ON EXTREME VALUE PROCESSES AND THE FUNCTIONAL D-NORM

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Abstract. We introduce some mathematical framework for functional extreme value theory and provide basic definitions and tools. In particular we introduce a functional domain of attraction approach for stochastic processes, which is more general than the usual one based on weak convergence.

The distribution function G of a continuous max-stable process on $[0,1]$ is introduced and it is shown that G can be represented via a norm on functional space, called D-norm. This is in complete accordance with the multivariate case and leads to the definition of functional generalized Pareto distributions (GPD) W. These satisfy $W = 1+\log(G)$ in their upper tails, again in complete accordance with the uni- or multivariate case.

Applying this framework to copula processes we derive characterizations of the domain of attraction condition for copula processes in terms of tail equivalence with a functional GPD.

δ-neighborhoods of a functional GPD are introduced and it is shown that these are characterized by a polynomial rate of convergence of functional extremes, which is well-known in the multivariate case.

1. Introduction

Since the publication of the pathbreaking articles by Pickands [15] and Balkema and de Haan [2] extreme value theory (EVT) has undergone a fundamental change. Instead of investigating the maxima in a set of observations, the focus is now on exceedances above a high threshold. The key result obtained in the above articles is the fact that the maximum of n iid univariate observations, linearly standardized, follows an extreme value distribution (EVD) as n increases if, and only if, the exceedances above an increasing threshold follow a generalized Pareto distribution (GPD). The multivariate analogon is due to Rootzén and Tajvidi [16]. For a recent account of multivariate EVT and GPD we refer to Falk et al. [11].

The most complex setup is demanded by *functional* EVT, which investigates maxima (taken pointwise) of stochastic processes, as initiated by de Haan [5] and de Haan and Pickands [14]. We refer to de Haan and Ferreira [6] for a detailed presentation of up-to-date theory.

In particular de Haan and Lin [7] provided a domain of attraction condition in terms of weak convergence of stochastic processes in the space $C[0,1]$ of continuous functions on $[0,1]$, which implies weak convergence of the maximum of n iid stochastic processes, linearly standardized, towards an extreme value process in $C[0,1]$.

1991 *Mathematics Subject Classification.* Primary 60G70.

Key words and phrases. Extreme value process, functional D-norm, functional domain of attraction, copula process, generalized Pareto process, Takahashi's Theorem, spectral decomposition, rate of convergence of functional extremes.

The first author was supported by DFG Grant FA 262/4-1.

This condition consists, essentially, of the ordinary univariate weak convergence of the marginal maxima to a univariate EVD together with weak convergence of the corresponding copula process in functional space.

This is in accordance with multivariate EVT, where it is well-known that the maximum (taken coordinatewise) of n iid random vectors converges weakly to a multivariate EVT if, and only if, this is true for the univariate maxima together with convergence of the corresponding copulas (Deheuvels [8, 9], Galambos [12]).

In the present paper we develop a framework for a functional domain of attraction theory, which is in even higher conformity with the multivariate case.

This paper is organized as follows. In Section 2 we introduce some mathematical framework for functional EVT and provide basic definitions and tools. In particular we give a characterization of the distribution of max-stable processes via a norm on $E[0,1]$, the space of bounded functions on $[0,1]$ which have finitely many discontinuities. This norm is called D-norm. In Section 3 we introduce a functional domain of attraction approach for stochastic processes which is more general than the usual one based on weak convergence. The results of the foregoing sections are applied in Subsection 3.1 to derive characterizations of the domain of attraction condition for copula processes. The idea of a functional GPD is introduced in Section 4 and, finally, well-known results of the multivariate case are carried over, particularly $\delta-$neighborhoods of a functional GPD are considered in Section 5.

To improve the readability of this paper we use bold face such as $\boldsymbol{\xi}$, $\boldsymbol{Y}$ for stochastic processes and default font f, a_n etc. for nonstochastic functions. Operations on functions such as $\boldsymbol{\xi} < f$ or $(\boldsymbol{\xi} - b_n)/a_n$ are meant componentwise. The usual abbreviations *df, fidis, iid* and *rv* for the terms *distribution function, finite dimensional distributions, independent and identically distributed* and *random variable*, respectively, are used.

2. Extreme Value Processes in $C[0,1]$

An *extreme value process* (EVP) $\boldsymbol{\zeta} = (\zeta_t)_{t\in[0,1]}$ in $C[0,1] := \{f : [0,1] \to \mathbb{R} : f \text{ continuous}\}$, equipped with the sup-norm $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$, is a stochastic process with the characteristic property that its distribution is max-stable, i.e., $\boldsymbol{\zeta}$ has the same distribution as $\max_{1\le i\le n}(\boldsymbol{\zeta}_i - b_n)/a_n$ for independent copies $\boldsymbol{\zeta}_1, \boldsymbol{\zeta}_2, \dots$ of $\boldsymbol{\zeta}$ and some $a_n, b_n \in C[0,1]$, $a_n > 0$, $n \in \mathbb{N}$ (c.f. de Haan and Ferreira [6]), i.e.,

$$\boldsymbol{\zeta} =_D \max_{1\le i\le n} (\boldsymbol{\zeta}_i - b_n)/a_n, \tag{2.1}$$

the maxima being taken componentwise. Since the fidis determine the distribution of a process $\boldsymbol{\zeta}$, we have in particular that $\boldsymbol{\zeta}(t)$ is a max-stable real valued rv for every $t \in [0,1]$. This one dimensional case is very well-known (c.f. Falk et al. [11], de Haan and Ferreira [6]). A nondegenerate max-stable df on $\mathbb{R}$ is (apart from a location and a scale parameter) a member of the parametric family

$$F_\gamma(x) := \exp\left(-(1+\gamma x)^{-1/\gamma}\right), \qquad \gamma x \ge -1, \qquad \gamma \in \mathbb{R}, \tag{2.2}$$

where for different values of γ

$$F_\gamma(x) = \begin{cases} 0 & \text{for } \gamma > 0 \text{ and } x \le -1/\gamma, \\ 1 & \text{for } \gamma < 0 \text{ and } x \ge -1/\gamma, \\ \exp\left(-\exp(-x)\right) & \text{for } \gamma = 0 \text{ and } x \in \mathbb{R}. \end{cases}$$

Therefore, for every $t \in [0,1]$, there exist $a(t) > 0$ and $b(t), \gamma(t) \in \mathbb{R}$, such that

$$P\left(\frac{\boldsymbol{\zeta}(t) - b(t)}{a(t)} \le x\right) = F_{\gamma(t)}(x). \tag{2.3}$$

As already mentioned in Giné et al. [13], for a max-stable process $\boldsymbol{\zeta}$ there is a straightforward relationship between the continuous norming functions $a_n > 0$, b_n in (2.1) and $a(t) > 0$ and $b(t), \gamma(t) \in \mathbb{R}$ from (2.3), as for every $n \in \mathbb{N}$ and $t \in [0,1]$

$$a_n(t) = n^{\gamma(t)}, \qquad b_n(t) = \begin{cases} \left(n^{\gamma(t)} - 1\right)\left(\frac{a(t)}{\gamma(t)} - b(t)\right) & \text{for } \gamma(t) \neq 0 \\ a(t)\log(n) & \text{for } \gamma(t) = 0. \end{cases} \tag{2.4}$$

Relation (2.4) together with some analytical arguments imply Lemma 3.5.(i) in Giné et al. [13]. Because this result is crucial for what follows we state this result here.

