

One says that G is an isoperimetric set if $\nu(G) = c$ and $I_\nu(c) = P_\nu(G,\Omega)$.
About these questions we provide two type of results.

Firstly we give various necessary conditions on the function ϕ for having $B_R \cap \Omega$ as isoperimetric set for the measure $d\nu$. Here and throughout B_R and $B_R(x)$ denote the ball of radius R centered at zero and at x respectively. In Theorem 2.1 we prove that if $B_R \cap \Omega$ is an isoperimetric set then

$$\phi = A(r)B(\Theta),$$

where $r = |x|$ and $\Theta = \frac{x}{|x|}$.

In some specific cases we are able to give the explicit expression of the density of the measure ϕ. For instance in Theorem 2.2, when Ω is the half-space

$$\Omega = \mathbb{R}^N_+ = \left\{x = (x_1, ..., x_N) \in \mathbb{R}^N : x_N > 0\right\}, \tag{1.4}$$

we prove that if ϕ is a smooth factorized function satisfying

$$\phi(x) = \prod_{i=1}^N \phi_i(x) \tag{1.5}$$

and if $B_R \cap \mathbb{R}^N_+$ is an isoperimetric set, then

$$\phi(x) = a x_N^k \exp(c\,|x|^2), \tag{1.6}$$

for some numbers $a > 0$, $k \geq 0$ and $c \geq 0$.

In order to prove these results we derive, among other things, some optimal bounds for weighted Neumann eigenvalues on the sphere that may have an interest of their own (see Proposition 2.1).

At this point a natural problem to face would be to check if $B_R \cap \Omega$ is actually an isoperimetric set for measures of the type (1.6). In Section 3 we provide an affirmative answer to this question. More precisely the following result holds.

Theorem 1.1. *Let D be a measurable subset of $\mathbb{R}^N_+$ and μ the measure defined by*

$$d\mu = x_N^k \exp(c\,|x|^2)dx, \quad x \in \mathbb{R}^N_+, \tag{1.7}$$

with $k, c \geq 0$. Then

$$P_\mu(D) \geq P_\mu(D^\star)\,,$$

where $D^\star = B_{r^\star} \cap \mathbb{R}^N_+$, with $r^\star$ such that $\mu(D) = \mu(D^\star)$,

Our results are inserted in the wide bibliography related to the isoperimetric problems for "manifolds with density" (see, for instance, [7], [9], [10], [11], [13], [26], [27], [31], [32]). We just recall that in [21] it was shown that the isoperimetric set for measures of the type $y^k dxdy$, with $k \geq 0$ and $(x,y) \in \mathbb{R}^2_+$, is $B_R \cap \mathbb{R}^2_+$, while in [9] and [31] it is proved that the B_R are isoperimetric sets for measures of the type $\exp(c\,|x|^2)dx$ in $\mathbb{R}^N$ with $c \geq 0$.

The isoperimetric inequality given by Theorem 1.1 can be applied to the study of the following class of degenerate elliptic equations

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = x_N^k \exp\left(c\,|x|^2\right) f(x) & \text{in } D \\ \\ u = 0 & \text{on } \Gamma_+, \end{cases} \tag{1.8}$$

where D is a bounded domain of $\mathbb{R}^N_+$, whose boundary is decomposed in a part Γ_0, lying on the hyperplane $\{x_N = 0\}$ and a part Γ_+ contained in $\mathbb{R}^N_+$. Furthermore we assume that $c, k \geq 0$, $A(x) = (a_{ij}(x))_{ij}$ is an $N \times N$ symmetric matrix with measurable coefficients satisfying

$$x_N^k \exp(c\,|x|^2)\,|\zeta|^2 \leq a_{ij}(x)\zeta_i\zeta_j \leq \Lambda x_N^k \exp\left(c\,|x|^2\right)|\zeta|^2, \quad \Lambda \geq 1, \tag{1.9}$$

for almost every $x \in D$ and for all $\zeta \in \mathbb{R}^N$. Finally the datum f belongs to the weighted Lebesgue space $L^2(D, d\mu)$ where $d\mu$ is the measure defined in (1.7).

The type of degeneracy in (1.9) occurs, for $k \in \mathbb{N}$, when one looks for solutions to linear PDE's which are symmetric with respect to a group of $(k+1)$ variables (see, e.g., [8], [21], [33] and the references therein). The case of noninteger k has been object of investigation, for instance, in the generalized axially symmetric potential theory (see, e.g., [36] and the subsequent works of A. Weinstein).

Our aim is to provide optimal bounds for the solution to problem (1.8) by means of symmetrization techniques. These methods, through a procedure due to G. Talenti (see [34]), allow us to estimate the solution of the original problem in terms of the one to a problem characterized by some symmetry and hence simpler. One of the main tools required for such a procedure is the isoperimetric inequality with respect to a measure (1.7) which is related to the ellipticity of the differential operator (see, e.g., [1], [3], [5], [8], [21], [22], [30] and [34]).

Once known the isoperimetric set, we introduce a suitable rearrangement $f^\star$ of a measurable function $f : D \to \mathbb{R}$ with respect to the measure $d\mu$. The function $f^\star : D^\star \to [0, +\infty[$ is uniquely defined by the following condition on its level sets

$$\left\{x \in D^\star : f^\star(x) > t\right\} = \{x \in D : |f(x)| > t\}^\star \quad \forall t \geq 0.$$

This means that the level sets of $f^\star$ are half-spheres centered at the origin (therefore isoperimetric sets w.r.t. $d\mu$) having the same $\mu-$measure of the corresponding level sets of $|f|$. A few definitions and properties concerning the notion of rearrangement will be recalled in Section 4. For exhaustive treatment of this subject see [2], [12], [15], [17], [18] and [20].

Now we can state our pointwise comparison result which allow us to estimate the rearrangement $u^\star$ of the weak solution u to problem (1.8) with the weak solution w to problem (1.10) whose data are symmetric. A pointwise comparison result stated for a general measure can be found in [22].

Theorem 1.2. *Let u be the weak solution to problem (1.8). Let w be the function*

$$w(x)=w^\star(x)=\frac{1}{C_k}\int_{|x|}^{r^\star}\left(\int_0^\rho f^\star(\sigma)\sigma^{N-1+k}\exp\left(c\sigma^2\right)d\sigma\right)\rho^{-N+1-k}\exp\left(-c\rho^2\right)d\rho,$$

where C_k denotes the μ-measure of $B_1\cap\mathbb{R}^N_+$, which is the weak solution to the problem

$$\left\{\begin{array}{lll}-div\left(x_N^k\exp\left(c\,|x|^2\right)\nabla w\right)=x_N^k\exp\left(c\,|x|^2\right)f^\star & in & D^\star\\ \\ w=0 & on & \partial D^\star\cap\mathbb{R}^N_+.\end{array}\right.\tag{1.10}$$

Then

$$u^\star(x)\le w(x)\ \ a.e.\ \ in\ D^\star,\tag{1.11}$$

and

$$\int_D|\nabla u|^q\,d\mu\le\int_{D^\star}|\nabla w|^q\,d\mu,\ \ for\ all\ 0<q\le 2.\tag{1.12}$$

2. Weighted isoperimetric inequalities in a cone of $\mathbb{R}^N$

In this section we study isoperimetric problems w.r.t. measures, relative to cones in $\mathbb{R}^N$. Notice that such problems have been studied for instance in [2], [14], [19], [28] and [29]. Our aim here is to get information on the structure of those measures for which an isoperimetric set is given by the intersection of a cone with the ball having center at the vertex of the cone. We begin by fixing some notation that will be used throughout: ω_N is the N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$. For points $x\in\mathbb{R}^N$ we will often use N-dimensional polar coordinates (r,Θ), where $r=|x|$ and $\Theta=x|x|^{-1}\in\mathscr{S}^{N-1}$. By $\mathscr{S}^{N-1}_+$ we denote the half sphere,

$$\mathscr{S}^{N-1}_+=\mathscr{S}^{N-1}\cap\mathbb{R}^N_+.$$

Consider the isoperimetric problem (1.3) where Ω is the cone defined in (1.1) and ν the measure given by (1.2)

The first result of this section says that, if the isoperimetric set of (1.3) is $B_R\cap\Omega$ for a suitable R, then the density of the measure $d\nu$ is a product of two functions A and B of the variables r and Θ, respectively.

