Pfaffian-type Sugawara operators

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Abstract

We show that the Pfaffian of a generator matrix for the affine Kac–Moody algebra $\hat{\mathfrak{o}}_{2n}$ is a Segal–Sugawara vector. Together with our earlier construction involving the symmetrizer in the Brauer algebra, this gives a complete set of Segal–Sugawara vectors in type D.

1 Introduction

For each affine Kac-Moody algebra $\hat{\mathfrak{g}}$ associated with a simple Lie algebra \mathfrak{g} , the corresponding vacuum module $V(\mathfrak{g})$ at the critical level is a vertex algebra. The structure of the center $\mathfrak{z}(\hat{\mathfrak{g}})$ of this vertex algebra was described by a remarkable theorem of Feigin and Frenkel in [3], which states that $\mathfrak{z}(\hat{\mathfrak{g}})$ is the algebra of polynomials in infinitely many variables which are associated with generators of the algebra of \mathfrak{g} -invariants in the symmetric algebra $S(\mathfrak{g})$. For a detailed proof of the theorem, its extensions and significance for the representation theory of the affine Kac-Moody algebras see [4].

In a recent paper [5] we used the symmetrizer in the Brauer algebra to construct families of elements of $\mathfrak{z}(\hat{\mathfrak{g}})$ (Segal–Sugawara vectors) for the Lie algebras \mathfrak{g} of types B, C and D in an explicit form. In types B and C they include complete sets of Segal–Sugawara vectors generating the center $\mathfrak{z}(\hat{\mathfrak{g}})$, while in type D one vector in the complete set of [5] was not given explicitly. The aim of this note is to produce this Segal–Sugawara vector in $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$ which is associated with the Pfaffian invariant in $S(\mathfrak{o}_{2n})$.

Simple explicit formulas for generators of the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{gl}}_n)$ were given recently in [1], [2], following Talalaev's construction of higher Gaudin Hamiltonians [6]. So, together with the results of [5] we get a construction of generators of the Feigin–Frenkel centers for all classical types.

2 Pfaffian-type generators

Denote by E_{ij} , $1 \leq i, j \leq 2n$, the standard basis vectors of the Lie algebra \mathfrak{gl}_{2n} . Introduce the elements F_{ij} of \mathfrak{gl}_{2n} by the formulas

$$F_{ij} = E_{ij} - E_{ji}. (2.1)$$

The Lie subalgebra of \mathfrak{gl}_{2n} spanned by the elements F_{ij} is isomorphic to the even orthogonal Lie algebra \mathfrak{o}_{2n} . The elements of \mathfrak{o}_{2n} are skew-symmetric matrices. Introduce the standard normalized invariant bilinear form on \mathfrak{o}_{2n} by

$$\langle X, Y \rangle = \frac{1}{2} \operatorname{tr} XY, \qquad X, Y \in \mathfrak{o}_{2n}.$$

Now consider the affine Kac–Moody algebra $\hat{\mathfrak{o}}_{2n} = \mathfrak{o}_{2n}[t, t^{-1}] \oplus \mathbb{C}K$ and set $X[r] = Xt^r$ for any $r \in \mathbb{Z}$ and $X \in \mathfrak{o}_{2n}$. The element K is central in $\hat{\mathfrak{o}}_{2n}$ and

$$\left[X[r], Y[s]\right] = [X, Y][r+s] + r\,\delta_{r,-s}\langle X, Y\rangle\,K.$$

Therefore, for the generators we have

$$\begin{bmatrix} F_{ij}[r], F_{kl}[s] \end{bmatrix} = \delta_{kj} F_{il}[r+s] - \delta_{il} F_{kj}[r+s] - \delta_{ki} F_{jl}[r+s] + \delta_{jl} F_{ki}[r+s] + r \delta_{r,-s} (\delta_{kj} \delta_{il} - \delta_{ki} \delta_{jl}) K.$$

The vacuum module at the critical level $V(\hat{\mathfrak{o}}_{2n})$ can be defined as the quotient of the universal enveloping algebra $U(\hat{\mathfrak{o}}_{2n})$ by the left ideal generated by $\mathfrak{o}_{2n}[t]$ and K + 2n - 2 (note that the dual Coxeter number in type D_n is $h^{\vee} = 2n - 2$). The Feigin–Frenkel center $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$ is defined by

$$\mathfrak{z}(\widehat{\mathfrak{o}}_{2n}) = \{ v \in V(\widehat{\mathfrak{o}}_{2n}) \mid \mathfrak{o}_{2n}[t] \, v = 0 \}.$$