Lemma 2.1. *Let $\boldsymbol{\zeta} \in C[0,1]$ be a max-stable process and for $t \in [0,1]$ let $a(t)$, $b(t)$, $\gamma(t)$ be the norming numbers from equation* (2.3)*. Then the functions $a : [0,1] \to (0,\infty)$, $t \mapsto a(t)$, $b : [0,1] \to \mathbb{R}$, $t \mapsto b(t)$ and $\gamma : [0,1] \to \mathbb{R}$, $t \mapsto \gamma(t)$ are continuous in $t \in [0,1]$.*

In the finite-dimensional case, the characterization of the max-stable distributions is typically done by characterizing some standard case (with certain margin restrictions) and reaching all other cases by (margin) transformation. We will proceed in an analogous way by characterizing *standard* max-stable processes and stating thereafter that an arbitrary max-stable process can always be transformed to such a standard max-stable process.

2.1. Standard Max-Stable Processes and the D-Norm in Function Spaces.

We call a process $\boldsymbol{\eta} \in C[0,1]$ a *standard* EVP, if it is an EVP with standard negative exponential (one-dimensional) margins, $P(\eta_t \le x) = \exp(x)$, $x \le 0$, $t \in [0,1]$.

Note that in case of a standard EVP the norming functions in above notation are $\gamma(t) \equiv -1 \equiv b(t)$, $a(t) \equiv 1$ and, consequently, $a_n(t) \equiv 1/n$ and $b_n(t) \equiv 0$, for all $t \in [0,1]$.

According to Giné et al. [13] and de Haan and Ferreira [6], a process $\boldsymbol{\xi} \in C[0,1]$ is called a *simple* EVP, if it is an EVP with standard Fréchet (one-dimensional) margins, $P(\xi_t \le x) = \exp(-1/x)$, $x > 0$, $t \in [0,1]$. We will see later that each EVP $\boldsymbol{\xi}$ can be transformed to a standard EVP $\boldsymbol{\eta}$ by just transforming the univariate margins $\eta_t := -1/\xi_t$, $0 \le t \le 1$, and, vice versa, $\xi_t := -1/\eta_t$. With this one-to-one correspondence one might consider the spaces of simple EVP and standard EVP as *dual spaces*.

A crucial observation is the fact that neither a simple EVP $\boldsymbol{\xi}$ nor a standard EVP $\boldsymbol{\eta}$ attains the value 0 (with probability one), which is the content of the following two auxiliary results, which are of interest of their own. Lemma 2.3 was already established by Giné et al. [13] using the theory of random sets. Furthermore, Theorem 9.4.1 in de Haan and Ferreira [6] contains this assertion, too, proven by elementary probabilistic arguments.

Lemma 2.2. *Let K be a compact subset of $[0,1]$ and let $\boldsymbol{\eta}_K = (\eta_t)_{t \in K}$ be a max-stable process on K with standard negative exponential margins, which realizes in the space of continuous functions $\bar{C}^-(K) := \{f : K \to (-\infty, 0],\ f \text{ is continuous}\}$.*

Then we have

$$P\left(\max_{t\in K}\eta_t<0\right)=1.$$

Proof. The crucial argument in this proof is the following fact. We have for an arbitrary interval $[a,b]\subset[0,1]$

$$P\left(\max_{t\in[a,b]\cap K}\eta_t<0\right)\in\{0,1\}\,. \tag{2.5}$$

This can be seen as follows. Define for $n\in\mathbb{N}$ and arbitrary $\varepsilon>0$ the function $f_{n,\varepsilon}(t):=(-\varepsilon/n)1_{[a,b]\cap K}(t)$, $t\in K$. Let $\boldsymbol{\eta}_K^{(1)},\boldsymbol{\eta}_K^{(2)},\dots$ be independent copies of $\boldsymbol{\eta}_K$. From the max-stability of η we obtain

$$\begin{aligned}
P\left(\max_{t\in[a,b]\cap K}\eta_t\le\frac{-\varepsilon}{n}\right)&=P\left(\boldsymbol{\eta}_K\le f_{n,\varepsilon}\right)\\
&=\left(P\left(\boldsymbol{\eta}_K\le f_{n,\varepsilon}\right)^n\right)^{1/n}\\
&=P\left(\max_{1\le i\le n}\boldsymbol{\eta}_K^{(i)}\le f_{n,\varepsilon}\right)^{1/n}\\
&=P\left(n\max_{1\le i\le n}\boldsymbol{\eta}_K^{(i)}\le nf_{n,\varepsilon}\right)^{1/n}\\
&=P\left(\boldsymbol{\eta}_K\le nf_{n,\varepsilon}\right)^{1/n}\\
&=P\left(\max_{t\in[a,b]\cap K}\eta_t\le-\varepsilon\right)^{1/n}\\
&\to_{n\to\infty}1
\end{aligned}$$

unless $P\left(\max_{t\in[a,b]\cap K}\eta_t\le-\varepsilon\right)=0$. Equation (2.5) now follows from the upper continuity of a probability measure:

$$\begin{aligned}
P\left(\max_{t\in[a,b]\cap K}\eta_t<0\right)&=P\left(\bigcup_{n\in\mathbb{N}}\left\{\max_{t\in[a,b]\cap K}\eta_t\le\frac{-\varepsilon}{n}\right\}\right)\\
&=\lim_{n\to\infty}P\left(\max_{t\in[a,b]\cap K}\eta_t\le\frac{-\varepsilon}{n}\right).
\end{aligned}$$

Equation (2.5) implies

$$1-P\left(\max_{t\in K}\eta_t<0\right)=P\left(\max_{t\in K}\eta_t=0\right)=P\left(\max_{t\in[0,1]\cap K}\eta_t=0\right)\in\{0,1\}\,.$$

We show by a contradiction that this probability is actually zero. Assume that it is 1. We divide the interval $[0,1]$ into the two subintervals $[0,1/2]$, $[1/2,1]$. Now we obtain from equation (2.5) that $P\big(\max_{t\in[0,1/2]\cap K}\eta_t=0\big)=1$ or $P\big(\max_{t\in[1/2,1]\cap K}\eta_t=0\big)=1$. Suppose without loss of generality that the first probability is 1 (if one of the two intersections with K is empty, the probability concerning the other one has to be equal to 1). Then we divide the interval $[0,1/2]$ into the two subintervals $[0,1/4]$, $[1/4,1/2]$ and repeat the preceding arguments. By iterating, this generates a sequence of nested intervals $I_n=[t_n,\tilde{t}_n]$ in $[0,1]$ with $P\left(\max_{t\in I_n\cap K}\eta_t=0\right)=1$, $\tilde{t}_n-t_n=2^{-n}$, $n\in\mathbb{N}$, and $t_n\uparrow t_0$, $\tilde{t}_n\downarrow t_0$ as $n\to\infty$ for some $t_0\in K$. From the lower continuity of a probability measure we now conclude

$$
\begin{aligned}
0 &= P\left(\eta_{t_0} = 0\right) \\
&= P\left(\bigcap_{n\in\mathbb{N}} \left\{\max_{t\in I_n\cap K} \eta_t = 0\right\}\right) \\
&= \lim_{n\to\infty} P\left(\left\{\max_{t\in I_n\cap K} \eta_t = 0\right\}\right) \\
&= 1,
\end{aligned}
$$

since η_{t_0} is negative exponential distributed; but this is the desired contradiction. □

The dual result is the following:

Lemma 2.3. *Let K be a compact subset of $[0,1]$ and let $\boldsymbol{\xi}_K = (\xi_t)_{t\in K}$ be a max-stable process on K with standard Fréchet margins, which realizes in the space of continuous functions $\bar{C}^+(K) := \{f : K \to [0,\infty),\ f \text{ is continuous}\}$. Then we have*

$$
P\left(\inf_{t\in K} \boldsymbol{\xi}(t) > 0\right) = 1.
$$

Proof. Repeat the arguments of the proof given in de Haan and Ferreira [6, p. 306]. Note that the restriction of the process to a compact subset of $[0,1]$ does not affect the conclusion therein. □

The following crucial characterization of max-stable processes is a consequence of Giné et al. [13, Proposition 3.2]; we refer also to de Haan and Ferreira [6, Theorem 9.4.1].