Theorem 2.1. *Consider problem (1.3), with $\phi\in C^1(\Omega)\cap C(\overline{\Omega})$, $\phi(x)>0$ for $x\in\Omega$. Suppose that $I_\nu(c)=P_\nu(B_R\cap\Omega)$ for $c=\nu(B_R\cap\Omega)$. Then*

$$\phi=A(r)B(\Theta),\tag{2.1}$$

where $A\in C^1((0,+\infty))\cap C([0,+\infty))$, $A(r)>0$ if $r>0$, and $B\in C^1(\omega)$, $B(\Theta)>0$ for $\Theta\in\omega$. Moreover, if $\phi\in C^2(\Omega)$, then

$$\lambda(B,\omega)\ge N-1+r^2\left[\frac{(A'(r))^2}{(A(r))^2}-\frac{A''(r)}{A(r)}\right]\quad\forall r>0\,,\tag{2.2}$$

where

$$\lambda(B,\omega):=\inf\left\{\frac{\int_\omega|\nabla_\Theta u|^2B\,d\Theta}{\int_\omega u^2B\,d\Theta}:\ u\in C^1(\omega),\ \int_\omega uB\,d\Theta=0,\ u\neq 0\right\}.\tag{2.3}$$

Here ∇_Θ denotes the gradient on the sphere.

Remark 2.1. Observe that $\lambda(B,\omega)$ is the first nontrivial eigenvalue of the Neumann problem

$$\begin{cases} -\nabla_\Theta\left(B\nabla_\Theta u\right) = \lambda B u & in \quad \omega \\ \\ \dfrac{\partial u}{\partial \nu} = 0 & on \quad \partial\omega \end{cases}$$

where $u \in W^{1,2}(\omega)$, and ν is the exterior unit normal to $\partial\omega$.

Proof of Theorem 2.1: Let $R > 0$. For $\varepsilon \in \mathbb{R}$ we define the following measure-preserving perturbations G_ε from $B_R \cap \Omega$:

$$G_\varepsilon := \{(r,\Theta) :\ 0 < r < R + \varepsilon h(\Theta) + s(\varepsilon),\ \Theta \in \omega\}, \quad |\varepsilon| \le \varepsilon_0$$

where $h \in C^1(\omega)$, and s is to be chosen such that $s \in C^2([-\varepsilon_0,\varepsilon_0])$, $s(0) = 0$, and $\nu(G_\varepsilon) = \nu(B_R)$ for $|\varepsilon| \le \varepsilon_0$. Writing $\phi = \phi(r,\Theta)$, and

$$R^\varepsilon := R + \varepsilon h + s(\varepsilon),$$

we have, for $|\varepsilon| \le \varepsilon_0$,

$$\nu(G_\varepsilon) = \int_\omega \int_0^{R^\varepsilon} r^{N-1}\phi(r,\Theta)\, dr\, d\Theta = \nu(B_R) \tag{2.4}$$

and

$$P_\nu(G_\varepsilon,\Omega) = \int_\omega (R^\varepsilon)^{N-2}\phi(R^\varepsilon,\Theta)\sqrt{(R^\varepsilon)^2 + |\nabla_\Theta R^\varepsilon|^2}\, d\Theta \ge P_\nu(B_R\cap\Omega,\Omega). \tag{2.5}$$

Denote $s_1 := s'(0)$ and $s_2 := s''(0)$. Equality (2.4) implies

$$0 = \int_\omega \phi(R,\Theta)(h(\Theta) + s_1)\, d\Theta, \tag{2.6}$$

and

$$0 = \int_\omega \left((N-1)\phi(R,\Theta) + R\phi_r(R,\Theta)\right)(h(\Theta) + s_1)^2\, d\Theta + s_2 R \int_\omega \phi(R,\Theta)\, d\Theta. \tag{2.7}$$

Using (2.5) we get

$$\begin{cases} (\partial P_\nu(G_\varepsilon,\Omega)/\partial\varepsilon)|_{\varepsilon=0} = 0 \\ \\ (\partial^2 P_\nu(G_\varepsilon,\Omega)/\partial\varepsilon^2)|_{\varepsilon=0} \ge 0. \end{cases} \tag{2.8}$$

The first condition in (2.8) gives

$$\int_\omega \left((N-1)\phi(R,\Theta) + R\phi_r(R,\Theta)\right)(h(\Theta) + s_1)\, d\Theta = 0. \tag{2.9}$$

In other words, (2.9) holds for all functions $h \in C^1(\omega)$ satisfying (2.6). Then the Fundamental Lemma in the Calculus of Variations tells us that there is a number $k(R) \in \mathbb{R}$ such that

$$\phi_r(R,\Theta) = k(R)\phi(R,\Theta) \quad \forall\Theta \in \omega. \tag{2.10}$$

Integrating this w.r.t. R implies (2.1). Hence (2.6) and (2.7) give

$$0 = \int_\omega B(\Theta)(h(\Theta)+s_1)\,d\Theta, \tag{2.11}$$

$$0 = \left\{\frac{N-1}{R}+\frac{A'(R)}{A(R)}\right\}\cdot\int_\omega B(\Theta)(h(\Theta)+s_1)^2\,d\Theta + s_2\int_\omega B(\Theta)\,d\Theta. \tag{2.12}$$

Next assume that $\phi \in C^2(\Omega)$. Then, using (2.1) and the second condition in (2.8) a short computation shows that

$$\begin{aligned} 0 \leq\ & \left\{(N-2)(N-1)R^{N-3}A(R)+2(N-1)R^{N-2}A'^{N-1}A''(R)\right\}\cdot \\ & \cdot\int_\omega B(\Theta)(h(\Theta)+s_1)^2\,d\Theta + \\ & s_2\left\{(N-1)R^{N-2}A(R)+R^{N-1}A'(R)\right\}\int_\omega B(\Theta)\,d\Theta + \\ & R^{N-3}A(R)\int_\omega B(\Theta)\,|\nabla_\Theta(h(\Theta)+s_1|^2\,d\Theta. \end{aligned}$$

Together with (2.12) this implies

$$\begin{aligned} 0 \leq\ & \left\{-(N-1)R^{N-3}A(R)-R^{N-1}\frac{A'^2(R)}{A(R)}+R^{N-1}A''(R)\right\}\cdot \\ & \cdot\int_\omega B(\Theta)(h(\Theta)+s_1)^2\,d\Theta + \\ & R^{N-3}A(R)\int_\omega B(\Theta)|\nabla_\Theta(h(\Theta)+s_1)|^2\,d\Theta. \end{aligned}$$

This implies (2.2), in view of (2.11), and the definition of $\lambda(B,\omega)$. □

Remark 3.2. The value of $\lambda(B,\omega)$ is explicitly known in some special cases. For instance, if $B\equiv 1$, and $\omega = \mathscr{S}^{N-1}$, we have

$$\lambda(1,\mathscr{S}^{N-1}) = N-1, \tag{2.13}$$

the eigenvalue has multiplicity N, with corresponding eigenfunctions $u_i(x)=x_i$, $(i=1,\dots,N)$, so that (2.2) reads as

$$A'^2 \leq A''(r)A(r), \tag{2.14}$$

or equivalently, A is log-convex, that is,

$$A(r) = e^{g(r)},$$

with a convex function g. It has been conjectured in [31], Conjecture 3.12, that for weights $\phi = A(r)$, with log-convex A, balls B_R, $(R>0)$, solve the isoperimetric problem in $\mathbb{R}^2$.
After finishing this paper, S. Howe kindly informed us that he has proven this conjecture, together with some extensions of it (see [16]).
It is interesting to note that Theorem 1.1, whose proof will be the object of the next section, and Theorem 2.1 imply the following result