Any element of $\mathfrak{g}(\hat{\mathfrak{o}}_{2n})$ is called a *Segal–Sugawara vector*. A complete set of *Segal–Sugawara* vectors $\phi_{22}, \phi_{44}, \ldots, \phi_{2n-22n-2}, \phi'_n$ was produced in [5], where all of them, except for ϕ'_n , were given explicitly. We will produce ϕ'_n in Theorem 2.1 below.

Combine the generators $F_{ij}[-1]$ into the skew-symmetric matrix $F[-1] = \lfloor F_{ij}[-1] \rfloor$ and define its *Pfaffian* by

Pf
$$F[-1] = \frac{1}{2^n n!} \sum_{\sigma \in \mathfrak{S}_{2n}} \operatorname{sgn} \sigma \cdot F_{\sigma(1) \sigma(2)}[-1] \dots F_{\sigma(2n-1) \sigma(2n)}[-1].$$

Note that the elements $F_{ij}[-1]$ and $F_{kl}[-1]$ of $\hat{\mathfrak{o}}_{2n}$ commute, if the indices i, j, k, l are distinct. Therefore, we can write the formula for the Pfaffian in the form

$$\operatorname{Pf} F[-1] = \sum_{\sigma} \operatorname{sgn} \sigma \cdot F_{\sigma(1) \sigma(2)}[-1] \dots F_{\sigma(2n-1) \sigma(2n)}[-1], \qquad (2.2)$$

summed over the elements σ of the subset $\mathcal{B}_{2n} \subset \mathfrak{S}_{2n}$ which consists of the permutations with the properties $\sigma(2k-1) < \sigma(2k)$ for all $k = 1, \ldots, n$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$. **Theorem 2.1.** The element $\phi'_n = \operatorname{Pf} F[-1]$ is a Segal-Sugawara vector for $\widehat{\mathfrak{o}}_{2n}$.

Proof. We need to show that $\mathfrak{o}_{2n}[t] \phi'_n = 0$ in the vacuum module $V(\widehat{\mathfrak{o}}_{2n})$. It suffices to verify that for all i, j,

$$F_{ij}[0] \operatorname{Pf} F[-1] = F_{ij}[1] \operatorname{Pf} F[-1] = 0.$$
(2.3)

Note that for any permutation $\pi \in \mathfrak{S}_{2n}$ the mapping

$$F_{ij}[r] \mapsto F_{\pi(i) \pi(j)}[r], \qquad K \mapsto K$$

defines an automorphism of the Lie algebra $\hat{\mathfrak{o}}_{2n}$. Moreover, the image of Pf F[-1] under its extension to $U(\hat{\mathfrak{o}}_{2n})$ coincides with sgn $\pi \cdot$ Pf F[-1]. Hence, it is enough to verify (2.3) for i = 1 and j = 2.

Observe that $F_{12}[0]$ commutes with all summands in (2.2) with $\sigma(1) = 1$ and $\sigma(2) = 2$. Suppose now that $\sigma \in \mathcal{B}_{2n}$ is such that $\sigma(2) > 2$. Then $\sigma(3) = 2$ and $\sigma(4) > 2$. In $V(\hat{\mathfrak{o}}_{2n})$ we have

$$F_{12}[0] F_{1\sigma(2)}[-1] F_{2\sigma(4)}[-1] \dots F_{\sigma(2n-1)\sigma(2n)}[-1] = -F_{2\sigma(2)}[-1] F_{2\sigma(4)}[-1] \dots F_{\sigma(2n-1)\sigma(2n)}[-1] + F_{1\sigma(2)}[-1] F_{1\sigma(4)}[-1] \dots F_{\sigma(2n-1)\sigma(2n)}[-1].$$

Set $i = \sigma(2)$ and $j = \sigma(4)$. Note that the permutation $\sigma' = \sigma(24)$ also belongs to the subset \mathcal{B}_{2n} , and $\operatorname{sgn} \sigma' = -\operatorname{sgn} \sigma$. We have

$$-F_{2i}[-1]F_{2j}[-1] + F_{1i}[-1]F_{1j}[-1] + F_{2j}[-1]F_{2i}[-1] - F_{1j}[-1]F_{1i}[-1] = F_{ij}[-2] - F_{ij}[-2] = 0.$$

This implies that the terms in the expansion of $F_{12}[0] \operatorname{Pf} F[-1]$ corresponding to pairs of the form (σ, σ') cancel pairwise. Thus, $F_{12}[0] \operatorname{Pf} F[-1] = 0$.