Proposition 2.4. *Let $\boldsymbol{Z} \in \bar{C}^+[0,1] := \{f \in C[0,1] : f \geq 0\}$ be a stochastic process with the properties*

$$
\max_{t\in[0,1]} Z_t = m \in [1,\infty) \text{ a.s. and } E(Z_t) = 1, \quad t \in [0,1]. \tag{2.6}
$$

(i) *A process $\boldsymbol{\xi} \in \bar{C}^+[0,1]$ is a* simple *EVP if, and only if there exists a stochastic process $\boldsymbol{Z}$ as above such that for compact subsets $K_1,\dots,K_d$ of $[0,1]$ and $x_1,\dots,x_d > 0$, $d \in \mathbb{N}$,*

$$
\begin{aligned}
&P\left(\max_{t\in K_j} \boldsymbol{\xi}_t \leq x_j, 1 \leq j \leq d\right) \\
&= \exp\left(-E\left(\max_{1\leq j\leq d}\left(\frac{\max_{t\in K_j} Z_t}{x_j}\right)\right)\right).
\end{aligned} \tag{2.7}
$$

(ii) *A process $\boldsymbol{\eta} \in \bar{C}^-[0,1] := \{f \in C[0,1] : f \leq 0\}$ is a* standard *EVP if, and only if there exists stochastic process $\boldsymbol{Z}$ as above such that for compact subsets $K_1,\dots,K_d$ of $[0,1]$ and $x_1,\dots,x_d \leq 0$, $d \in \mathbb{N}$,*

$$
\begin{aligned}
&P\left(\max_{t\in K_j} \boldsymbol{\eta}_t \leq x_j, 1 \leq j \leq d\right) \\
&= \exp\left(-E\left(\max_{1\leq j\leq d}\left(|x_j| \max_{t\in K_j} Z_t\right)\right)\right).
\end{aligned} \tag{2.8}
$$

Conversely, every stochastic process $\boldsymbol{Z} \in \bar{C}^+[0,1]$ *satisfying* (2.6) *gives rise to a simple and to a standard EVP. The connection is via* (2.7) *and* (2.8)*, respectively. We call* $\boldsymbol{Z}$ generator *of* $\boldsymbol{\eta}$ *and* $\boldsymbol{\xi}$.

Proof. Identify the finite measure σ in Giné et al. [13, Proposition 3.2] with $m(P * \tilde{\boldsymbol{Z}})$, where $(P * \tilde{\boldsymbol{Z}})$ denotes the distribution of some process $\tilde{\boldsymbol{Z}} \in \bar{C}_1^+ := \{f \in C[0,1] : f \geq 0, \|f\|_\infty = 1\}$ and set $\boldsymbol{Z} = m\tilde{\boldsymbol{Z}}$, where m is the total mass of the measure σ. Assertion *(ii)* now follows by setting $\boldsymbol{\eta} = -1/\boldsymbol{\xi}$, which is well defined by Lemma 2.3. □

According to de Haan and Ferreira [6, Corollary 9.4.5], condition (2.6) can be weakened to the condition $E(Z_t) = 1$, $t \in [0,1]$, together with $E\left(\max_{t\in[0,1]} Z_t\right) < \infty$. While a generator $\boldsymbol{Z}$ is in general not uniquely determined, the number $m = E\left(\max_{t\in[0,1]} Z_t\right)$ is uniquely determined, see below. We, therefore, call m the *generator constant* of $\boldsymbol{\eta}$.

The preceding characterization implies in particular that the fidis of $\boldsymbol{\eta}$ are multivariate EVD with standard negative exponential margins: We have for $0 \leq t_1 < t_2 \cdots < t_d \leq 1$

$$-\log(G_{t_1,\dots,t_d}(\boldsymbol{x})) = E\left(\max_{1\leq i\leq d}(|x_i|\, Z_{t_i})\right) =: \|\boldsymbol{x}\|_{D_{t_1,\dots,t_d}}, \ \boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d, \tag{2.9}$$

where $\|\cdot\|_{D_{t_1,\dots,t_d}}$ is a *D-norm* on $\mathbb{R}^d$ (see Falk et al. [11]).

Denote by $E[0,1]$ the set of all functions on $[0,1]$ that are bounded and which have only a finite number of discontinuities. Furthermore, denote by $\bar{E}^-[0,1]$ those functions in $E[0,1]$ which do not attain positive values.

Definition 2.5. For a generator process $\boldsymbol{Z} \in \bar{C}^+[0,1]$ with properties (2.6) set

$$\|f\|_D := E\left(\sup_{t\in[0,1]}(|f(t)|\, Z_t)\right), \qquad f \in E[0,1].$$

Then $\|\cdot\|_D$ obviously defines a norm on $E[0,1]$, called a *D-norm with generator* $\boldsymbol{Z}$.

The sup-norm $\|f\|_\infty := \sup_{t\in[0,1]} |f(t)|$, $f \in E[0,1]$, is a particular D-norm with constant generator $Z_t = 1$, $t \in [0,1]$. It is, moreover, the least D-norm, as

$$\|f\|_\infty \leq \|f\|_D \leq m\, \|f\|_\infty, \qquad f \in E[0,1], \tag{2.10}$$

for any D-norm $\|\cdot\|_D$ whose generator satisfies $E\left(\max_{t\in[0,1]} Z_t\right) = m$. For the constant function $f = 1$ we obtain $\|1\|_D = m$.

Note that inequality (2.10) implies that each functional D-norm is equivalent with the sup-norm. This, in turn, implies that no L_p-norm $\|f\|_p = \left(\int_0^1 |f(t)|^p\, dt\right)^{1/p}$ with $p \in (0,\infty)$ is a functional D-norm.

Lemma 2.6. *Let* $\boldsymbol{\eta}$ *be a standard EVP with generator* $\boldsymbol{Z}$*. Then we have for each* $f \in \bar{E}^-[0,1]$

$$P(\boldsymbol{\eta} \leq f) = \exp\left(-\|f\|_D\right) = \exp\left(-E\left(\sup_{t\in[0,1]}(|f(t)|\, Z_t)\right)\right). \tag{2.11}$$

Conversely, if there is some $\boldsymbol{Z}$ *with properties* (2.6) *and some* $\boldsymbol{\eta} \in C^-[0,1]$ *which satisfies* (2.11)*, then* $\boldsymbol{\eta}$ *is standard max-stable with generator* $\boldsymbol{Z}$.

Proof. Let $Q = \{q_1, q_2 \dots\}$ be a denumerable and dense subset of $[0,1]$ which contains the finitely many points, at which $f \in \bar{E}^-[0,1]$ has a discontinuity. Hence, the set $\{\boldsymbol{\eta} \le f\}$ can be identified as

$$\{\boldsymbol{\eta}(t) \le f(t),\ t \in [0,1]\} = \bigcap_{d\in\mathbb{N}} \left\{\eta_{q_j} \le f(q_j),\, 1 \le j \le d\right\}.$$

As the finite dimensional sets are part of the Borel-σ-Algebra on $C[0,1]$, equipped with the sup-norm, the right hand side is a measurable set and, thus, $\{\boldsymbol{\eta} \le f\}$ is also measurable.