Proposition 2.1. *Let $k \geq 0$, and*

$$B = B_k(\Theta) = \left(\frac{x_N}{|x|}\right)^k, \quad (x \in \mathbb{S}_+^{N-1}). \tag{2.15}$$

Then

$$\lambda(B_k, \mathscr{S}_+^{N-1}) = N - 1 + k, \tag{2.16}$$

with corresponding eigenfunctions

$$u_i = x_i, \quad (i = 1, \dots, N-1). \tag{2.17}$$

Proof: Theorem 1.1, whose proof will be the object of the next section, and Theorem 2.1 imply that (2.2) holds, with $\omega = \mathscr{S}_+^{N-1}$, $A(r) = r^k e^{cr^2}$, $(c \geq 0)$, and $B(\Theta) = B_k(\Theta)$. Hence $\lambda(B_k, \mathscr{S}_+^{N-1}) \geq N - 1 + k - 2cr^2$ for all $r > 0$, which implies that $\lambda(B_k, \mathscr{S}_+^{N-1}) \geq N - 1 + k$. The assertion follows from the identities $u_i = x_i$, $(i = 1, \dots, N-1)$ satisfy

$$\int_{\mathscr{S}_+^{N-1}} |\nabla_\Theta u_i|^2 B_k \, d\Theta = \int_{\mathscr{S}_+^{N-1}} (u_i)^2 B_k \, d\Theta,$$

$$\int_{\mathscr{S}_+^{N-1}} u_i B_k \, d\Theta = 0, \quad (i = 1, \dots, N-1). \qquad \square$$

We add two isoperimetric results that are also interesting in their own right. The first one will be used in the proof of Theorem 2.2 below.

Lemma 2.1. *Let $B \in C^2(\mathscr{S}^{N-1})$, $B(\Theta) > 0$ on $\mathscr{S}^{N-1}$. Then*

$$\lambda(B, \mathscr{S}^{N-1}) \leq \lambda(1, \mathscr{S}^{N-1}) = N - 1, \tag{2.18}$$

with equality only if $B = const.$

Proof: Let $v(\Theta)$ be any eigenfunction for the problem with $B \equiv 1$, and set $v^\xi(\Theta) := v(\xi + \Theta)$, and $u^\xi(\Theta) := v(\xi + \Theta)/B(\Theta)$, $(\xi \in \mathscr{S}^{N-1})$. Then $\int_{\mathscr{S}^{N-1}} u^\xi B \, d\Theta = 0$, and using integration by parts and the fact that $-\Delta_\Theta v^\xi = (N-1)v^\xi$, ($\Delta_\Theta$ denotes the Laplace-Beltrami operator on $\mathscr{S}^{N-1}$), we have,

$$\begin{aligned}
I(\xi) := \quad & \int_{\mathscr{S}^{N-1}} B \left(|\nabla_\Theta u^\xi|^2 - (N-1)[u^\xi]^2 \right) d\Theta \\
= \quad & \int_{\mathscr{S}^{N-1}} \left(\frac{|\nabla_\Theta v^\xi|^2}{B} - \frac{2v^\xi \nabla_\Theta v^\xi \cdot \nabla_\Theta B}{B^2} + \frac{(v^\xi)^2 |\nabla_\Theta B|^2}{B^3} - \frac{(N-1)(v^\xi)^2}{B} \right) d\Theta \\
= \quad & \int_{\mathscr{S}^{N-1}} \left(-\frac{v^\xi \Delta_\Theta v^\xi}{B} - \frac{v^\xi \nabla_\Theta v^\xi \cdot \nabla_\Theta B}{B^2} + \frac{(v^\xi)^2 |\nabla_\Theta B|^2}{B^3} - \frac{(N-1)(v^\xi)^2}{B} \right) d\Theta \\
= \quad & \int_{\mathscr{S}^{N-1}} \left(-\frac{v^\xi \nabla_\Theta v^\xi \cdot \nabla_\Theta B}{B^2} + \frac{(v^\xi)^2 |\nabla_\Theta B|^2}{B^3} \right) d\Theta \\
= \quad & \int_{\mathscr{S}^{N-1}} \left(\frac{(v^\xi)^2}{2} \nabla_\Theta \cdot \left(\frac{\nabla_\Theta B}{B^2} \right) + \frac{(v^\xi)^2 |\nabla_\Theta B|^2}{B^3} \right) d\Theta \\
= \quad & (1/2) \int_{\mathscr{S}^{N-1}} \frac{(v^\xi)^2 \Delta_\Theta B}{B^2} \, d\Theta. \qquad (2.19)
\end{aligned}$$

Setting

$$c := \int_{\mathscr{S}^{N-1}} v^2 \, d\Theta,$$

an integration of (2.19) gives

$$\int_{\mathscr{S}^{N-1}} I(\xi)\, d\xi = \frac{c}{2} \int_{\mathscr{S}^{N-1}} \frac{\Delta_\Theta B}{B^2}\, d\Theta = -c \int_{\mathscr{S}^{N-1}} \frac{|\nabla_\Theta B|^2}{B^3}\, d\Theta \ \leq 0,$$

with equality if and only if $B(\Theta) = \text{const}$. Hence there is a $\xi_0 \in \mathscr{S}^{N-1}$ such that $I(\xi_0) \leq 0$, with equality if and only if $B(\Theta) = \text{const}$. □

Theorem 2.1 has some further consequences when the cone Ω contains the wedge

$$W_+ := \{x = (x_1, \dots, x_N) : \, x_i > 0, \, i = 1, \dots, N\},$$

and if

$$\phi(x) = \prod_{i=1}^{N} \phi_i(x_i), \tag{2.20}$$

for some smooth functions ϕ_i, $(i = 1, \dots, N)$.
In the following, let

$$\omega_+ := W_+ \cap \mathscr{S}^{N-1}.$$

We first show

Lemma 2.2. *Let $\phi \in C^2(W_+)$, and suppose there are functions $A, \phi_i \in C^2((0,+\infty)) \cap C([0,+\infty))$, $B \in C^2(\omega_+) \cap C(\overline{\omega_+})$, such that $\phi_i(x_i) > 0$ for $x_i > 0$, $(i = 1, \dots, N)$, $A(r) > 0$ for $r > 0$, and $B(\Theta) > 0$ for $\Theta \in \omega_+$. Then*

$$\phi(x) = a \prod_{i=1}^{N} x_i^{k_i} e^{c|x|^2}, \quad x \in W_+, \tag{2.21}$$

where $a > 0$, $k_i \geq 0$, $(i = 1, \dots, N)$, and $c \in R$.

Proof : Differentiating the equation $A(r)B(\Theta) = \prod_{i=1}^{N} \phi_i(x_i)$ w.r.t. r gives

$$\frac{rA'(r)}{A(r)} = \sum_{i=1}^{N} \frac{x_i \phi_i'(x_i)}{\phi_i(x_i)}.$$

Differentiating once more w.r.t. x_i yields

$$\frac{A'(r)}{rA(r)} + \frac{A''(r)}{A(r)} - \frac{A'^2(r)}{A^2(r)} = \frac{\phi_i'(x_i)}{x_i \phi_i(x)} + \frac{\phi_i''(x_i)}{\phi_i(x_i)} - \frac{(\phi_i'(x_i))^2}{(\phi_i(x_i))^2} = 4c, \quad (i = 1, \dots, N),$$

for some number $c \in \mathbb{R}$. An integration of these equalities shows that

$$\phi_i(x_i) = a_i x_i^{k_i} e^{c x_i^2}$$

for some numbers $a_i > 0$, $k_i \geq 0$, and (2.21) follows.

As pointed out in the Introduction, we can specify the expression of the density ϕ of the measure, when the cone Ω is $\mathbb{R}^N_+$ and ϕ is factorized.

