On non-uniformly simple groups

Hiroki KODAMA

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Abstract

Suppose G is a simple group. For any nontrivial elements g and h, g can be written as a finite product of conjugates of h or the inverse of h. G is called uniformly simple if the length of such an expression is uniformly bounded. We show that the infinite alternating group is non-uniformly simple and evaluate how the length of such an expression is unbounded.

For an element g of a group G, we denote by C_g the conjugacy class of g, and define $[g] := C_g \cup C_{g^{-1}}$. Suppose G is a simple group. For a nontrivial element $h \in G$, the normal subgroup generated by h,

$$N_h = \{h_1 h_2 \cdots h_n \,|\, n \in \mathbb{N},\ h_i \in [h]\}$$

coincides with G. Therefore, for any nontrivial elements $g, h \in G$, g can be written as a finite product of elements of $[h]$. Denote by $\lambda_h(g)$ the minimum number of n in such expressions $g = h_1 h_2 \cdots h_n$. A simple group G is called **uniformly simple** if $\lambda_h(g)$ is bounded by a constant independent of g, h and **non-uniformly simple** if it is not.

If h_1, $h_2 \in [h]$ and g_1, $g_2 \in [g]$ then $\lambda_{h_1}(g_1) = \lambda_{h_2}(g_2)$. Therefore, we can consider λ as a function of $[h]$ and $[g]$ to use an abuse of notation

$$\lambda_{[h]}([g]) = \lambda_{[h]}(g) = \lambda_h([g]) = \lambda_h(g).$$

Example 1 The group S_∞ of bijections on the set $\mathbb{N}$ of natural numbers with finite support is called the **infinite symmetric group**. The even permutations of S_∞ form the **infinite alternating group** A_∞, which is a non-uniformly simple group.

For any nontrivial elements $g, h \in G$, we set

$$d([g],[h]) := \log(\max\{\lambda_{[g]}([h]), \lambda_{[h]}([g])\}).$$

From an easy inequality $\lambda_{[f]}([h]) \leqq \lambda_{[f]}([g])\lambda_{[g]}([h])$ $([f],[g],[h] \in G \setminus \{e\})$ $\cdots(1)$, it follows that d is a metric on $\widehat{G} := \{[g] \,|\, g \in G \setminus \{e\}\}$. Remark that the followings are equivalent; 1. the group G is uniformly simple, 2. the metric space $(\widehat{G}, d)$ is quasi-isometric to a point.

What if the group G is non-uniformly simple? For the case $G = A_\infty$ we show the following theorem.

Theorem 2 *the metric space* $(\widehat{A_\infty}, d)$ *is quasi-isometric to the half-line* $\mathbb{R}_+$

In the rest of this paper we consider only the following two kind of groups, namely infinite symmetric group S_∞ and the infinite alternating group A_∞. The following claim is important;

Claim 3 For any two elements $h, h' \in A_\infty$, they are conjugate in A_∞ if and only if they are conjugate in S_∞.

Proof. Suppose $h' = ghg^{-1}$ for an odd permutation $g \in S_\infty$. Take two distinguished elements $a, b \in \mathbb{N}$ away from the support of h'. Then $h' = ((a\ b)g)h((a\ b)g)^{-1}$ and $((a\ b)g)$ is an even permutation. □

We define $\lambda_h(g)$ similarly on the infinite symmetric group S_∞ while $\lambda_h(g) = \infty$ if $g \notin N_h$ i.e. g is odd and f is even. Claim 3. assures that this definition is an extension of one on A_∞.

Consider a transposition $\iota_1 = (1\ 2) \in S_\infty$. For a permutation $g \in S_\infty$, the number $\lambda_{\iota_1}(g)$ is called the **word length** of g and simply denoted by $\lambda(g)$. g is an even permutation if and only if $\lambda(g)$ is an even number.

Theorem 2 follows from the following evaluation.

Proposition 4 *For any nontrivial elements* $g, h \in A_\infty$*, the following inequality holds.*

$$\frac{\lambda(g)}{\lambda(h)} \leqq \lambda_h(g) \leqq 4\frac{\lambda(g)}{\lambda(h)} + 4.$$

The left hand side evaluation follows directly from the inequality (1).

To acquire the upper evaluation, we show the following lemma. We denote the product of k transposition $(1\ 2)(3\ 4)\cdots(2k-1\ 2k)$ by ι_k.

Lemma 5 *For any nontrivial permutation* h *in the infinite symmetric group* S_∞*, there exists an integer* $\ell \geq \lambda(h)/4$ *such that* $\iota_{2\ell} = (1\ 2)(3\ 4)\cdots(4\ell{-}1\ 4\ell)$ *can be written as a product of two permutations which are conjugate to* h*.*

Proof. The following table shows the lemma holds if h is a cyclic permutation. Any nontrivial permutation h can be written as $h = h_1 \cdots h_n$ where h_i are cyclic permutations and $\lambda(h) = \lambda(h_1) + \cdots + \lambda(h_n)$, therefore the lemma follows.