Moreover, we obtain from the continuity of $\boldsymbol{\eta}$, the continuity from above of each probability measure and equation (2.9)

$$\begin{aligned}
P(\boldsymbol{\eta} \le f) &= P\left(\bigcap_{d\in\mathbb{N}} \left\{\eta_{q_j} \le f(q_j),\, 1 \le j \le d\right\}\right)\\
&= \lim_{d\to\infty} P\left(\eta_{q_j} \le f(q_j),\, 1 \le j \le d\right)\\
&= \lim_{d\to\infty} \exp\left(-E\left(\max_{1\le j\le d}\left(|f(q_j)|\, Z_{q_j}\right)\right)\right)\\
&= \exp\left(-\lim_{d\to\infty} E\left(\max_{1\le j\le d}\left(|f(q_j)|\, Z_{q_j}\right)\right)\right)\\
&= \exp\left(-E\left(\lim_{d\to\infty}\max_{1\le j\le d}\left(|f(q_j)|\, Z_{q_j}\right)\right)\right)\\
&= \exp\left(-E\left(\sup_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right)\right)\\
&= \exp\left(-\left\|f\right\|_D\right),
\end{aligned}$$

where the third to last equation follows from the dominated convergence theorem.

If some $\boldsymbol{Z}$ has properties (2.6) it gives rise to some standard max-stable process $\widehat{\boldsymbol{\eta}}$ due to Proposition 2.4. But the fidis of $\widehat{\boldsymbol{\eta}}$ given by (2.8) and those of $\boldsymbol{\eta}$ given by (2.11) coincide, so $\widehat{\boldsymbol{\eta}} =_D \boldsymbol{\eta}$ follows. □

The representation

$$P(\boldsymbol{\eta} \le f) = \exp\left(-\left\|f\right\|_D\right), \qquad f \in \bar{E}^-[0,1],$$

of a standard EVP is in complete accordance with the df of a multivariate EVD with standard negative exponential margins via a D-norm on $\mathbb{R}^d$ as developed in Falk et al. [11, Section 4.4].

The extension to $f \in \bar{E}^-[0,1]$ allows the incorporation of the fidis of $\boldsymbol{\eta}$ into the preceding representation: Choose indices $0 \le t_1 < \cdots < t_d \le 1$ and numbers $x_i < 0$, $1 \le i \le d$, $d \in \mathbb{N}$. The function

$$f(t) = \sum_{i=1}^d x_i 1_{\{t_i\}}(t)$$

is an element of $\bar{E}^{-}[0,1]$ with the property

$$\begin{aligned}P(\boldsymbol{\eta} \le f) &= \exp\left(-\|f\|_D\right)\\ &= \exp\left(-E\left(\sup_{t\in[0,1]}\left(|f(t)|\,Z_t\right)\right)\right)\\ &= \exp\left(-E\left(\max_{1\le i\le d}\left(|x_i|\,Z_{t_i}\right)\right)\right)\\ &= \exp\left(-\|\boldsymbol{x}\|_{D_{t_1,\dots,t_d}}\right).\end{aligned}$$

We can now, for example, extend Takahashi's [17] characterization of the maximum-norm in $\mathbb{R}^d$ to the functional space $E[0,1]$.

Lemma 2.7 (Functional Takahashi)**.** *Let $\|\cdot\|_D$ be an arbitrary D-norm on $E[0,1]$ with generator $\boldsymbol{Z}$. Then*

$$\begin{aligned}&\|f\|_D = \|f\|_\infty \text{ for at least one } f \in E[0,1] \text{ with } f(t) \ne 0,\ t \in [0,1]\\ &\iff \|\cdot\|_D = \|\cdot\|_\infty \text{ on } E[0,1].\end{aligned}$$

Proof. Let $f \in E[0,1]$ have the property $\|f\|_D = \|f\|_\infty$. Suppose first that $\|f\|_\infty$ is attained on $[0,1]$, i.e., there exists $t_0 \in [0,1]$ such that $|f(t_0)| = \sup_{t\in[0,1]}|f(t)|$. Then we obtain for arbitrary indices $0 \le t_1 < \dots < t_d \le 1$

$$|f(t_0)|\,Z_{t_0} \le \max_{i=0,1,\dots,d}\left(|f(t_i)|\,Z_{t_i}\right) \le \sup_{t\in[0,1]}\left(|f(t)|\,Z_t\right)$$

and, thus,

$$\begin{aligned}\|f\|_\infty &= E\left(|f(t_0)|\,Z_{t_0}\right)\\ &\le E\left(\max_{i=0,1,\dots,d}\left(|f(t_i)|\,Z_{t_i}\right)\right)\\ &= \|(f(t_0),\dots,f(t_d))\|_{D_{t_0,\dots,t_d}}\\ &\le E\left(\sup_{t\in[0,1]}\left(|f(t)|\,Z_t\right)\right)\\ &= \|f\|_D\\ &= \|f\|_\infty,\end{aligned}$$

i.e.,

$$\|(f(t_0),\dots,f(t_d))\|_{D_{t_0,\dots,t_d}} = \|(f(t_0),\dots,f(t_d))\|_\infty.$$

Takahashi's Theorem [17] for the finite-dimensional Euclidean space now implies

$$\|\boldsymbol{x}\|_{D_{t_0,\dots,t_d}} = \|\boldsymbol{x}\|_\infty, \qquad \boldsymbol{x} \in \mathbb{R}^{d+1}, \tag{2.12}$$

for arbitrary $0 \le t_1 < \dots < t_d \le 1$, $d \in \mathbb{N}$. This, in turn implies that $Z_t = Z_{t_0}$, $t \in [0,1]$, a.s., which can be seen as follows. Choose f the constant function 1 and let $\mathbb{Q} \cap [0,1] = \{t_1, t_2, \dots\}$. Then we have by equation (2.12) for arbitrary $s \in \{t_1, t_2, \dots\}$ if d is large

$$1 = E(Z_s) \le E\left(\max_{i=0,\dots,d} Z_{t_i}\right) = 1$$

and, thus, by the dominated convergence theorem and the continuity of $(Z_t)_{t\in[0,1]}$

$$1 = E(Z_s) \le E\left(\sup_{t\in[0,1]} Z_t\right) = 1.$$

But this implies $0 = E\left(\sup_{t\in[0,1]} Z_t - Z_s\right)$ and, hence, $Z_s = \sup_{t\in[0,1]} Z_t$ a.s., which yields $Z_t = Z_{t_0}$, $t \in [0,1]$, a.s. by the continuity of the process $\boldsymbol{Z}$. This implies $\|f\|_D = \|f\|_\infty$, $f \in E[0,1]$.

Suppose next that $\|f\|_\infty$ is not attained. Then there exists a sequence of indices t_n, $n \in \mathbb{N}$, in $[0,1]$ with $t_n \to_{n\to\infty} t_0 \in [0,1]$ and $\lim_{n\to\infty} |f(t_n)| = \|f\|_\infty$. From the continuity of the process $\boldsymbol{Z}$ we obtain

$$\|f\|_\infty Z_{t_0} = \lim_{n\to\infty} |f(t_n)|\, Z_{t_n}$$

and, thus,

$$\|f\|_\infty Z_{t_0} \le \sup_{t\in[0,1]} \left(|f(t)|\, Z_t\right).$$

Choose arbitrary indices $0 \le t_1 < \dots, t_d \le 1$. Then

$$\begin{aligned}
\|f\|_\infty &= E\left(\|f\|_\infty Z_{t_0}\right)\\
&\le E\left(\max\left\{\|f\|_\infty Z_{t_0}, |f(t_1)|\, Z_{t_1}, \dots, |f(t_d)|\, Z_{t_d}\right\}\right)\\
&\le E\left(\sup_{t\in[0,1]} \left(|f(t)|\, Z_t\right)\right)\\
&= \|f\|_\infty .
\end{aligned}$$

From Takahashi's Theorem [17] we now deduce that

$$E\left(\max_{i=0,\dots,n} \left(|f(t_i)|\, Z_{t_i}\right)\right) = \|f\|_\infty$$

for arbitrary $f \in E[0,1]$. Concluding as above yields the assertion. □

Just like in the uni- or multivariate case, we might consider

$$H(f) := P(\boldsymbol{\zeta} \le f), \qquad f \in \bar{E}^-[0,1],$$

as the df of a stochastic process $\boldsymbol{\zeta}$ in $\bar{C}^-[0,1] = \{f \in C[0,1] : f(t) \le 0,\ t \in [0,1]\}$. Note that the df $G(f) = \exp\left(-\|f\|_D\right)$, $f \in \bar{E}^-[0,1]$, of an EVP $\boldsymbol{\eta}$ is continuous with respect to the sup-norm, i.e.,

$$G(f_n) \to G(f) \quad \text{as} \quad \|f_n - f\|_\infty \to 0, \qquad f_n, f \in \bar{E}^-[0,1],$$

cf. property (2.10).