Theorem 2.2. *Assume $\Omega = \mathbb{R}^N_+$ and consider problem (1.3), where $\phi \in C^1(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$, and satisfies (2.20), for some functions $\phi_i \in C^2(\mathbb{R})$, $\phi_i(t) > 0$ for $t \in \mathbb{R}$, $(i = 1, \dots, N-1)$, and $\phi_N \in C^2((0,+\infty)) \cap C([0,\infty))$, $\phi_N(t) > 0$ for $t > 0$. Suppose that $I_\nu(c) = P_\nu(B_R \cap \mathbb{R}^N_+, \mathbb{R}^N_+)$ for $c = \nu(B_R \cap \mathbb{R}^N_+)$. Then*

$$\phi(x) = a x_N^k e^{c|x|^2}, \tag{2.22}$$

for some numbers $a > 0$, $k \geq 0$ and $c \geq 0$.

Proof : By Theorem 2.1 we have $\phi = A(r)B(\Theta)$ with smooth positive functions A and B, and

$$\lambda(B, \mathscr{S}^{N-1}_+) \geq N - 1 + r^2 \left[\frac{(A')^2}{A(r)^2} - \frac{A''(r)}{A(r)} \right] \quad \forall r > 0. \tag{2.23}$$

By Lemma 2.2, it follows (2.22) for some numbers $a > 0$, $k \geq 0$ and $c \in \mathbb{R}$. Hence, $B(\Theta) = [x_N|x|^{-1}]^k$, and $A(r) = ar^k e^{cr^2}$. Therefore (2.23) and 2.16 imply that

$$N - 1 + k \geq N - 1 + k - 2cr^2 \quad \forall r > 0.$$

Hence we must have $c \geq 0$. □

We end this section by analyzing the case where the cone Ω is $\mathbb{R}^N \setminus \{0\}$.

Theorem 2.3. *Assume $\Omega = \mathbb{R}^N \setminus \{0\}$ and consider problem (1.3), with $\phi \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N)$, $\phi(x) > 0$ for $x \neq 0$, and satisfies (2.20), where $\phi_i \in C^2(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$, and $\phi_i(t) > 0$ for $t \neq 0$, $(i = 1, \dots, N)$. Suppose that $I_\nu(c) = P_\nu(B_R)$ for $c = \nu(B_R)$. Then*

$$\phi(x) = a e^{c|x|^2}, \tag{2.24}$$

for some numbers $a > 0$, and $c \geq 0$.

Proof : By Theorem 2.1 we have $\phi = A(r)B(\Theta)$ with smooth positive functions A and B, and

$$\lambda(B, \mathscr{S}^{N-1}) \geq N - 1 + r^2 \left[\frac{(A')^2}{A(r)^2} - \frac{A''(r)}{A(r)} \right] \quad \forall r > 0. \tag{2.25}$$

By Lemma 2.2, we obtain (2.24) with some numbers $a > 0$, and $c \in \mathbb{R}$, that is, $B(\Theta) \equiv 1$ and $A(r) = ae^{cr^2}$. Hence, (2.25) and (2.13) imply that A is log-convex, that is, we must have $c \geq 0$. □

3. A Dido's problem

In this section we provide the proof of Theorem 1.1. As pointed out in the Introduction, we have to find the set having minimum $\mu-$perimeter among all the subsets of $\mathbb{R}^N_+$ having prescribed $\mu-$measure, where μ is the measure defined in (1.7). In order to face such a problem we firstly show a simple inequality for measures defined on the real line related to $d\mu$. Then the isoperimetric problem is addressed in the plane: the one-dimensional results allow to restrict the search of the optimal sets among the ones which are starlike with respect to the origin. Finally Theorem 1.1 is achieved in its full generality.

3.1. **Dido's problem on the real line.** Let $\mathbb{R}_+ = (0, +\infty)$. The following isoperimetric inequality holds.

Proposition 3.1. *Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing continuous function, $d\nu = \phi(x)dx$ and M be a measurable subset of $\mathbb{R}_+$. Then*

$$P_\nu(M) \geq P_\nu(D(M)), \tag{3.1}$$

where $D(M)$ denotes the interval $(0, d)$, with $d \geq 0$ chosen such that $\nu(M) = \nu(D(M))$.

Proof : By standard approximation arguments, it is sufficient to consider sets M in the form

$$M = \cup_{j=1}^{k} (a_j, b_j),$$

with

$$0 \leq a_j < a_{j+1}, \quad a_j < b_j, \quad b_j < b_{j+1} < +\infty,$$

for all $j \in \{1, ..., k-1\}$.

By the properties of the weight function φ we have that $b_k \geq d$ and hence

$$P_\nu(M) = \sum_{j=1}^{k} [\varphi(a_j) + \varphi(b_j)] \geq \varphi(0) + \varphi(d) = P_\nu(D(M)).$$

□

3.2. **Dido's problem in two dimensions.** In our study of the measure $d\mu$, an important role will be played by the following isoperimetric theorem (see [9] and [31]) relative to the measure

$$d\tau = \exp(c\,|x|^2)dx, \quad x \in \mathbb{R}^m, \text{ with } m \geq 1 \text{ and } c \geq 0.$$

Theorem 3.1. *If G is any measurable subset of $\mathbb{R}^m$ and $G^{\bigstar,\tau}$ is the ball of $\mathbb{R}^m$ centered at the origin having the same $\tau-$measure of G, then*

$$P_\tau(G) \geq P_\tau(G^{\bigstar,\tau}). \tag{3.2}$$

We write (x, y) for points in $\mathbb{R}^2$. We consider in $\mathbb{R}^2_+$ the measure

$$d\mu = y^k \exp\left(c(x^2 + y^2)\right)\, dxdy,$$

where $c \geq 0$ and $k \geq 0$. If M is a measurable subset of $\mathbb{R}^2_+$, given any number $m > 0$, the isoperimetric problem on $\mathbb{R}^2_+$ reads as:

$$I_\mu(m) := \inf\{P_\mu(M), \text{ with } M : \mu(M) = m\}. \tag{3.3}$$

The following result holds true.

Theorem 3.2. *Let $m > 0$. Then $I_\mu(m)$ is attained for the half circle $B_r \cap \mathbb{R}^2_+$, centered at zero, having $\mu-$measure m. Equivalently there exists $r > 0$ such that*

$$I_\mu(m) = P_\mu(B_r \cap \mathbb{R}^2_+) = \exp\left(cr^2\right) r^{k+1} \int_0^\pi \sin^k \theta d\theta. \tag{3.4}$$

Proof Step 1: An equivalent isoperimetric problem

Let $\{T_n\}_n \subset \mathbb{R}^2_+$ be a minimizing sequence for problem (3.3), i.e.

$$\mu(T_n) = m \ \forall n \in \mathbb{N} \ \text{ and } \lim_{n\to+\infty} P_\mu(T_n) = I_\mu(m),$$

where, without loss of generality, we may assume that the sets T_n are smooth.

Let T be a smooth set of $\mathbb{R}^2_+$. We denote by $S(T)$ and $D(T)$ the Steiner symmetrization in x-direction, with respect to the measure $d\mu_x = \exp\left(cx^2\right) dx$, of T and the Steiner symmetrization in y-direction, with respect to the measure $d\mu_y = e^{cy^2} y^k\, dy$, of T, respectively.

By this we mean that $S(T)$ is the set in $\mathbb{R}^2_+$ whose cross sections parallel to the x-axis are open intervals centered at the $y-$axis whose μ_x-lengths are equal to those of the corresponding cross sections of T.

The set $D(T)$ is defined in a similar way: its intersection with the cross sections parallel to the y-axis are open intervals with an endpoint lying on the x-axis whose μ_y- lengths are equal to those of the corresponding cross sections of M_n.

Now consider the sequence of sets $M_n = D(S(T_n))$. By Proposition 3.1 and Theorem 3.1 we have that $P_\mu(D(S(T_n))) \le P_\mu(T_n)$ and by Cavalieri's principle $\mu(D(S(T_n))) = \mu(T_n)$. Therefore M_n is still a minimizing sequence for (3.3). Under the effect of the symmetrization, the sets M_n can lose regularity on one hand, in general they are not more then locally Lipschitz continuous, on the other they acquire some nice geometrical properties. Firstly they are all starlike with respect to the origin. So, introducing polar coordinates (r,θ) by $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$M_n = \{(r,\theta) : 0 \le r < \rho_n(\theta),\, \theta \in (0,\pi)\},\ \forall n \in \mathbb{N}, \tag{3.5}$$

for some functions $\rho_n(\theta) : (0,\pi) \to (0,+\infty)$.