Now we verify that

$$F_{12}[1] \operatorname{Pf} F[-1] = 0. \tag{2.4}$$

Consider first the summands in (2.2) with $\sigma(1) = 1$ and $\sigma(2) = 2$. In $V(\hat{\mathfrak{o}}_{2n})$ we have

$$F_{12}[1] F_{12}[-1] F_{\sigma(3) \sigma(4)}[-1] \dots F_{\sigma(2n-1) \sigma(2n)}[-1] = -K F_{\sigma(3) \sigma(4)}[-1] \dots F_{\sigma(2n-1) \sigma(2n)}[-1].$$

Furthermore, let $\tau \in \mathcal{B}_{2n}$ with $\tau(2) > 2$. Then $\tau(3) = 2$ and $\tau(4) > 2$. We have

$$F_{12}[1] F_{1\tau(2)}[-1] F_{2\tau(4)}[-1] \dots F_{\tau(2n-1)\tau(2n)}[-1]$$

= $-F_{2\tau(2)}[0] F_{2\tau(4)}[-1] \dots F_{\tau(2n-1)\tau(2n)}[-1]$
= $F_{\tau(2)\tau(4)}[-1] \dots F_{\tau(2n-1)\tau(2n)}[-1].$

Note that for any given σ , the number of elements τ such that the product

$$F_{\tau(2)\,\tau(4)}[-1]\ldots F_{\tau(2n-1)\,\tau(2n)}[-1]$$

coincides, up to a sign, with the product

$$F_{\sigma(3)\sigma(4)}[-1]\dots F_{\sigma(2n-1)\sigma(2n)}[-1]$$

equals 2n - 2. Indeed, for each k = 2, ..., n the unordered pair $\{\tau(2), \tau(4)\}$ can coincide with the pair $\{\sigma(2k-1), \sigma(2k)\}$. Hence, taking the signs of permutations into account, we find that

$$F_{12}[1] \operatorname{Pf} F[-1] = (-K - 2n + 2) \sum_{\sigma} \operatorname{sgn} \sigma \cdot F_{\sigma(3) \sigma(4)}[-1] \dots F_{\sigma(2n-1) \sigma(2n)}[-1],$$

summed over $\sigma \in \mathcal{B}_{2n}$ with $\sigma(1) = 1$ and $\sigma(2) = 2$. Since -K - 2n + 2 = 0 at the critical level, we get (2.4).

Introduce formal Laurent series

$$F_{ij}(z) = \sum_{r \in \mathbb{Z}} F_{ij}[r] \, z^{-r-1}$$
 and $F_{ij}(z)_+ = \sum_{r < 0} F_{ij}[r] \, z^{-r-1}$

and expand the Pfaffians of the matrices $F(z) = [F_{ij}(z)]$ and $F(z)_+ = [F_{ij}(z)_+]$ by

Pf
$$F(z) = \sum_{p \in \mathbb{Z}} S_p z^{-p-1}$$
 and Pf $F(z)_+ = \sum_{p < 0} S_p^+ z^{-p-1}$.

Invoking the vertex algebra structure on the vacuum module $V(\mathfrak{o}_{2n})$ (see [4]), we derive from Theorem 2.1 that the coefficients S_p are Sugawara operators for $\hat{\mathfrak{o}}_{2n}$; they commute with the elements of $\hat{\mathfrak{o}}_{2n}$ (note that normal ordering is irrelevant here, as the coefficients of the series pairwise commute). Moreover, the coefficients S_p^+ are elements of the Feigin– Frenkel center $\mathfrak{z}(\hat{\mathfrak{o}}_{2n})$.

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