2.2. **Transformation to Arbitrary Margins.** Next we recall that the characterization in Proposition 2.4 is sufficient to cover all max-stable processes in $C[0,1]$. To this end we have to make sure that properly normed max-stable processes never take the value zero with probability one. This assertion was already shown by Giné et al. [13] using the theory of random closed sets. We state this result in our setup, as we use only elementary probabilistic arguments for its proof.

Lemma 2.8. *Let $\boldsymbol{\zeta}$ be an arbitrary max-stable process in $C[0,1]$ and $a > 0$, b, γ the continuous functions which fulfill*

$$P\left(\frac{\boldsymbol{\zeta}(t) - b(t)}{a(t)} \le x\right) = F_{\gamma(t)}(x), \qquad t \in [0,1],$$

cf. equations (2.2)*,* (2.3) *and Lemma 2.1. Then the process* $\tilde{\boldsymbol{\zeta}} := 1 + \frac{\gamma}{a}(\boldsymbol{\zeta} - b) \in C[0,1]$ *will not take the value zero with probability one, i.e.*

$$P(\tilde{\boldsymbol{\zeta}}(t) \neq 0 \textit{ for all } t \in [0,1]) = 1$$

Proof. First observe that we have for every $t \in [0,1]$

$$P(\tilde{\boldsymbol{\zeta}}(t) \geq 0) = \left\{ \begin{array}{ll} 1 - P\big((\boldsymbol{\zeta}(t) - b(t))/a(t) \leq -1/\gamma(t)\big) & \text{if } \gamma(t) > 0 \\ P\big(1 \geq 0\big) & \text{if } \gamma(t) = 0 \\ P\big((\boldsymbol{\zeta}(t) - b(t))/a(t) \leq -1/\gamma(t)\big) & \text{if } \gamma(t) < 0 \end{array} \right\} = 1$$

because of (2.2) and the explanation thereafter.

Now define

$$K_1 := \{t \in [0,1] : \gamma(t) \geq 0\}, \quad K_2 := \{t \in [0,1] : \gamma(t) \leq 0\},$$

which are compact subsets of $[0,1]$ by the continuity of γ. Because of

$$P(\boldsymbol{\zeta}(t) = 0 \text{ for some } t \in [0,1]) \leq \sum_{i=1}^{2} P(\boldsymbol{\zeta}(t) = 0 \text{ for some } t \in K_i)$$

the assertion is proved, if both summands on the right hand side vanish. Define for $t \in K_2$

$$\widehat{\boldsymbol{\eta}}(t) := -\left(1 + \frac{\gamma(t)}{a(t)}(\boldsymbol{\zeta}(t) - b(t))\right)^{-1/\gamma(t)},$$

whereas for $\gamma(t) = 0$ this is meant to be $\widehat{\boldsymbol{\eta}}(t) = -\exp(-(\boldsymbol{\zeta}(t) - b(t))/a(t))$.

Then $\widehat{\boldsymbol{\eta}}$ is a well-defined max-stable process on K_2 with standard negative exponential margins, which can be seen by elementary computations. Thus, Lemma 2.2 gives $P(\boldsymbol{\zeta}(t) = 0 \text{ for some } t \in K_2) = 0$.

Now consider for $t \in K_1$ the process

$$\widetilde{\boldsymbol{\xi}}(t) := \left(1 + \frac{\gamma(t)}{a(t)}(\boldsymbol{\zeta}(t) - b(t))\right)^{1/\gamma(t)}$$

and, as above, $\widetilde{\boldsymbol{\xi}}(t) = \exp((\zeta(t) - b(t))/a(t))$ for $\gamma(t) = 0$.

Then $\widetilde{\boldsymbol{\xi}}$ is well-defined max-stable process on K_2 with standard Fréchet margins, which can be seen by analogue computations as before. So $P(\widetilde{\boldsymbol{\xi}} \neq 0) = 1$ by Lemma 2.3 and the result follows. □

Proposition 2.9. *Let* $\boldsymbol{\zeta}$ *an arbitrary max-stable process in* $C[0,1]$ *and* $a > 0$*,* b, γ *the continuous functions which satisfy*

$$P\left(\frac{\boldsymbol{\zeta}(t) - b(t)}{a(t)} \leq x\right) = F_{\gamma(t)}(x), \qquad t \in [0,1],$$

cf. equations (2.2)*,* (2.3) *and Lemma 2.1. Define*

$$\boldsymbol{\eta}(t) := \left\{ \begin{array}{ll} -\left(1 + \frac{\gamma(t)}{a(t)}(\boldsymbol{\zeta}(t) - b(t))\right)^{-1/\gamma(t)} & \textit{for } \gamma(t) \neq 0 \\ -\exp\left(-(\boldsymbol{\zeta}(t) - b(t))/a(t)\right) & \textit{for } \gamma(t) = 0. \end{array} \right. \tag{2.13}$$

Then $\boldsymbol{\eta}$ *is a* standard max-stable *process in* $C[0,1]$*.*

Proof. Because of Lemma 2.8 the process $\boldsymbol{\eta}$ is well defined and it is continuous because of Lemma 2.1. Moreover, elementary computations show that the one-dimensional distributions are negative exponential distributions and that the fidis of $n \max_{1\le i\le n} \boldsymbol{\eta}_i$ for iid copies $\boldsymbol{\eta}_i$ of $\boldsymbol{\eta}$, $n \in \mathbb{N}$, are the same as those of $\boldsymbol{\eta}$, so the process is standard max-stable. □

By inverting equation (2.13) we get for an arbitrary max-stable process $\boldsymbol{\zeta} \in C[0,1]$, coming along with its norming functions $a > 0$, b, γ as before, some standard max-stable process $\boldsymbol{\eta} \in C^-[0,1]$ with

$$\boldsymbol{\zeta}(t) := \begin{cases} \frac{-a(t)}{\gamma(t)}\left(1-(-\boldsymbol{\eta}(t))^{-\gamma(t)}\right)+b(t) & \text{for } \gamma(t)\neq 0 \\ -a(t)\log\left(-\boldsymbol{\eta}(t)\right)+b(t) & \text{for } \gamma(t)=0. \end{cases}$$

for $t \in [0,1]$.