Moreover it holds that

(i) the functions $\rho_n(\theta)$ are locally Lipschitz in $(0,\pi)$,

(ii) $\rho_n(\theta) = \rho_n(\pi-\theta), \forall n \in \mathbb{N}, \forall\theta \in (0,\pi)$,

(iii) $x_n(\theta) = \dfrac{\rho_n(\theta)}{\cos\theta}$ and $y_n(\theta) = \dfrac{\rho_n(\theta)}{\sin\theta}$ are nonincreasing and nondecreasing, respectively, in $(0,\pi/2)$.

From these considerations, in view of finding the infimum in (3.3), we may assume that the minimizing sequence is in the form (3.5) with conditions (i)-(iii) in force. Clearly, under these conditions, the set M_n, its μ-measure and μ-perimeter are uniquely determined by the function $\rho_n(\theta)$. Indeed, setting $z := \sin^k\theta$, $\theta \in [0,\pi]$,

$$F(r) := \int_0^r \exp\left(ct^2\right) t^{k+1}\, dt$$

and

$$G(r,p) := \exp\left(cr^2\right) r^k \sqrt{r^2+p^2}, \quad (r>0,\ p \in \mathbb{R}),$$

we find that

$$\mu(M_n) = \int_0^\pi F(\rho_n) z\, d\theta =: \mu(\rho_n),$$

and

$$P_\mu(M_n) = \int_0^\pi G(\rho_n, \rho_n')z\,d\theta =: P_\mu(\rho_n).$$

With this notations, the isoperimetric problem (3.3) then reads as

$$\text{Minimize } P_\mu(\rho) \text{ over } K = \{\rho : (0,\pi) \to (0,+\infty) : \rho \text{ satisfies (i)-(iii) and } \mu(\rho) = m\}. \tag{3.6}$$

Step 2: The minimum is achieved

Let $\{\rho_n\}_n$ be a minimizing sequence for problem (3.6). From condition (iii) we easily get that

$$-\rho_n(\theta)\cot\theta \le \rho_n'(\theta) \le \rho_n(\theta)\tan\theta \quad \text{a.e. on } (0,\pi/2),\ n \in \mathbb{N}. \tag{3.7}$$

Set

$$y_n^0 := \sup_{\theta\in(0,\pi/2)} y_n(\theta) = y_n(\pi/2).$$

We claim that

$$\sup_{n\in\mathbb{N}} y_n^0 := y^0 < +\infty. \tag{3.8}$$

We have for any fixed $n \in \mathbb{N}$

$$\begin{aligned} P_\mu(\rho_n) &= 2\int_0^{\pi/2} \exp\left[c\left(x_n^2(\theta) + y_n^2(\theta)\right)\right] y_n^k(\theta)\sqrt{x_n'^2(\theta) + y_n'^2(\theta)}d\theta \\ &\ge 2\int_0^{\pi/2} \exp\left[cy_n^2(\theta)\right] y_n^k(\theta) y_n'(\theta) d\theta = 2\int_0^{y_n^0} \exp\left(ct^2\right) t^k dt. \end{aligned}$$

We get (3.8) observing that $\{P_\mu(\rho_n)\}_n$ is a bounded sequence.

From (3.7) and (3.8) we deduce that for any θ in $(0,\pi/2)$ it holds

$$\rho_n(\theta) = \frac{y_n(\theta)}{\sin\theta} \le \frac{y_n(\pi/2)}{\sin\theta} \le \frac{y^0}{\sin\theta} \quad \forall n \in \mathbb{N}. \tag{3.9}$$

Using (3.9) and again (3.7), for any fixed $\delta \in (0,\pi/2)$ we get

$$\sup_{\theta\in(\delta,\pi/2-\delta)} \left\{\rho_n(\theta), \left|\rho_n'(\theta)\right|\right\} \le C_\delta, \tag{3.10}$$

where C_δ is a constant depending only on δ.

Hence there exists a function ρ locally Lipschitz continuous on $(0,\pi/2)$ and a, not relabeled, subsequence of ρ_n, such that

$$\rho_n \to \rho \text{ uniformly on compact subsets of } (0,\pi/2), \tag{3.11}$$

$$\rho(\theta) = \rho(\pi-\theta)\ \forall\theta \in (0,\pi/2), \tag{3.12}$$

$$-\rho(\theta)\cot\theta \le \rho'(\theta) \le \rho(\theta)\tan\theta \text{ a.e. on } (0,\pi/2). \tag{3.13}$$

Now we want to show that the limit set $M = \{(r,\theta) : 0 \le r < \rho(\theta),\ \theta \in (0,\pi)\}$ satisfies

$$\mu(\rho) = m, \text{ and } P_\mu(\rho) = I_\mu(m), \tag{3.14}$$

or in other words that the infimum in (3.6) is achieved on M.

In the next subsection we will conclude the proof of the Theorem by showing that M is an half-circle centered at the origin.

We claim that, for any $\theta \in (0, \pi/2)$, the following uniform estimate holds

$$\phi_n(\theta) \equiv \exp\left(c y_n(\theta)^2\right) y_n(\theta)^k \int_0^{x_n(\theta)} \exp\left(ct^2\right) dt \le C \tag{3.15}$$

To this purpose we consider the set

$$\widetilde{M}_n = \{(x, y) \in M_n : y \le y_n(\theta)\}.$$

It is easy to verify that

$$\begin{aligned} & \frac{1}{2}\left(P_\mu(M_n) - P_\mu(\widetilde{M}_n)\right) \\ = & \int_\theta^{\pi/2} \exp\left(c\left(x_n^2(\theta) + y_n^2(\theta)\right)\right) y_n(\theta)^k \sqrt{x_n'^2(\theta) + y_n'^2(\theta)} d\theta - \int_0^{x_n(\theta)} \exp\left(c\left(t^2 + y_n^2(\theta)\right)\right) y_n^k(\theta) dt \\ \ge & \int_\theta^{\pi/2} \exp\left(c\left(x_n^2(\theta) + y_n^2(\theta)\right)\right) y_n(\theta)^k \left(-x_n'(\theta)\right) d\theta - \int_0^{x_n(\theta)} \exp\left(c\left(t^2 + y_n^2(\theta)\right)\right) y_n^k(\theta) dt \\ = & \; 0. \end{aligned}$$

Therefore

$$C \ge P_\mu(M_n) \ge P_\mu(\widetilde{M}_n) \ge 2 y_n^k(\theta) \exp\left(c y_n^2(\theta)\right) \int_0^{x_n(\theta)} \exp\left(ct^2\right) dt.$$

Now (3.8) and (3.15) allow to deduce

$$m = \lim_{n\to+\infty} \mu(\rho_n) = \mu(\lim_{n\to+\infty} \rho_n) = \mu(\rho). \tag{3.16}$$

The detailed proof of (3.16), in spite of its elementary character, is a bit long. Therefore we prefer to postpone it in the Appendix.

On the other hand the lower semicontinuity of the perimeter gives

$$I_\mu(m) = \lim_{n\to+\infty} P_\mu(\rho_n) \ge P_\mu(\rho),$$

but $\rho \in K$ therefore $I_\mu(m) = P_\mu(\rho)$.

Step 3: The optimal set is a half-circle

Standard calculus of variations then shows that there is a number $\gamma \in \mathbb{R}$ - a Lagrangian multiplier - such that (see also [21])

$$\lim_{\theta\to 0} G_p \sin^k \theta = \lim_{\theta\to\pi} G_p \sin^k \theta = 0, \tag{3.17}$$

$$z G_r - \frac{d}{d\theta}\left(z G_p\right) = \gamma z F' \quad \text{in } \mathcal{D}'(0, \pi). \tag{3.18}$$

Here and in the following, values of F, G and their derivatives are taken at ρ and (ρ, ρ'), respectively.