$\lambda(h)$		ℓ
1	$(1\ 2)^{-1}(3\ 4) = \iota_2$	1
2	$(1\ 2\ 3)^{-1}(1\ 3\ 4) = \iota_2$	1
3	$(1\ 2\ 3\ 5)^{-1}(1\ 3\ 4\ 5) = \iota_2$	1
4	$(1\ 2\ 3\ 5\ 6)^{-1}(1\ 3\ 4\ 5\ 6) = \iota_2$	1
5	$(1\ 2\ 3\ 5\ 6\ 7)^{-1}(1\ 3\ 4\ 5\ 7\ 8) = \iota_4$	2
6	$(1\ 2\ 3\ 5\ 6\ 7\ 9)^{-1}(1\ 3\ 4\ 5\ 7\ 8\ 9) = \iota_4$	2
7	$(1\ 2\ 3\ 5\ 6\ 7\ 9\ 10)^{-1}(1\ 3\ 4\ 5\ 7\ 8\ 9\ 10) = \iota_4$	2
8	$(1\ 2\ 3\ 5\ 6\ 7\ 9\ 10\ 11)^{-1}(1\ 3\ 4\ 5\ 7\ 8\ 9\ 11\ 12) = \iota_6$	3
9	$(1\ 2\ 3\ 5\ 6\ 7\ 9\ 10\ 11\ 13)^{-1}(1\ 3\ 4\ 5\ 7\ 8\ 9\ 11\ 12\ 13) = \iota_6$	3
10	$(1\ 2\ 3\ 5\ 6\ 7\ 9\ 10\ 11\ 13\ 14)^{-1}(1\ 3\ 4\ 5\ 7\ 8\ 9\ 11\ 12\ 13\ 14) = \iota_6$	3
$\vdots$	$\vdots$	$\vdots$
$3k-1$	$(1\ 2\ 3\ \cdots\ 4k-3\ 4k-2\ 4k-1)^{-1}(1\ 3\ 4\ \cdots\ 4k-3\ 4k-1\ 4k) = \iota_{2k}$	k
$3k$	$(1\ 2\ 3\ \cdots\ 4k-3\ 4k-2\ 4k-1\ 4k+1)^{-1}(1\ 3\ 4\ \cdots\ 4k-3\ 4k-1\ 4k\ 4k+1) = \iota_{2k}$	k
$3k+1$	$(1\ 2\ 3\ \cdots\ 4k-3\ 4k-2\ 4k-1\ 4k+1\ 4k+2)^{-1}(1\ 3\ 4\ \cdots\ 4k-3\ 4k-1\ 4k\ 4k+1\ 4k+2) = \iota_{2k}$	k
$\vdots$	$\vdots$	$\vdots$

□

Lemma 6 *For $k = 2n\ell$, ι_k can be written as a product of n permutations which are conjugate to $\iota_{2\ell}$.*

Proof. Trivial. □

Lemma 7 *For any even permutation $g \in A_\infty$ and any integer $k \geqq \lambda(g)/2$, g can be written as a product of two permutations which are conjugate to ι_k.*

Proof. Since g can be written as a product of cyclic permutations, we consider the following sublemma.

Sublemma 8 *Any cyclic permutation with word length $2n$ can be written as a product of two permutations which are conjugate to ι_n. Any cyclic permutation with word length $2n+1$ can be written as a product of two permutations which are conjugate to ι_n and ι_{n+1} (or ι_{n+1} and ι_n).*

Proof for the sublemma.

$$
\begin{aligned}
&((2\ 3)(4\ 5)\cdots(2n\ 2n+1))\,((1\ 2)(3\ 4)\cdots(2n-1\ 2n)) \\
&\qquad\qquad = (1\ 3\ \cdots\ 2n+1\ 2n\ 2n-2\ \cdots\ 2), \\
&((2\ 3)(4\ 5)\cdots(2n\ 2n+1))\,((1\ 2)(3\ 4)\cdots(2n+1\ 2n+2)) \\
&\qquad\qquad = (1\ 3\ \cdots\ 2n+1\ 2n+2\ 2n\ \cdots\ 2), \\
&((1\ 2)(3\ 4)\cdots(2n+1\ 2n+2))\,((2\ 3)(4\ 5)\cdots(2n\ 2n+1)) \\
&\qquad\qquad = (1\ 2\ 4\ \cdots\ 2n+2\ 2n+1\ 2n-1\ \cdots\ 3).
\end{aligned}
$$

□

Because of the sublemma, we can write $g = f_1 f_2$ where f_1 and f_2 are conjugate to $\iota_{\lambda(g)/2}$. Take $a_1, \ldots, a_{2r}$ away from $\operatorname{supp} g$,

$$g = (f_1(a_1\ a_2)\cdots(a_{2r-1}\ a_{2r}))((a_1\ a_2)\cdots(a_{2r-1}\ a_{2r})f_2),$$

which shows the lemma 7. □

Proposition 4 follows from Lemmas 5, 6 and 7. We define the map $\psi\colon \widehat{A_\infty} \to \mathbb{R}_+$ by $\psi([g]) = \log \lambda([g])$, then ψ is quasi-isometric and Theorem 2 is proved.

Problem 9 For other non-uniformly simple group G, What is the shape of the metric space $(\widehat{G}, d)$? Is it quasi-isometric to the half-line?

Graduate School of Mathematical Sciences, University of Tokyo,
3-8-1 Komaba, Meguro-ku, Tokyo 153-9814, Japan.
kodama@ms.u-tokyo.ac.jp