Finally, the functional df of an arbitrary max-stable process $\boldsymbol{\zeta} \in C[0,1]$ can be written by means of the D-norm: Let $a > 0$, b, γ be the norming functions of $\boldsymbol{\zeta}$ as before and $\boldsymbol{\eta} \in C^-[0,1]$ as in (2.2). Note that we have for every $t \in [0,1]$ with $\gamma(t) \neq 0$

$$P(\boldsymbol{\zeta}(t) \le f(t)) \in (0,1) \iff \frac{\gamma(t)}{a(t)}(f(t)-b(t)) > -1 \tag{2.14}$$

due to (2.2). Thus, we get for every $f \in E[0,1]$ satisfying the inequality in (2.14) for every $t \in [0,1]$:

$$\begin{aligned} P(\boldsymbol{\zeta}\le f) &= P\left(\boldsymbol{\eta}(t) \le -\left(1+\frac{\gamma(t)}{a(t)}(f(t)-b(t))\right)^{-1/\gamma(t)},\ t\in[0,1]\right) \\ &= \exp\left(-\left\|\left(1+\frac{\gamma}{a}(f-b)\right)^{-1/\gamma}\right\|_D\right), \end{aligned}$$

where for $\gamma(t) = 0$ this is as usual meant to be

$$P(\boldsymbol{\zeta} \le f) = \exp\left(-\left\|\exp\left(-(f-b)/a\right)\right\|_D\right).$$

3. Functional Domain of Attraction

We say that a stochastic process $\boldsymbol{Y} \in C[0,1]$ is *in the functional domain of attraction* of a standard EVP $\boldsymbol{\eta}$, denoted by $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$, if there are functions $a_n \in C^+[0,1] := \{f \in C[0,1] : f > 0\}$, $b_n \in C[0,1]$, $n \in \mathbb{N}$, such that

$$\lim_{n\to\infty} P\left(\frac{\boldsymbol{Y}-b_n}{a_n} \le f\right)^n = P(\boldsymbol{\eta}\le f) = \exp\left(-\|f\|_D\right) \tag{FuDA}$$

for any $f \in \bar{E}^-[0,1]$. Note that this condition is equivalent with

$$\lim_{n\to\infty} P\left(\max_{1\le i\le n} \frac{\boldsymbol{Y}_i-b_n}{a_n} \le f\right) = P(\boldsymbol{\eta}\le f) \tag{FuDA'}$$

for any $f \in \bar{E}^-[0,1]$, where $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots$ are independent copies of $\boldsymbol{Y}$.

There should be no risk of confusion with the notation of domain of attraction in the sense of weak convergence of stochastic processes as investigated in de Haan and Lin [7]. But to distinguish between these two approaches we will consistently speak of *functional* domain of attraction in this paper, if the above definition is meant. Actually, this definition of domain of attraction is less restrictive as the next lemma shows.

Lemma 3.1. *Suppose that* $\boldsymbol{Y} \in \bar{C}^-[0,1]$ *and let* $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots$ *be independent copies of* $\boldsymbol{Y}$. *If the sequence* $\boldsymbol{X}_n := \max_{1\le i\le n}((\boldsymbol{Y}_i - b_n)/a_n))$ *of continuous processes converges weakly in* $\bar{C}^-[0,1]$, *equipped with the sup-norm* $\|\cdot\|_\infty$, *to the standard EVP* $\boldsymbol{\eta}$, *then* $\boldsymbol{Y} \in \mathcal{D}(\boldsymbol{\eta})$.

Proof. The set $F := \left\{g \in \bar{C}^-[0,1] : g \le f\right\}$ is for any $f \in \bar{E}^-[0,1]$ a closed set in the space $\left(\bar{C}^-[0,1], \|\cdot\|_\infty\right)$, whereas $G := \bigcup_{k\in\mathbb{N}}\{g \in \bar{C}^-[0,1] : g \le f - 1/k\}$ is an open set.

From the Portmanteau Theorem (see, e.g., Billingsley [3]), we obtain

$$\begin{aligned}
&P\left(\boldsymbol{\eta} \in \bigcup_{k\in\mathbb{N}} \left\{g \in \bar{C}^-[0,1] : g \le f - 1/k\right\}\right)\\
&\le \liminf_{n\in\mathbb{N}} P\left(\boldsymbol{X}_n \in \bigcup_{k\in\mathbb{N}} \left\{g \in \bar{C}^-[0,1] : g \le f - 1/k\right\}\right)\\
&\le \liminf_{n\in\mathbb{N}} P\left(\boldsymbol{X}_n \le f\right)\\
&\le \limsup_{n\in\mathbb{N}} P\left(\boldsymbol{X}_n \le f\right)\\
&\le P(\boldsymbol{\eta} \le f).
\end{aligned}$$

The continuity from below of an arbitrary probability measure together with the dominated convergence imply

$$\begin{aligned}
&P\left(\boldsymbol{\eta} \in \bigcup_{k\in\mathbb{N}} \left\{g \in \bar{C}^-[0,1] : g \le f - 1/k\right\}\right)\\
&= \lim_{k\to\infty} P\left(\boldsymbol{\eta} \le f - \frac{1}{k}\right)\\
&= \lim_{k\to\infty} \exp\left(-\left\|f - \frac{1}{k}\right\|_D\right)\\
&= \lim_{k\to\infty} \exp\left(-E\left(\sup_{t\in[0,1]}\left(\left|f(t) - \frac{1}{k}\right| Z_t\right)\right)\right)\\
&= \exp\left(-E\left(\sup_{t\in[0,1]}\left(|f(t)|\, Z_t\right)\right)\right)\\
&= \exp\left(-\|f\|_D\right)\\
&= P(\boldsymbol{\eta} \le f),
\end{aligned}$$

which yields $\lim_{n\to\infty} P(\boldsymbol{X}_n \le f) = P(\boldsymbol{\eta} \le f)$ and, thus, condition (FuDA'). □

Examples of continuous processes in $\bar{C}^-[0,1]$, whose properly normed maxima of iid copies converge in distribution to an EVP and which obviously satisfy condition (FuDA), are the *GPD-processes* introduced by Buishand et al. [4]. We consider these generalized Pareto processes in Section 4 below.

3.1. **Domain of Attraction for Copula Processes.** Let $\boldsymbol{Y} = (Y_t)_{t\in[0,1]} \in C[0,1]$ be a stochastic process with identical continuous marginal df F. Set

$$\boldsymbol{U} = (U_t)_{t\in[0,1]} := (F(Y_t))_{t\in[0,1]}, \tag{3.1}$$

which is the *copula process* corresponding to $\boldsymbol{Y}$. Note that each onedimensional marginal distribution of $\boldsymbol{U}$ is the uniform distribution on $[0,1]$.

We conclude from de Haan and Lin [7] that the process $\boldsymbol{Y}$ is in the domain of attraction of an EVP if, and only if each Y_t is in the domain of attraction of a univariate extreme value distribution together with the condition that the copula process converges in distribution to a standard EVP $\boldsymbol{\eta}$, that is

$$\left(\max_{1\le i\le n} n(U_t^{(i)}-1)\right)_{t\in[0,1]} \to_D \boldsymbol{\eta}$$

in $C[0,1]$, where $\boldsymbol{U}^{(i)}$, $i\in\mathbb{N}$, are independent copies of $\boldsymbol{U}$. Note that the univariate margins determine the norming constants, so the norming functions are necessarily the constant functions $a_n = 1/n$, $b_n = 1$, $n\in\mathbb{N}$.

Suppose that the copula process corresponding to $\boldsymbol{Y}$ is in the functional domain of attraction of a standard EVP $\boldsymbol{\eta}$, representable as in Proposition 2.4. Then we know from Aulbach et al. [1] that for $d\in\mathbb{N}$ the copula C_d corresponding to the rv $(Y_{i/d})_{i=1}^d$ satisfies the equation

$$C_d(\boldsymbol{y}) = 1 - \|\mathbf{1}-\boldsymbol{y}\|_{D_d} + o\left(\|\mathbf{1}-\boldsymbol{y}\|_\infty\right), \tag{3.2}$$

as $\|\mathbf{1}-\boldsymbol{y}\|_\infty \to \mathbf{0}$, uniformly in $\boldsymbol{y}\in[0,1]^d$, where the D-norm is given by

$$\|\boldsymbol{x}\|_{D_d} = E\left(\max_{1\le i\le d}\left(|x_i|\, Z_{i/d}\right)\right), \qquad \boldsymbol{x}\in\mathbb{R}^d.$$

We are going to establish an analogous result for the functional domain of attraction.