We note that the fact that $x(\theta) = \frac{\rho(\theta)}{\cos\theta}$ and $y(\theta) = \frac{\rho(\theta)}{\sin\theta}$ are nonincreasing and nondecreasing, respectively, in $(0, \pi)$ easily implies that

$$\rho(\theta) \ge C > 0 \text{ in } (0, \pi). \tag{3.19}$$

From (3.11), (3.19) and the equation (3.18) we easily deduce that ρ is smooth on $(0, \pi/2)$. Integrating equation (3.18) over $(\pi - t, t)$, (with $t \in (0, \pi/2)$), gives

$$2G_p(\rho(t), \rho'(t)) \sin^k t = \int_{\pi - t}^{t} \left(G_r - \gamma F'\right) z d\theta$$

Passing to the limit $t \nearrow (\pi/2)$ this shows that $\rho'(\pi/2) = 0$. This means that ρ is smooth on the whole interval $(0, \pi)$.

Now we want to show that ρ is bounded in $(0, \pi)$. By the previous considerations it suffices to examine the behavior of ρ at zero. To this aim we integrate equation (3.18) over (ε, t), where $0 < \varepsilon < t < (\pi/2)$, obtaining

$$G_p(\rho(t), \rho'(t)) \sin^k t - G_p(\rho(\varepsilon), \rho'(\varepsilon)) \sin^k \varepsilon = \int_{\varepsilon}^{t} \left(G_r - \gamma F'\right) z \, d\theta.$$

Passing to the limit $\varepsilon \searrow 0$ and taking into account (3.17) this means that

$$G_p(\rho(t), \rho'(t)) \sin^k t = \int_0^t \left(G_r - \gamma F'\right) z \, d\theta. \tag{3.20}$$

Assume that ρ is unbounded at zero. Then there exists a sequence $t_n \searrow 0$ such that $\rho(t_n) \to +\infty$ and $\rho'(t_n) \leq 0$, $n \in \mathbb{N}$. From (3.20) we then have that

$$0 \geq \int_0^{t_n} \left(-\gamma\rho + \left(2c\rho + \frac{k+1}{\rho}\right)\sqrt{\rho^2 + \rho'^2} + \frac{\rho}{\sqrt{\rho^2 + \rho'^2}}\right) \rho^k \exp\left(c\rho^2\right) z \, d\theta, \ n \in \mathbb{N}. \tag{3.21}$$

On the other hand, (3.13) tells us that $\rho(t_n) \cos t_n \leq \rho(\theta) \cos\theta$ for $\theta \in (0, t_n)$. Hence the integrand in (3.21) is positive for large n. This gives a contradiction and the claim is proved.

Having proved that ρ is bounded, the identity (3.20) shows that

$$\frac{\rho'(t)}{t} \text{ is bounded on } (0, \pi), \text{ and } \rho'(0) = 0. \tag{3.22}$$

Finally, using the equation (3.18), we find that

$$\rho'' \text{ is bounded on } (0, \pi).$$

Now, since ρ is a minimizer of (3.6), the second variation of P_μ at ρ in K is nonnegative. This means that

$$0 \leq \int_0^{\pi} \left(G_{rr}\kappa^2 + 2G_{rp}\kappa\kappa' + G_{pp}\kappa^{'2} - \gamma F^{''}\kappa^2\right) z \, d\theta, \tag{3.23}$$

for any $k \in C^1([0, \pi])$ such that

$$\int_0^{\pi} F'\kappa z \, d\theta = 0, \tag{3.24}$$

and, by density, for any $\kappa \in W^{1,\infty}(0, \pi)$.

Observe that, by the symmetry property (3.12), the function ρ', which has been shown to belong to $W^{1,\infty}(0, \pi)$, satisfies (3.24).

On the other hand, dividing (3.18) by z and then differentiating yields

$$G_{rr}\rho' + G_{rp}\rho'' - \frac{d}{d\theta}\left(G_{rp}\rho' + G_{pp}\rho''\right) - \left(G_{pr}\rho' + G_{pp}\rho''\right)\frac{z'}{z} - G_p\left(\frac{z'}{z}\right)' = \gamma F''\rho' \quad \text{in } (0,\pi). \tag{3.25}$$

Multiplying (3.25) by $\rho' z$ and then integrating by parts we obtain

$$\int_0^\pi G_p\rho'\left(\frac{z'}{z}\right)' d\theta = \int_0^\pi \left(G_{rr}\rho'^2 + 2G_{rp}\rho'\rho'' + G_{pp}\rho''^2 - \gamma F''\rho'^2 z\, d\theta\right) z d\theta. \tag{3.26}$$

Together with (3.23), with $\kappa = \rho'$, this shows that

$$0 \le \int_0^\pi G_p\rho'\left(\frac{z'}{z}\right)' z\, d\theta = -k\pi \int_0^\pi \exp\left(c\rho^2\right)\rho^k\left(\rho^2 + {\rho'}^2\right)^{-1/2} {\rho'}^2 \sin^{k-2}\theta\, d\theta. \tag{3.27}$$

This implies that $\rho' = 0$ in $[0,\pi]$, hence ρ is constant in $[0,\pi]$, and the result follows. □

3.3. **The N-dimensional case. Proof of Theorem 1.1.** Let $(x', x_N) = (x_1, .., x_{N-1}, x_N)$ denote a point in $\mathbb{R}^N$ and let D be a smooth subset of $\mathbb{R}^N_+$ having finite μ-measure.

For any fixed $t > 0$ we consider the $(N-1)-$dimensional slices D_t of D parallel to the hyperplane $\{x_N = 0\}$

$$D_t = \left\{x' \in \mathbb{R}^{N-1} : (x', t) \in D\right\}.$$

By transforming each D_t into $D_t^{\star,\tau}$ the set D becomes a set, D^c, with cylindrical symmetry. At this point we recall (see for instance Theorem 4.2 of [4]) that the isoperimetric property of the slices carries over the whole set, therefore

$$P_\mu(D) \ge P_\mu(D^c).$$

Now we consider the set T of $\mathbb{R}^2_+$ obtained by intersecting D^c with any $(N-1)-$dimensional hyperplane containing the x_N-axis. The measure μ and the set D^c are both rotational invariant with respect to the first $(N-1)$ coordinates. That allows us to apply Theorem 3.2 to the set T obtaining an half-circle, whose complete rotation about the x_N-axis provides the set $D^\star$. By the consideration above we have that

$$P_\mu(D^c) \ge P_\mu(D^\star),$$

i.e. the claim.

4. Application to a class of degenerate elliptic equations

4.1. **Notation and preliminary results.** We introduce the notion of weighted rearrangement with respect to the measure

$$d\mu = x_N^k \exp(c\,|x|^2)dx,$$

with $c, k \ge 0$. Let D be a measurable subset of $\mathbb{R}^N_+$. The distribution function of $u : D \to \mathbb{R}$, with respect to $d\mu$, is the function m_μ defined by

$$m_\mu(t) = \mu\left(\{x \in D : |u(x)| > t\}\right),\ \forall t \ge 0.$$

The decreasing rearrangement of u is the function u^* defined by

$$u^*(s) = \inf\left\{t \ge 0 : m_\mu(t) \le s\right\},\ \forall s \in (0, \mu(D)].$$

Let C_k be the μ-measure of $B_1 \cap \mathbb{R}^N_+$ and $\psi(r)$ be the function defined by

$$\psi(r) = \int_0^r \exp\left(ct^2\right) t^{N+k-1} dt.$$

We recall that $D^\bigstar$ is the half-sphere of $\mathbb{R}^N$ centered at the origin having the same $\mu-$measure as D. Or equivalently

$$D^\bigstar = B_{r^\bigstar} \cap \mathbb{R}^N_+, \tag{4.1}$$

where

$$r^\bigstar = \psi^{-1}\left(\frac{\mu(D)}{C_k}\right). \tag{4.2}$$

The rearrangement $u^\bigstar$ of u is the following function

$$u^\bigstar(x) = u^*\left(C_k \psi\left(|x|\right)\right), \ \forall x \in D^\bigstar.$$