Let $\boldsymbol{\eta}$ be a standard EVP with functional df G, and let $\boldsymbol{Y}$ be an arbitrary stochastic process in $C[0,1]$. By taking logarithms, we obtain the following equivalences with some norming functions $a_n\in C^+[0,1]$, $b_n\in C[0,1]$, $n\in\mathbb{N}$:

$$\begin{aligned}
&\boldsymbol{Y}\in\mathcal{D}(\boldsymbol{\eta}) \text{ in the sense of condition (FuDA)}\\
&\iff P\left(\frac{\boldsymbol{Y}-b_n}{a_n}\le f\right)^n = \exp\left(-\|f\|_D\right) + o(1),\ f\in\bar{E}^-[0,1],\ \text{as } n\to\infty,\\
&\iff P\left(\frac{\boldsymbol{Y}-b_n}{a_n}\le f\right) = 1-\frac{1}{n}\|f\|_D + o\left(\frac{1}{n}\right),\ f\in\bar{E}^-[0,1],\ \text{as } n\to\infty.
\end{aligned}$$

Let $\boldsymbol{U}\in C[0,1]$ be a copula-process as defined in (3.1) above and set $H_f(t) := P(\boldsymbol{U}-1\le t\,|f|)$, $t\le 0$, $f\in\bar{E}^-[0,1]$. Note that $H_f(\cdot)$ defines a univariate df on $(-\infty,0]$. The family $\mathcal{P} := \left\{H_f : f\in\bar{E}^-[0,1]\right\}$ of univariate df is the *spectral decomposition* of the df $H(f) = P(\boldsymbol{U}-1\le f)$, $f\in\bar{E}^-[0,1]$ of $\boldsymbol{U}-1$. This extends the spectral decomposition of a multivariate df in Falk et al. [11, Section 5.4]. Standard arguments yield the next result.

Proposition 3.2. *The following equivalences hold:*

$$\begin{aligned} &\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta}) \text{ in the sense of condition (FuDA)} \\ &\Leftrightarrow \quad P\left(\boldsymbol{U} - 1 \le \frac{f}{n}\right) = 1 - \left\|\frac{f}{n}\right\|_D + o\left(\frac{1}{n}\right),\ f \in \bar{E}^-[0,1], \text{ as } n \to \infty, \\ &\Leftrightarrow \quad H_f(t) = 1 + t\,\|f\|_D + o(t),\ f \in \bar{E}^-[0,1], \text{ as } t \uparrow 0, \end{aligned} \tag{3.3}$$

Remark 3.3. Characterization (3.3) entails in particular, that $H_f(t)$ is differentiable from the left in $t = 0$ with derivative $h_f(0) := \frac{d}{dt} H_f(t)|_{t=0} = \|f\|_D$, $f \in \bar{E}^-[0,1]$.

Remark 3.4. A sufficient condition for $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$ is given by

$$P(\boldsymbol{U} - 1 \le g) = 1 - \|g\|_D + o(\|g\|_\infty) \tag{3.3'}$$

as $\|g\|_\infty \to 0$, uniformly for all $g \in \bar{E}^-[0,1]$ with $\|g\|_\infty \le 1$, i.e., for all g in the unit ball of $\bar{E}^-[0,1]$.

Take, for example, $\boldsymbol{U} = \exp(\boldsymbol{\eta})$. Then $\boldsymbol{U}$ is a copula process, and we obtain uniformly for $g \in \bar{E}^-[0,1]$ with $\|g\|_\infty \le 1 - \varepsilon$ by using the approximation $\log(1 + x) = x + O\left(x^2\right)$ as $x \to 0$

$$\begin{aligned} P(\boldsymbol{U} - 1 \le g) &= P(\boldsymbol{\eta} \le \log(1 + g)) \\ &= \exp\left(-E\left(\sup_{t \in [0,1]} \left(|\log(1 + g(t))|\, Z_t\right)\right)\right) \\ &= \exp\left(-E\left(\sup_{t \in [0,1]} \left(\left|g(t) + O\left(g(t)^2\right)\right|\, Z_t\right)\right)\right) \\ &= \exp\left(-E\left(\sup_{t \in [0,1]} \left(|g(t)|\, Z_t\right)\right) + O\left(\|g\|_\infty^2\right)\right) \\ &= 1 - \|g\|_D + O\left(\|g\|_\infty^2\right), \end{aligned} \tag{3.4}$$

i.e., the copula process $\boldsymbol{U} = \exp(\boldsymbol{\eta})$ satisfies condition (3.3').

4. Functional GPD

A univariate GPD W is simply given by $W(x) = 1 + \log(G(x))$, $G(x) \ge 1/e$, where G is a univariate EVD. It was established by Pickands [15] and Balkema and de Haan [2] that the maximum of n iid univariate observations, linearly standardized, converges in distribution to an EVD as n increases if, and only if, the exceedances above an increasing threshold follow a generalized Pareto distribution (GPD). The multivariate analogon is due to Rootzén and Tajvidi [16]. It was observed by Buishand et al. [4] that a d-dimensional GPD W with ultimately standard Pareto margins can be represented in its upper tail as $W(\boldsymbol{x}) = P(U^{-1}\boldsymbol{Z} \le \boldsymbol{x})$, $\boldsymbol{x}_0 \le \boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, where the rv U is uniformly on $(0,1)$ distributed and independent of the rv $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ with $0 \le Z_i \le c$ for some $c \ge 1$ and $E(Z_i) = 1$, $1 \le i \le d$. We extend this approach to functional spaces. For a recent account of multivariate EVT and GPD we refer to Falk et al. [11].

Definition 4.1. Let U be a uniformly on $[0,1]$ distributed rv, which is independent of a generator process $\boldsymbol{Z} \in \bar{C}^+[0,1]$ with properties (2.6). Then the stochastic process

$$\boldsymbol{Y} := \frac{1}{U}\boldsymbol{Z} \in \bar{C}^+[0,1].$$

is called a *GPD-process* (cf. Buishand et al. [4]).

The onedimensional margins Y_t of $\boldsymbol{Y}$ have ultimately standard Pareto tails:

$$\begin{aligned} P(Y_t \le x) &= P\left(\frac{1}{x}Z_t \le U\right) \\ &= \int_0^m P\left(\frac{1}{x}z \le U\right)(P * Z_t)(dz) \\ &= 1 - \frac{1}{x}\int_0^m z\,(P * Z_t)(dz) \\ &= 1 - \frac{1}{x}E(Z_t) \\ &= 1 - \frac{1}{x}, \qquad x \ge m,\ 0 \le t \le 1. \end{aligned}$$

Put $\boldsymbol{V} := -1/\boldsymbol{Y}$. Then, by Fubini's Theorem,

$$\begin{aligned} P(\boldsymbol{V} \le f) &= P\left(\sup_{t\in[0,1]}(|f(t)|\,Z_t) \le U\right) \\ &= 1 - \int_0^1 P\left(\sup_{t\in[0,1]}(|f(t)|\,Z_t) > u\right) du \\ &= 1 - E\left(\sup_{t\in[0,1]}(|f(t)|\,Z_t)\right) \\ &= 1 - \|f\|_D \end{aligned}$$

for all $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le 1/m$, i.e., $\boldsymbol{V}$ has the property that its distribution function is in its upper tail equal to

$$\begin{aligned} W(f) &:= P(\boldsymbol{V} \le f) \\ &= 1 - \|f\|_D \\ &= 1 + \log\left(\exp\left(-\|f\|_D\right)\right) \\ &= 1 + \log(G(f)), \qquad f \in \bar{E}^-[0,1],\ \|f\|_\infty \le 1/m, \end{aligned} \tag{4.1}$$

where $G(f) = P(\boldsymbol{\eta} \le f)$ is the functional df of the EVP $\boldsymbol{\eta}$ with D-norm $\|\cdot\|_D$ and generator $\boldsymbol{Z}$.