The isoperimetric inequality we have proved in the previous section can be also stated as follows

$$P_\mu(D) \geq I_\mu(\mu(D)),$$

where $I_\mu(\tau)$ is the function such that $P_\mu(D^\bigstar) = I_\mu(\mu(D^\bigstar))$, or equivalently

$$I_\mu(\tau) = C_k \exp\left(c\left[\psi^{-1}\left(\frac{\tau}{C_k}\right)\right]^2\right)\left[\psi^{-1}\left(\frac{\tau}{C_k}\right)\right]^{N+k-1}. \tag{4.3}$$

Now we introduce the weighted Sobolev $W^{1,2}(D, d\mu)$ the set of the measurable functions u satisfying the following two conditions

(i) $$\int_D |\nabla u|^2 \, d\mu + \int_D u^2 d\mu < +\infty,$$

(ii) There exists a sequence of functions $u_n \in C^1(D)$ such that $u_n(x) = 0$ on the set $\partial D \setminus \{x_N = 0\}$ and

$$\lim_{n\to\infty}\left(\int_D |\nabla(u_n - u)|^2 \, d\mu + \int_D (u_n - u)^2 \, d\mu\right) = 0.$$

Any nonnegative function belonging to the space $W^{1,2}(D, d\mu)$ satisfies the following Pólya-Szegö - type inequality (see, e.g., [35])

Theorem 4.1. *Let u be a nonnegative function in $W^{1,2}(D, d\mu)$. Then the following inequality holds true*

$$\int_D |\nabla u|^2 d\mu \geq \int_{D^\bigstar} |\nabla u^\bigstar|^2 d\mu. \tag{4.4}$$

Since, by Cavalieri's principle, rearrangement preserves the L^p norms, we have that the Rayleigh-Ritz quotient decreases under rearrangement i.e.

$$\frac{\int_D |\nabla u|^2 d\mu}{\int_D u^2 d\mu} \geq \frac{\int_{D^\bigstar} |\nabla u^\bigstar|^2 d\mu}{\int_{D^\bigstar} \left(u^\bigstar\right)^2 d\mu}, \ \forall u \in W^{1,2}(D, d\mu).$$

The following Poincarè type inequality, which clearly states the continuous embedding of $W^{1,2}(D,d\mu)$ in $L^2(D,d\mu)$, is a consequence of some one-dimensional inequalities contained in [25].

Corollary 4.1. *There exists a constant C, depending on D only, such that for any $u \in W^{1,2}(D,d\mu)$ it holds*

$$\int_D u^2 d\mu \leq C \int_D |\nabla u|^2 \, d\mu.$$

The functional space $W^{1,2}(D,d\mu)$ will be equipped with the norm

$$\|u\|^2_{W^{1,2}(D,d\mu)} = \int_D |\nabla u|^2 \, d\mu.$$

4.2. **Comparison result.** Now we apply the above results in order to get sharp estimates for the solution to problem (1.8). By a weak solution to such a problem we mean a function u belonging to $W^{1,2}(D,d\mu)$ such that

$$\int_D A(x)\nabla u \nabla \chi d\mu = \int_D f\chi d\mu, \tag{4.5}$$

for every $\chi \in C^1(\bar{D})$ such that $\chi = 0$ on the set $\partial D \setminus \{x_N = 0\}$.

Proof of Theorem 1.2 Note that the existence of a unique solution to problems (1.8) and (1.10) is ensured by the Lax and Milgram Theorem. Arguing, for instance, as in [34] or in [8] (see also [22]), we get

$$1 \leq \left\{ \left[I_\mu \left(m_u(t) \right) \right]^{-2} \int_0^{m_u(t)} f^*(\sigma) d\sigma \right\} \left(-m_u'(t) \right) \tag{4.6}$$

and

$$u^*(s) \leq \int_s^{\mu(D)} \left(I_\mu^{-2}(l) \int_0^l f^*(\sigma) d\sigma \right) dl, \tag{4.7}$$

Using (4.3) in (4.7) we obtain

$$\begin{aligned}
u^\bigstar(x) &\leq \frac{1}{{C_k}^2} \int_{C_k \psi(|x|)}^{\mu(D)} \left\{ \exp\left(-2c \left[\psi^{-1}\left(\frac{l}{C_k} \right) \right]^2 \right) \left(\psi^{-1}\left(\frac{l}{C_k} \right) \right)^{-2N-2k+2} \int_0^l f^*(\sigma) d\sigma \right\} d\tau \\
&= \frac{1}{C_k} \int_{|x|}^{r^\bigstar} \exp\left(-c\eta^2 \right) \eta^{-N-k+1} \left(\int_0^{C_k \psi(\eta)} f^*(\sigma) d\sigma \right) d\eta \qquad \left(\eta := \psi^{-1}\left(\frac{l}{C_k} \right) \right) \\
&= \int_{|x|}^{r^\bigstar} \exp\left(-c\eta^2 \right) \eta^{-N-k+1} \left(\int_0^\eta f^*(C_k\psi(\xi)) \xi^{N+k-1} \exp\left(c\xi^2 \right) d\xi \right) d\eta \qquad (\sigma := C_k \psi(\xi)) \\
&= \int_{|x|}^{r^\bigstar} \exp\left(-c\eta^2 \right) \eta^{-N-k+1} \left(\int_0^\eta f^\bigstar(\xi) \xi^{N+k-1} \exp\left(c\xi^2 \right) d\xi \right) d\eta \\
&= w(x).
\end{aligned}$$

Now let us show (1.12). By using the same chain of inequalities contained in [34] or in [8] we get

$$\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|\,d\mu\right)^2\le\left(-m_u'(t)\right)\left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,d\mu\right).$$

On the other hand Hölder inequality implies

$$\begin{aligned}-\frac{d}{dt}\int_{|u|>t}|\nabla u|^q\,d\mu &\le \left(-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,d\mu\right)^{q/2}\left(-\frac{d}{dt}\int_{|u|>t}d\mu\right)^{1-q/2}\\ &\le \left(\int_{|u|>t}|f|\,d\mu\right)^{q/2}\left(-m_u'(t)\right)^{1-q/2}.\end{aligned}$$

Then Hardy inequality and (4.6) give

$$\begin{aligned}-\frac{d}{dt}\int_{|u|>t}|\nabla u|^q\,d\mu &\le \left(\int_0^{m_u(t)}f^*(s)ds\right)^{q/2}\left(-m_u'(t)\right)^{1-q/2}\\ &\le (I(m_u(t)))^{-q}\left(\int_0^{m_u(t)}f^*(s)ds\right)^{q}\left(-m_u'(t)\right).\end{aligned}$$

Integrating the last inequality between 0 and $+\infty$, we get

$$\begin{aligned}\int_D|\nabla u|^q\,d\mu &= \int_0^{+\infty}\left[-\frac{d}{dt}\int_{|u|>t}|\nabla u|^2\,d\mu\right]dt\\ &\le \int_0^{+\infty}(I_\mu(m_u(t)))^{-q}\left(\int_0^{m_u(t)}f^*(\sigma)d\sigma\right)^q\left(-m_u'(t)\right)dt\\ &\le \int_0^{\mu(D)}(I_\mu(s))^{-q}\left(\int_0^{s}f^*(\sigma)d\sigma\right)^q ds.\end{aligned}$$

Now a straightforward calculation yields

$$
\begin{aligned}
\int_D |\nabla u|^q \, d\mu &\leq C_k \int_0^{\mu(D)} \exp\left(-qc\left[\psi^{-1}\left(\frac{s}{C_k}\right)\right]^2\right)\left[\psi^{-1}\left(\frac{s}{C_k}\right)\right]^{-q(N+k-1)}\left(\int_0^s f^*(\sigma)d\sigma\right)^q ds \\
&= C_k^2 \int_0^{R^\star} \exp\left(-qc\eta^2\right)\eta^{-q(N+k-1)}\left(\int_0^{C_k\psi(\eta)} f^*(\sigma)d\sigma\right)^q \exp\left(c\eta^2\right)\eta^{N+k-1}d\eta \\
&= C_k^2 \int_0^{R^\star} \exp\left((1-q)c\eta^2\right)\eta^{(1-q)(N+k-1)}\left(\int_0^{C_k\psi(\eta)} f^*(\sigma)d\sigma\right)^q d\eta \\
&= C_k^2 \int_0^{R^\star}\left(\int_0^\eta f^*(C_k\psi(\rho))C_k\exp\left(c\rho^2\right)\rho^{N+k-1}d\rho\right)^q \exp\left((1-q)c\eta^2\right)\eta^{(1-q)(N+k-1)}d\eta \\
&= C_k^{2+q} \int_0^{R^\star}\left(\int_0^\eta f^\bigstar(\rho)\exp\left(c\rho^2\right)\rho^{N+k-1}d\rho\right)^q \exp\left((1-q)c\eta^2\right)\eta^{(1-q)(N+k-1)}d\eta \\
&= \int_D |\nabla w|^q \, d\mu.
\end{aligned}
$$

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5. Appendix

Here we provide, with all the notation of Section 2 in force, the detailed proof of (3.16).

Proof of (3.16) Setting

$$\phi(\theta) \equiv \exp\left(cy(\theta)^2\right) y(\theta)^k \int_0^{x(\theta)} \exp\left(ct^2\right) dt,$$

from (3.10) we deduce that, up to a subsequence, a.e. θ in $(0, \pi/2)$ it holds

$$\left\{ \begin{array}{l} x_n(\theta) \to x(\theta) \\ y_n(\theta) \to y(\theta) \\ \phi_n(\theta) \to \phi(\theta), \end{array} \right. \tag{5.1}$$

where $\{\phi_n(\theta)\}_n$ is the sequence defined in (3.15).

Our claim is therefore

$$\left(m = \lim_{n\to\infty} \mu(\rho_n) =\right) \quad \lim_{n\to\infty} \int_0^{\pi/2} \phi_n y_n' d\theta = \int_0^{\pi/2} \phi y' d\theta \quad (= \mu(\rho)) \,. \tag{5.2}$$

Estimates (3.8) and (3.10) imply

$$y_n' \rightharpoonup y' \text{ weakly in } L^p(\delta, \pi/2-\delta) \ \forall p>1 \text{ and } \forall \delta > 0. \tag{5.3}$$

In order to prove (5.2), we firstly show that

$$\lim_{n\to\infty} \int_\delta^{\pi/2-\delta} \phi_n y_n' d\theta = \int_\delta^{\pi/2-\delta} \phi y' d\theta, \quad \forall \delta > 0. \tag{5.4}$$

We have

$$\int_\delta^{\pi/2-\delta} (\phi_n y_n' - \phi y') d\theta = \int_\delta^{\pi/2-\delta} (\phi_n - \phi) y_n' d\theta + \int_\delta^{\pi/2-\delta} \phi (y_n' - y') d\theta. \tag{5.5}$$

By Lebesgue's dominated convergence Theorem, taking into account of (5.3), (3.15) and (5.1), we get

$$\lim_{n\to\infty} \int_\delta^{\pi/2-\delta} (\phi_n - \phi) y_n' d\theta = 0, \quad \forall \delta > 0, \tag{5.6}$$

and

$$\lim_{n\to\infty} \int_\delta^{\pi/2-\delta} \phi (y_n' - y') d\theta = 0, \quad \forall \delta > 0. \tag{5.7}$$

At this point (5.4) is an immediate consequence of (5.5), (5.6) and (5.7).

Now we claim that

$$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0 : \int_0^\delta \phi_n y_n' d\theta < \epsilon \quad \forall \delta < \delta_\epsilon \text{ and } \forall n \in \mathbb{N} \tag{5.8}$$

and

$$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0 : \int_{\pi/2-\delta}^{\pi/2} \phi_n y_n' d\theta < \epsilon \quad \forall \delta < \delta_\epsilon \text{ and } \forall n \in \mathbb{N}. \tag{5.9}$$

Let us show, for instance, (5.8). Since $0 \le \phi_n \le C \ \forall n \in \mathbb{N}$, it suffices to show that

$$\forall \epsilon > 0 \ \exists \delta_\epsilon > 0 : \int_0^\delta y_n' d\theta < \epsilon \quad \forall \delta < \delta_\epsilon \ \text{ and } \forall n \in \mathbb{N}.$$

We argue by absurd. If (5.8) was not true then $\exists \overline{\epsilon} > 0$ and $\exists \delta_n \searrow 0^+$:

$$\rho_n(\delta_n) \sin \delta_n = y_n(\delta_n) \ge y_n(\delta_n) - y_n(0) \ge \int_0^{\delta_n} y_n' d\theta \ge \overline{\epsilon} \ \ \forall n \in \mathbb{N}.$$

It follows that $x_n(\delta_n) = \rho_n(\delta_n)\cos\delta_n > \overline{\epsilon} \cot \delta_n$ and finally $\lim_{n\to\infty} x_n(\delta_n) = +\infty$. The uniform estimate (3.15) would guarantee that

$$\exp\left(c y_n^2(\delta_n)\right) y_n^k(\delta_n) \int_0^{x_n(\delta_n)} \exp\left(ct^2\right) dt \le C \ \ \forall n \in \mathbb{N},$$

a contradiction since $y_n(\delta_n) \geq \overline{\epsilon}\ \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n(\delta_n) = +\infty$.

Now let $\delta_\epsilon > 0$ such that

$$\int_0^\delta \phi_n y_n' d\theta + \int_{\pi/2-\delta}^{\pi/2} \phi_n y_n' d\theta < \epsilon \ \forall \delta < \delta_\epsilon \quad \text{and } \forall n \in \mathbb{N}. \tag{5.10}$$

We set

$$\int_0^{\pi/2} \phi_n y_n' d\theta = \left(\int_0^\delta \phi_n y_n' d\theta + \int_{\pi/2-\delta}^{\pi/2} \phi_n y_n' d\theta \right) + \int_\delta^{\pi/2-\delta} \phi_n y_n' d\theta \qquad \text{with } \delta < \delta_\epsilon.$$

Recalling (5.2), we have

$$\begin{aligned}
\limsup_{n\to\infty} \int_0^{\pi/2} \phi_n y_n' d\theta &\leq \limsup_{n\to\infty} \left(\int_0^\delta \phi_n y_n' d\theta + \int_{\pi/2-\delta}^{\pi/2} \phi_n y_n' d\theta \right) + \limsup_{n\to\infty} \int_\delta^{\pi/2-\delta} \phi_n y_n' d\theta \\
&\leq \epsilon + \int_\delta^{\pi/2-\delta} \phi y' d\theta \leq \epsilon + \int_0^{\pi/2} \phi y' d\theta
\end{aligned}$$

Since ϵ is can be taken arbitrarily small we have

$$\limsup_{n\to\infty} \int_0^{\pi/2} \phi_n y_n' d\theta \leq \int_0^{\pi/2} \phi y' d\theta. \tag{5.11}$$

On the other hand

$$\begin{aligned}
\liminf_{n\to\infty} \int_0^{\pi/2} \phi_n y_n' d\theta &\geq \liminf_{n\to\infty} \left(\int_0^\delta \phi_n y_n' d\theta + \int_{\pi/2-\delta}^{\pi/2} \phi_n y_n' d\theta \right) + \liminf_{n\to\infty} \int_\delta^{\pi/2-\delta} \phi_n y_n' d\theta \\
&\geq \int_\delta^{\pi/2-\delta} \phi y' d\theta \quad \forall \delta < \delta_\epsilon.
\end{aligned}$$

It follows that

$$\liminf_{n\to\infty} \int_0^{\pi/2} \phi_n y_n' d\theta \geq \int_0^{\pi/2} \phi y' d\theta. \tag{5.12}$$

Finally the claim (5.2) follows from (5.11) and (5.12). □