The preceding representation of the upper tail of a functional GPD in terms of $1 + \log(G)$ is in complete accordance with the unit- and multivariate case (see, for example, Falk et al. [11, Chapter 5]). We write $W = 1 + \log(G)$ in short notation and call $\boldsymbol{V}$ a GPD-process as well.

Remark 4.2. Due to representation (4.1), the GPD process $\boldsymbol{V}$ is clearly in the functional domain of attraction of the standard EVP $\boldsymbol{\eta}$ with D-norm $\|\cdot\|_D$ and generator $\boldsymbol{Z}$ (in the sense of equation (FuDA); take $a_n \equiv 1/n$ and $b_n \equiv 0$).

Remark 4.3. As already mentioned by Buishand et. al [4], the GPD-process $\boldsymbol{Y}$ is in the domain of attraction of a *simple* max-stable process $\boldsymbol{\xi}$ in the sense of weak convergence on $C[0,1]$: for $\boldsymbol{Y}_1, \boldsymbol{Y}_2, \dots$ independent copies of $\boldsymbol{Y}$ we have

$$\frac{1}{n} \max_{1\le i\le n} \boldsymbol{Y}_i \ \to_D \boldsymbol{\xi} \qquad \text{in } C[0,1].$$

We deduce in particular a functional version of the well-known fact that the spectral df of a GPD random vector is equal to a uniform df in a neighborhood of 0.

Lemma 4.4. *We have for $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le m$ and some $t_0 < 0$*

$$W_f(t) := P\left(\boldsymbol{V} \le t\,|f|\right) = 1 + t\,\|f\|_D\,, \qquad t_0 \le t \le 0.$$

Let $\boldsymbol{U}$ be a copula process. Then the following variant of Proposition 3.2 holds.

Proposition 4.5. *The property $\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$ in the sense of condition* (FuDA) *is equivalent to*

$$\lim_{t\uparrow 0} \frac{1 - H_f(t)}{1 - W_f(t)} = 1, \qquad f \in \bar{E}^-[0,1], \tag{4.2}$$

i.e., the spectral df $H_f(t) = P(\boldsymbol{U} - 1 \le t\,|f|)$, $t \le 0$, of $\boldsymbol{U} - 1$ is tail equivalent with that of the GPD $W = 1 + \log(G)$.

We finish this section by defining a standard generalized Pareto process we are working with in the sequel.

Definition 4.6. A stochastic process $\boldsymbol{V} \in \bar{C}^-[0,1]$ is a *standard generalized Pareto process* (GPP), if there exists a D-norm $\|\cdot\|_D$ on $E[0,1]$ and some $c_0 > 0$ such that

$$P(\boldsymbol{V} \le f) = 1 - \|f\|_D$$

for all $f \in \bar{E}^-[0,1]$ with $\|f\|_\infty \le c_0$.

5. Spectral δ-Neighborhood of a Standard GPP

Using the spectral decomposition of a stochastic process in $\bar{C}^-[0,1]$, we can easily extend the definition of a spectral δ-neighborhood of a multivariate GPD as in Falk et al. [11, Section 5.5] to the spectral δ-neighborhood of a standard GPP. Denote by $\bar{E}_1^-[0,1] = \left\{f \in \bar{E}^-[0,1] : \|f\|_\infty = 1\right\}$ the unit sphere in $\bar{E}^-[0,1]$.

Definition 5.1. We say that a stochastic process $\boldsymbol{Y} \in \bar{C}^-[0,1]$ belongs to the *spectral δ-neighborhood* of the GPP $\boldsymbol{V}$ for some $\delta \in (0,1]$, if we have uniformly for $f \in \bar{E}_1^-[0,1]$ the expansion

$$\begin{aligned} 1 - P(\boldsymbol{Y} \le cf) &= (1 - P(\boldsymbol{V} \le cf)\left(1 + O\left(c^\delta\right)\right) \\ &= \|f\|_D \left(1 + O\left(c^\delta\right)\right) \end{aligned}$$

as $c \downarrow 0$.

A standard EVP is, for example, in the spectral δ-neighborhood of the corresponding GPP with $\delta = 1$, see expansion (3.4).

The following result on the rate of convergence extends Theorem 5.5.5 in Falk et al. [11] on the rate of convergence of multivariate extremes to functional extremes.

Proposition 5.2. *Let $\boldsymbol{Y}$ be a stochastic process in $\bar{C}^-[0,1]$, $\boldsymbol{V}$ a standard GPP with D-norm $\|\cdot\|_D$ and $\boldsymbol{\eta}$ a corresponding standard EVP.*

(i) *Suppose that $\boldsymbol{Y}$ is in the spectral δ-neighborhood of $\boldsymbol{V}$ for some $\delta \in (0,1]$. Then we have*

$$\sup_{f\in \bar{E}^-[0,1]} \left| P\left(\boldsymbol{Y} \le \frac{f}{n}\right)^n - P(\boldsymbol{\eta} \le f)\right| = O\left(n^{-\delta}\right).$$

(ii) *Suppose that $H_f(c) = P(\boldsymbol{Y} \le c\,|f|)$ is differentiable with respect to c in a left neighborhood of 0 for any $f \in \bar{E}_1^-[0,1]$, i.e., $h_f(c) := (\partial/\partial c)H_f(c)$ exists for $c \in (-\varepsilon, 0)$ and any $f \in \bar{E}_1^-[0,1]$. Suppose, moreover, that H_f satisfies the von Mises condition*

$$\frac{-ch_f(c)}{1-H_f(c)} =: 1 + r_f(c) \to_{c\uparrow 0} 1, \qquad f \in \bar{E}_1^-[0,1],$$

with remainder term r_f satisfying

$$\sup_{f\in \bar{E}_1^-[0,1]} \left|\int_c^0 \frac{r_f(t)}{t}\,dt\right| \to_{c\uparrow 0} 0.$$

If

$$\sup_{f\in \bar{E}^-[0,1]} \left| P\left(\boldsymbol{Y} \le \frac{f}{n}\right)^n - P(\boldsymbol{\eta} \le f)\right| = O\left(n^{-\delta}\right)$$

for some $\delta \in (0,1]$, then $\boldsymbol{Y}$ is in the spectral δ-neighborhood of the GPP $\boldsymbol{V}$.

Proof. Note that

$$\begin{aligned}
&\sup_{f\in \bar{E}^-[0,1]} \left| P\left(\boldsymbol{Y} \le \frac{f}{n}\right)^n - P(\boldsymbol{\eta} \le f)\right| \\
&= \sup_{f\in \bar{E}^-[0,1]} \left| P\left(\boldsymbol{Y} \le \frac{\|f\|_\infty}{n}\frac{f}{\|f\|_\infty}\right)^n - P\left(\boldsymbol{\eta} \le \frac{\|f\|_\infty}{n}\frac{f}{\|f\|_\infty}\right)\right| \\
&= \sup_{c<0}\sup_{f\in \bar{E}_1^-[0,1]} \left| P\left(\boldsymbol{Y} \le c\,|f|\right)^n - P\left(\boldsymbol{\eta} \le c\,|f|\right)\right| \\
&= \sup_{f\in \bar{E}_1^-[0,1]}\sup_{c<0} \left| P\left(\boldsymbol{Y} \le c\,|f|\right)^n - P\left(\boldsymbol{\eta} \le c\,|f|\right)\right|.
\end{aligned}$$

The assertion now follows by repeating the arguments in the proof of Theorem 1.1 in Falk and Reiss [10], where the bivariate case has been established. □

Acknowledgment

The second author gave an invited talk on an earlier version of this paper at the meeting 'On Some Current Research Topics in Extreme Value Theory', organized by the FCT/MCTES Research Project *Spatial Extremes* and CEAUL on December 13, 2010, University of Lisbon, Portugal. The authors are indebted to the organizers, Ana Ferreira and Laurens de Haan, for the constructive discussion from which the present version of the paper has benefitted a lot.

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